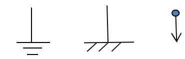
Chapter 3 Methods of Analysis: Nodal analysis and Mesh analysis

Basic concepts for nodal analysis:

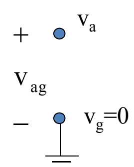
<u>Node voltage and ground</u>

Symbols for the ground:



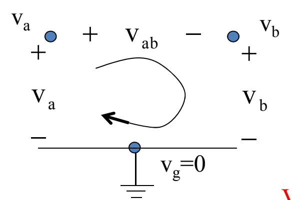
Definition: The <u>ground</u> for a circuit is defined as the point (or node), where the voltage = 0.

<u>Node voltage</u> at any other point is defined as the voltage drop from this point to the ground.



$v_a \triangleq v_{ag}$

With node voltage defined this way, the voltage drop between any two points is simply the difference of two node voltages.



By KVL, $v_{ab}+v_b-v_a=0$

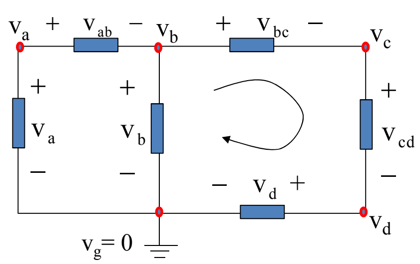
⇓

$v_{ab}= v_a-v_b$

Voltage at +

Voltage at −

KVL automatically satisfied:



$$v_{ab} = v_a - v_b$$
$$v_{bc} = v_b - v_c$$
$$v_{cd} = v_c - v_d$$

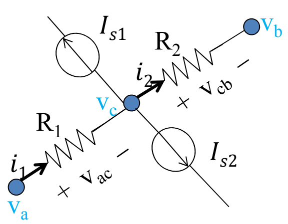
Along right loop: $v_{bc} + v_{cd} + v_d - v_b = v_b - v_c + v_c - v_d + v_d - v_b = 0$

All items cancelled. KVL automatically satisfied.
This is due to the assignment of node voltage and that the voltage between any two points is the difference of two node voltages.

Main Idea of nodal analysis:

Pick node voltages as key variables.
Express everything else in terms of node voltages,
such as branch voltage, currents.
Then apply KCL at the nodes to form equations for node voltages.

A typical node:



Express resistor currents in terms of node voltages

$$i_1 = \frac{v_{ac}}{R_1} = \frac{v_a - v_c}{R_1}; \quad i_2 = \frac{v_{cb}}{R_2} = \frac{v_c - v_b}{R_2}$$

KCL at center node:

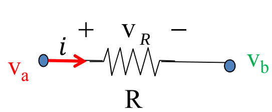
$$i_1 \; - \; i_2 \; - I_{s1} - I_{s2} = 0$$

$$\frac{v_a - v_c}{R_1} - \frac{v_c - v_b}{R_2} - I_{s1} - I_{s2} = 0$$

An equation for node voltages

One key step in nodal analysis:

Express the current through a resistor in terms of node voltages.



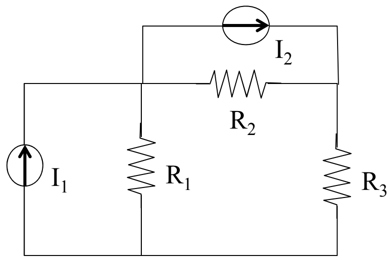
Assign v_R so that passive sign convention is satisfied. Then

$$i = \frac{v_R}{R} = \frac{v_a - v_b}{R}$$

Observation:
The current i enters at v_a and exits at v_b.
In the future, v_R will not be assigned.

General rule: $$i = \frac{v_{at\ entrance} - v_{at\ exit}}{\mathrm{R}}$$

Example: A circuit with only current sources. We need to find the current through each resistor.

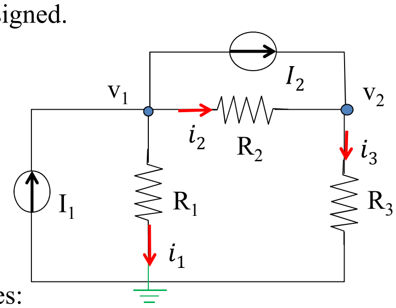


Solution: Additional variables need to be assigned.

How many nodes? only three nodes.

Step 1: pick the ground.
Assign node voltages, v_1, v_2.

