Chapter 4 Circuit Theorems

From Chapter 2: Basic laws -- Foundations.
From Chapter 3: Mesh analysis, Nodal analysis.

Systematic tools, can be used for complex circuits.

More useful tools to be learned in Chapter 4:

1. Linearity
2. Superposition
3. Source transformation
4. Thevenin's Theorem
5. Norton's Theorem
6. Maximum power transfer

Could be more tricky than Chapters 2,3.

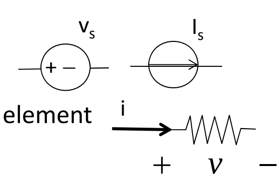
- There may exist several solutions to one problem
- Need to develop insight to plan solutions ahead and identify the simplest solution

§4.2 Linearity

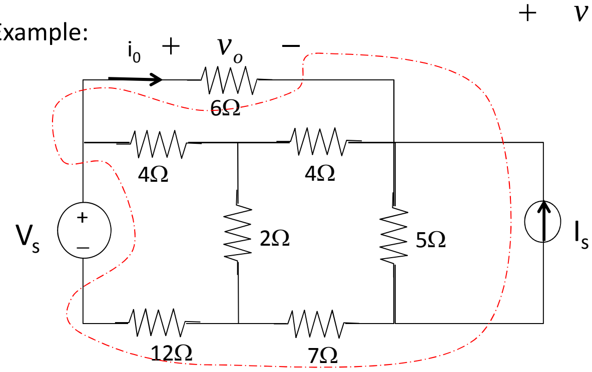
Basic concepts: Input, output

Input: the cause, or excitation,

Output: voltage or current of any element



Example:



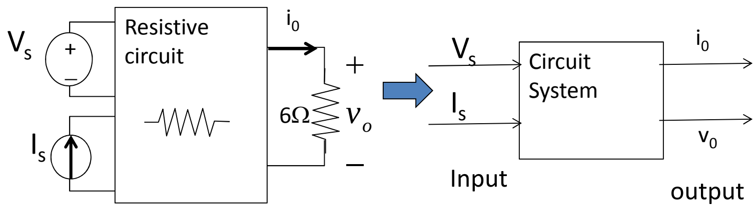
- Two inputs
  V_s, I_s
- Two outputs:
  i_0, v_0

You can imagine pulling out the inputs, outputs and put the rest inside a box

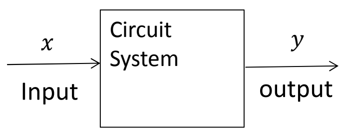
L12

Pull out the inputs, outputs and put the rest inside a box

More abstract form



First consider a circuit with one input, one output:



Input output relationship can be described by a function

$$y = f(x)$$

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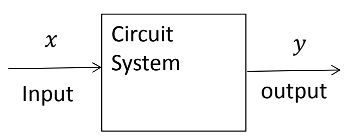
Input output relationship: $y = f(x)$:



Input x: can be V_s, I_s
Output y: can be v_o, i_o of any element

<u>Linearity:</u> The input output relationship is said to be linear if $y = k\,x$ for a certain constant k.

Two implications: Linearity = homogeneity + additivity

Homogeneity: If $x_1 \Rightarrow y_1$, then $\alpha x_1 \Rightarrow \alpha\, y_1$

Additivity: If $x_1 \Rightarrow y_1$, $x_2 \Rightarrow y_2$, then $x_1 + x_2 \Rightarrow y_1 + y_2$

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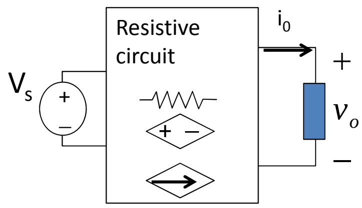
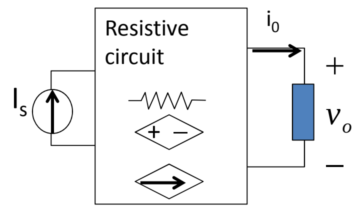
Linear circuit: the input-output relationship of a resistive circuit is linear.