Step 2: Assign resistor current with reference direction, i_1, i_2, i_3



Express the currents in terms of node voltages:

$$i_1 = \frac{v_1 - 0}{R_1} = \frac{v_1}{R_1}; \; i_2 = \frac{v_1 - v_2}{R_2}; \; i_3 = \frac{v_2}{R_3} \quad (*)$$

Step 3: Apply KCL at each node

At v_1: $-i_1 - i_2 + I_1 - I_2 = 0$

$\Rightarrow -i_1 - i_2 = I_2 - I_1 \quad (1)$

At v_2: $i_2 - i_3 + I_2 = 0$

$\Rightarrow i_2 - i_3 = -I_2 \quad (2)$

Plug (*) into (1), (2), respectively,

$$-\frac{v_1}{R_1} - \frac{v_1 - v_2}{R_2} = I_2 - I_1;$$

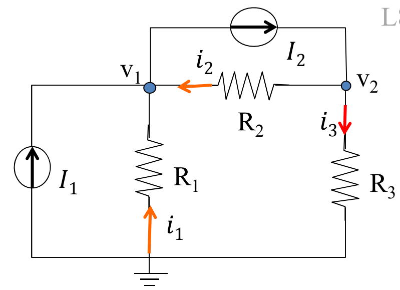
$$\frac{v_1 - v_2}{R_2} - \frac{v_2}{R_3} = -I_2$$

$$\begin{bmatrix} -\frac{1}{R_1} - \frac{1}{R_2} & \frac{1}{R_2} \\ \frac{1}{R_2} & -\frac{1}{R_2} - \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_2 - I_1 \\ -I_2 \end{bmatrix}$$

L8

The reference direction of currents can be arbitrarily assigned.

Reversing the direction does not change the equations for node voltages.
Let us reverse i_1 and i_2



Then the signs of i_1 and i_2 are reversed:

$$i_1 = \frac{0 - v_1}{R_1} = -\frac{v_1}{R_1}; \quad i_2 = \frac{+v_2 - v_1}{R_2} = -\frac{v_1 - v_2}{R_2}; \quad i_3 = \frac{v_2}{R_3} \qquad (*)$$

When apply KCL at each node, the sign for i_1, i_2 also reversed

At v_1: $+i_1 + i_2 + I_1 - I_2 \Rightarrow i_1 + i_2 = I_2 - I_1$ (1)

At v_2: $-i_2 - i_3 + I_2 = 0 \Rightarrow -i_2 - i_3 = -I_2$ (2)

Plug (*) into (1), (2), respectively, we obtain the same equations

$$-\frac{v_1}{R_1} - \frac{v_1 - v_2}{R_2} = I_2 - I_1;$$

$$\frac{v_1 - v_2}{R_2} - \frac{v_2}{R_3} = -I_2$$

L8

<u>Summary of steps in nodal analysis.</u>

Suppose that the circuit has n+1 nodes and has only current sources.

Step 1: pick the ground. Assign node voltages for the remaining n nodes: $v_1, v_2, \ldots, v_n$

Step 2: Assign resistor currents with reference direction, $i_1, i_2, \ldots$, Express them in terms of node voltages.

Step 3: Apply KCL at each of the node to obtain n equations for all branch currents. Then plug in the expressions from step 2, to obtain n equations for the n node voltages.

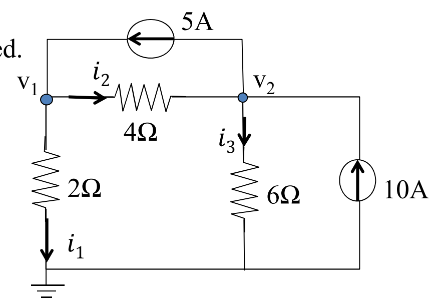
Step 4: clean up the equations, solve for the voltages $v_1, v_2, \ldots, v_n$ and compute whatever is asked. Since all variables can be expressed in terms of these node voltages.

For circuit with only current sources, every branch current is either given, or, can be expressed in terms of node voltages.

L8

Example: Find i_1, i_2, i_3 using nodal analysis

The ground and node voltages already assigned.



Express resistor currents:

$$i_1 = \frac{v_1}{2};\quad i_2 = \frac{v_1 - v_2}{4};\quad i_3 = \frac{v_2}{6} \qquad (*)$$

Apply KCL at the nodes:

At v_1: $\quad -i_1 - i_2 + 5 = 0 \qquad (1)$

At v_2: $\quad i_2 - i_3 + 10 - 5 = 0 \qquad (2)$

Plug (*) into (1), (2)

$$-\frac{v_1}{2} - \frac{v_1 - v_2}{4} + 5 = 0 \qquad \times 4$$

$$\frac{v_1 - v_2}{4} - \frac{v_2}{6} + 5 = 0 \qquad \times 12$$

$$-2v_1 - v_1 + v_2 + 20 = 0$$

$$3v_1 - 3v_2 - 2v_2 + 60 = 0$$

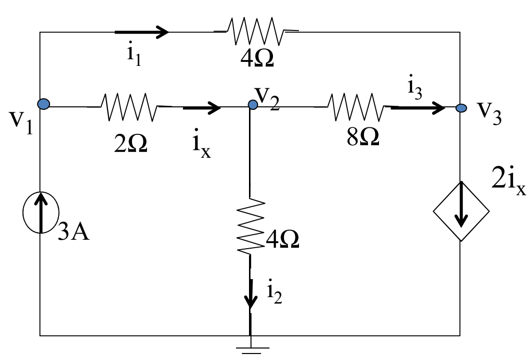
$$\boxed{3v_1 - v_2 = 20 \qquad (1a)}$$

$$\boxed{3v_1 - 5v_2 = -60 \qquad (2a)}$$

$$v_1 = \frac{40}{3}V;\ v_2 = 20V$$

$$i_1 = 6.667A;\quad i_2 = -1.667A;\ i_3 = 6.667A$$

Example: A circuit with dependent current source. Form 3 equations for v_1, v_2, v_3. L8



i_x has been assigned. Assign other resistor currents, i_1, i_2, i_3

Express all resistor currents:

$$i_x = \frac{v_1 - v_2}{2};\quad i_1 = \frac{v_1 - v_3}{4}$$

$$i_2 = \frac{v_2}{4};\qquad i_3 = \frac{v_2 - v_3}{8};$$

KCL at v_1: $\quad i_1 + i_x = 3$

KCL at v_2: $\quad i_x - i_2 - i_3 = 0$

KCL at v_3: $\quad i_1 + i_3 - 2i_x = 0$

Plug in current expression:

$$\frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2} = 3$$

$$\frac{v_1 - v_2}{2} - \frac{v_2}{4} - \frac{v_2 - v_3}{8} = 0$$

$$\frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} - 2\frac{v_1 - v_2}{2} = 0$$

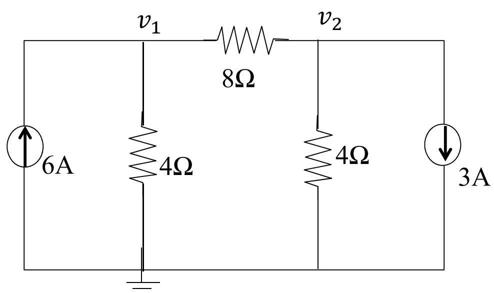
$$\boxed{3v_1 - 2v_2 - v_3 = 12 \qquad (1)}$$

$$\boxed{4v_1 - 7v_2 + v_3 = 0 \qquad (2)}$$

$$\boxed{-6v_1 + 9v_2 - 3v_3 = 0 \qquad (3)}$$

R8

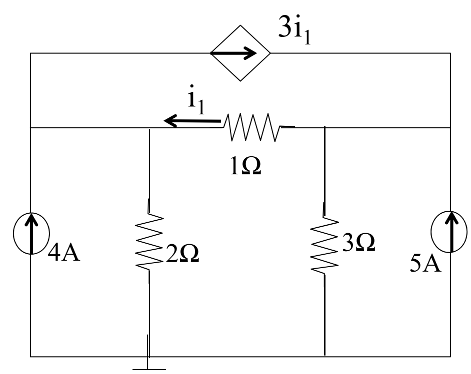
Practice problem 1:
Find v_1 *and* v_2 by using nodal analysis method



R8

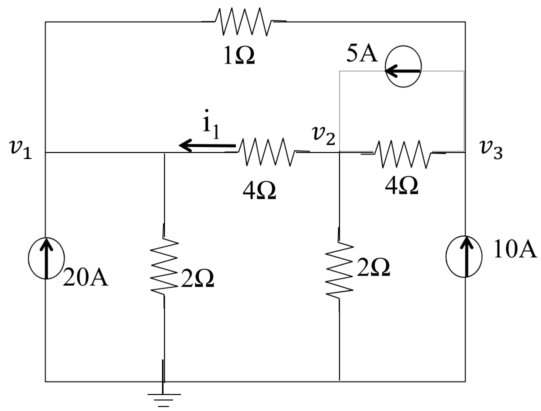
Practice problem 2:
Find i_1 by using nodal analysis method.