$v_o=k_1V_s$ $\quad$ $v_o=k_3I_s$

$i_0=k_2V_s$ $\quad$ $i_0=k_4I_s$

Linearity is the simplest relationship between input and output

It is predictable: double efforts → double results

If $V_S = 5V \rightarrow v_0 = 3V$

Then $V_S = 10V \rightarrow v_0 = 6V$

$V_S = 2.5V \rightarrow v_0 = 1.5V$

We first use one example to demonstrate linearity

Later on, we use this linear property to analyze circuits

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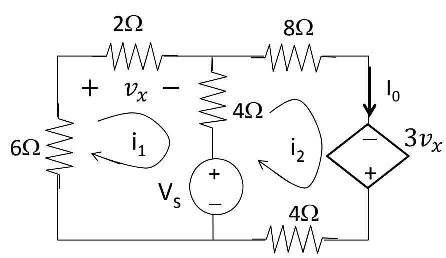
Example:



Input: V_s $\quad(x)$

Output: I_0 $\quad(y)$

A resistive circuit.

There exists a k such that

$I_0=kV_s$ $\quad(y = kx)$

What is k?

- Use mesh analysis to find k.

Assign mesh currents i_1, i_2.

$v_x=2i_1 \quad \Rightarrow 3v_x= 6i_1 \quad I_0= i_2$

KVL along mesh 1: $2i_1+4(i_1-i_2)+V_s+6i_1=0$

$\Rightarrow 12i_1-4i_2= -V_s \quad (1)$

KVL along mesh 2: $8i_2- 3v_x+4i_2-V_s+4(i_2-i_1) = 0$

$8i_2- 6i_1 +4i_2-V_s+4(i_2-i_1) = 0$

$\Rightarrow -10i_1+16i_2 = V_s \quad (2)$

(1)*5+(2)*6 ➔ $76i_2 = V_s$

$I_0 = i_2=\frac{1}{76}V_s$ $\quad$ Compare with $I_0=kV_s$

$k = \frac{1}{76}$

$V_s = 1V \rightarrow I_0 = \frac{1}{76}A$

$V_s = 4V \rightarrow I_0 = \frac{4}{76}A$

L12

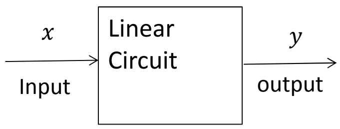
For this circuit, the input output relationship is derived by solving the output from the input

Given input $\Rightarrow$ find output, <u>The direct approach</u>

For some circuit, it is more convenient to find the input from the output.

Given output $\Rightarrow$ find input, <u>The backward approach</u>

In general,



$y = kx$ for a certain k

How to find k?

If you know any pair of input and output, $(x, y) = (x_0, y_0)$

Then $y_0 = kx_0$ $\Rightarrow$ $k = \frac{y_0}{x_0}$

Just need one pair (x_0, y_0) to determine k

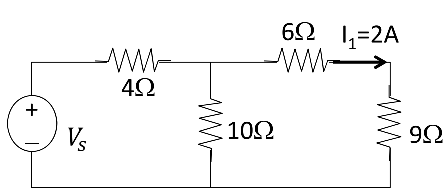
Direct approach:
Assume $x = x_0$, find $y = y_0$

Backward approach:
Assume y $= y_0$, find $x = x_0$

R7

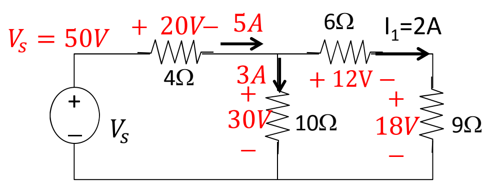
A practice problem from Chapter 2:

Practice 20: Find V_s for the circuit

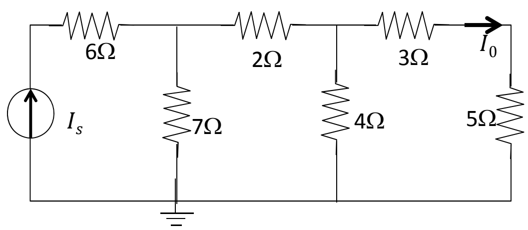


V_s is the input, the cause
I_1 is the output, the result

Assume $I_1 = 2A$, $V_s = ?$



Example:



Input: $x = I_s$
Output: $y = I_0$

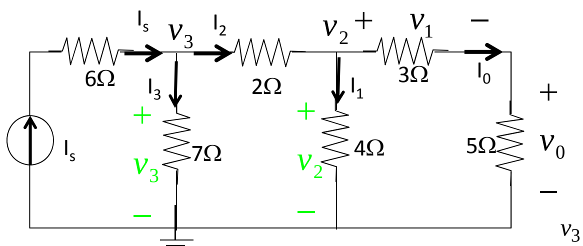
Find k such that
$I_0 = kI_s$

Use backward method

Assume $I_0 = 1A$, $I_s = ?$ You may assume I_0 to be any number.