How many node voltages?



R8

Practice problem 3: Form 3 equations for v_1, v_2, v_3



L9

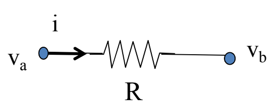
Nodal analysis

- Basic concepts: node voltage and ground
  - Voltage between any two points is the difference of two node voltages

v_a ● + v − ● v_b

$v = v_a - v_b$;
v_a: voltage at +
v_b: voltage at -

- Basic step: Express resistor current in terms of node voltages:

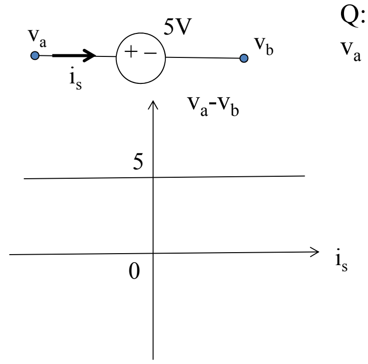


$$i = \frac{v_{\text{at the entrance}} - v_{\text{at the exit}}}{R}$$

$$= \frac{v_a - v_b}{R}$$

- Basic idea: use KCL to form equations for node voltages

What if we have a voltage source?



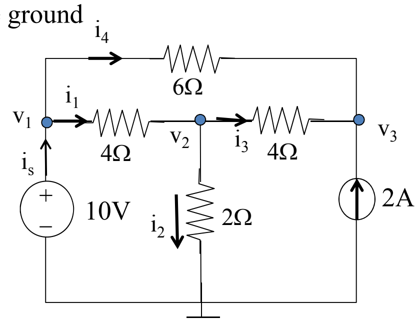
Q: How to express i_s in terms of v_a and v_b ?

A: i_s can not be expressed in terms of only v_a and v_b

i_s can be anything.

§ 3.3 Nodal Analysis with voltage sources

Case 1: The voltage source is connected to the ground



Three non-reference nodes: v_1, v_2, v_3

$$i_1 = \frac{v_1 - v_2}{4}; i_2 = \frac{v_2}{2}; i_3 = \frac{v_2 - v_3}{4}; i_4 = \frac{v_1 - v_3}{6}$$

At v_2, i_1-i_2-i_3= 0

$$\frac{v_1 - v_2}{4} - \frac{v_2}{2} - \frac{v_2 - v_3}{4} = 0 \qquad (1)$$

At v_3, i_3+i_4= -2

$$\frac{v_2}{2} + \frac{v_1 - v_3}{6} = -2 \qquad (2)$$

At v_1, i_s-i_1-i_4= 0

But i_s can't be expressed in terms of v_1

Get stuck here

Do we really need a third KCL equation at v_1?

How many unknown node voltages?

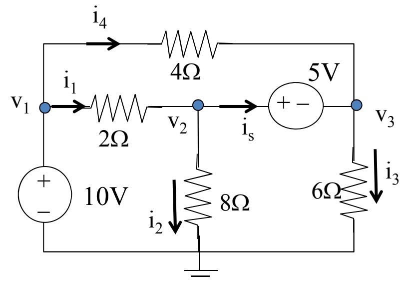
v_1=? v_1=10V

So we only have two unknowns, v_2 and v_3

(1) and (2) are sufficient to solve for v_2 and v_3

Case 2: One voltage source is not connected to the ground

-- A floating voltage source



Now we know v_1=10V

Only two equations are needed for v_2 and v_3

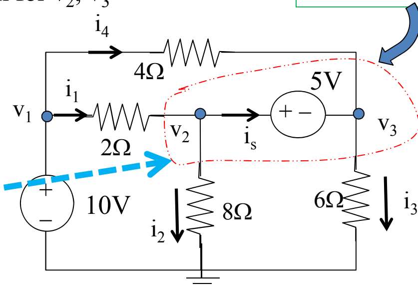
KCL at v_2: $i_1 - i_2 - i_s = 0$ (1)

KCL at v_3: $i_4 - i_3 + i_s = 0$ (2)

Same trouble with i_s,
can't be expressed in terms of v_2, v_3

Neither (1) nor (2) would yield an equation for v_2, v_3

A supernode



➢ Approach:

Add up (1) and (2): $i_1 - i_2 + i_4 - i_3 = 0$ (3)

What is this equation (3)?

❖ (3) is obtained by applying KCL at the closed boundary

It includes both nodes v_2 and v_3

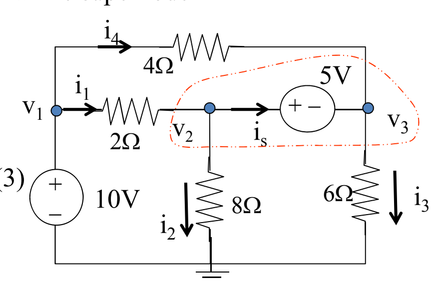
In the future, we don't apply KCL at each node connected to a voltage source (such as (1), (2)), then add them up.

We directly make a supernode and apply KCL at the supernode

The currents:

$$i_1 = \frac{10 - v_2}{2};\quad i_2 = \frac{v_2}{8};\quad i_3 = \frac{v_3}{6};\quad i_4 = \frac{10 - v_3}{4}$$

(3)



KCL at the super node: $i_1 - i_2 + i_4 - i_3 = 0$

Plug in current expressions

$$\frac{10 - v_2}{2} - \frac{v_2}{8} + \frac{10 - v_3}{4} - \frac{v_3}{6} = 0$$

➡ $\boxed{15v_2 + 10v_3 = 180 \qquad \text{(Eq1)}}$

Need a second equation for v_2 and v_3

It comes from the 5V voltage source:

$\boxed{v_2 - v_3 = 5 \qquad \text{(Eq2)}}$

Combining (Eq1) and (Eq2),
v_2 and v_3 can be solved.

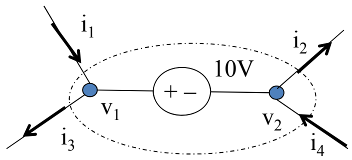
v_2 = 9.2V; v_3=4.2V

General steps for nodal analysis with floating voltage sources:

Step 1: Choose ground and assign node voltages

Step 2: Assign resistor currents with reference direction, express them in terms of node voltages

Step 3: Form super node to combine two or more nodes. The purpose is to avoid the current of a voltage source in a KCL equation. The super node is made such that the closed boundary does not intersect a voltage source.

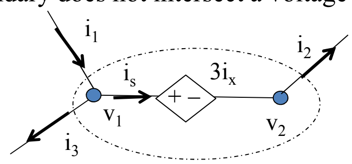


KCL at super node:

$i_1 - i_2 - i_3 + i_4 = 0$

Additional equation:

$v_1 - v_2 = 10$



KCL at super node: $i_1 - i_2 - i_3 = 0$

Additional equation: $v_1 - v_2 = 3i_x$

Be careful: $i_s \neq 3i_x$

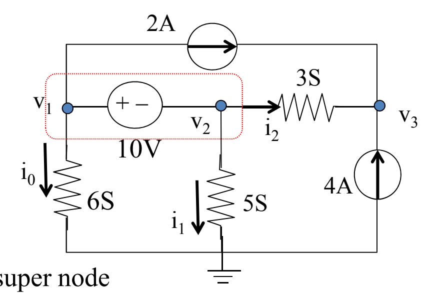
Never make equation like:

$\cancel{i_1 - 3i_x - i_3 = 0}$

$3i_x$ is a voltage, not current

Step 4: clean up and solve equations

Example: Find i_0 using nodal analysis. What is the power supplied by the 10V voltage source?



Solution:

Express resistor currents:

$i_0 = 6v_1;$

$i_1 = 5v_2$

$i_2 = 3(v_2 - v_3)$

Note that the conductance is given, not resistance

Need to combine nodes at v_1, v_2 to form a super node

KCL at super node v_1, v_2

$i_0 + i_1 + i_2 + 2 = 0$

$6v_1 + 5v_2 + 3(v_2 - v_3) = -2$

$\boxed{6v_1 + 8v_2 - 3v_3 = -2 \quad (1)}$

At v_3: $i_2 + 6 = 0$

$\boxed{3v_2 - 3v_3 = -6 \quad (2)}$

The 3rd equation comes from the 10V voltage source:

$\boxed{v_1 - v_2 = 10 \quad (3)}$

$$\begin{bmatrix} 6 & 8 & -3 \\ 0 & 3 & -3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 10 \end{bmatrix}$$

$v_1 = 4.909V; v_2 = -5.091V; v_3 = -3.091V$

How to find the power supplied by the 10V voltage source?

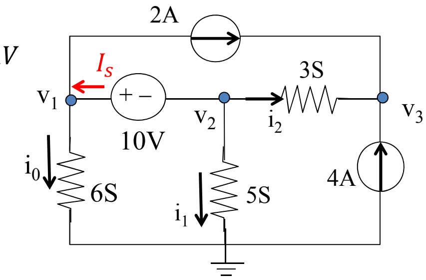
We have obtained

$v_1 = 4.909V; v_2 = -5.091V; v_3 = -3.091V$

With these node voltage,
everything can be solved

To find the power by the voltage source,
We need the current coming out of the
positive terminal

How to find I_s?



By KCL at v_1: $I_s = i_0 + 2$ $\quad i_0 = 6v_1 = 6 \times 4.909 = 29.454A$

$I_s = 31.454A,$

Power supplied by 10V = $10I_s = 314.54W$

Example: Form 3 equations for the 3 node voltages

Assign resistor currents:

$i_1 = 2(4 - v_1)$

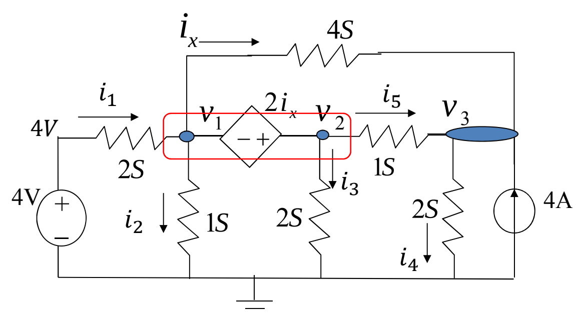
$i_2 = v_1$

$i_3 = 2v_2$

$i_4 = 2v_3$

$i_5 = v_2 - v_3$

$i_x = 4(v_1 - v_3)$



KCL at supernode v_1, v_2:

$i_1 - i_2 - i_3 - i_5 - i_x = 0$

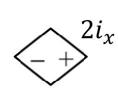
$2(4 - v_1) - v_1 - 2v_2 - (v_2 - v_3) - 4(v_1 - v_3) = 0$

$\boxed{-7v_1 - 3v_2 + 5v_3 = -8 \quad (1)}$

KCL at v_3: $\quad i_5 - i_4 + i_x + 4 = 0$

$v_2 - v_3 - 2v_3 + 4(v_1 - v_3) = -4$

$\boxed{4v_1 + v_2 - 7v_3 = -4 \quad (2)}$

By 

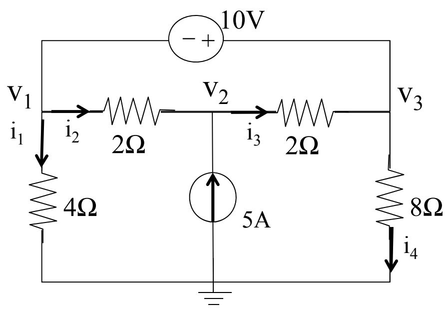
$v_2 - v_1 = 2i_x$

$v_2 - v_1 = 2(4(v_1 - v_3))$

$\boxed{9v_1 - v_2 - 8v_3 = 0 \quad (3)}$

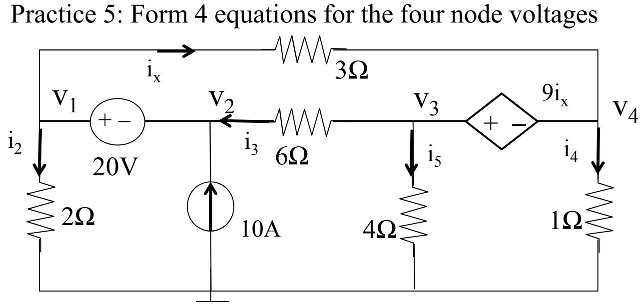
R9

Practice 4: Form 3 equations for the three node voltages



R9

Practice 5: Form 4 equations for the four node voltages



R9

Practice 6: Find v_1, v_2, v_3 *and* i_{dep}