With I_s solved, you have $\mathrm{k} = \frac{\mathrm{y}}{\mathrm{x}} = \frac{\mathrm{I_0}}{\mathrm{I_s}}$

Solution: Assign additional variables as in the following circuit:



v_2, v_3 are node voltages

$I_2 = \frac{v_3 - v_2}{2}$

⇕

$v_3 = v_2 + 2I_2$ (*)

Starting with $I_0=1A$
By Ohm's law: $v_0=5I_0=5V$; $v_1=3I_0=3V$
By KVL: $v_2=v_0+v_1=5+3=8V$
By Ohms Law: $I_1=v_2/4=2A$
By KCL at v_2: $I_2=I_0+I_1=1+2=3A$
By (*): $v_3=v_2+2I_2=8+2*3=14V$
By Ohms Law: $I_3=v_3/7=14/7=2A$
By KCL at v_3: $I_s=I_2+I_3=3+2=5A$

$I_0=1A$ ⇓ $I_s=5A$

⇨ $k = \frac{I_0}{I_s} = \frac{1}{5}$

Conclusion:
The input-output relationship is

$I_0=kI_s= 0.2I_s$

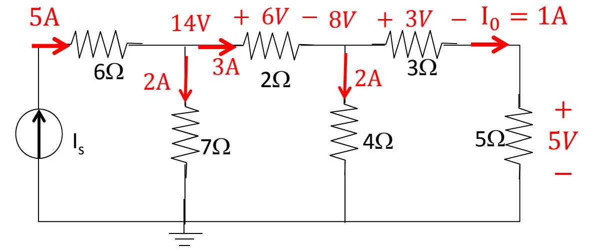
If $I_s=15A$, then
$I_0=0.2*15=3A$

If $I_s=10A$, then
$I_0=0.2*10=2A$

With $I_0=0.2I_s$, you can find output I_o for any input I_s
You can also find the input I_s for any output I_0

A simple demonstration:

Suppose $I_0 = 1A,\ I_s = ?$

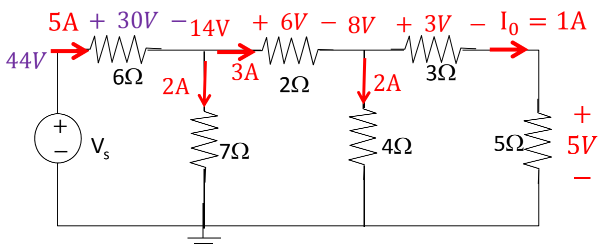


For $I_0 = 1A$,

$I_s = 5A$,

$$k = \frac{I_0}{I_s} = 0.2$$

What if I_s is replaced by V_s?



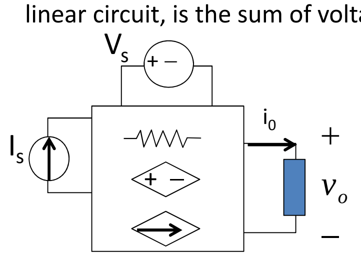
For $I_0 = 1A$,

$V_s = 44V$,

$$k = \frac{I_0}{V_s} = \frac{1}{44}$$

§4.3 Superposition -- About the relationship between one output and several inputs

Superposition principle: the voltage across (or current through) an element in a linear circuit, is the sum of voltage/current due to each <u>independent</u> source alone.

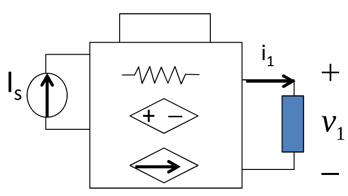


$v_0 = v_1 + v_2$

$i_0 = i_1 + i_2$

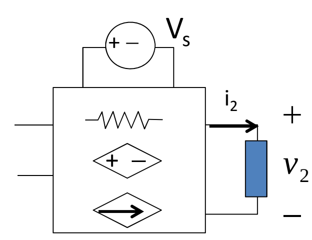
Due to I_s alone: turn off V_s

Set $V_s = 0 \Leftrightarrow$ replace voltage source with short circuit



Due to V_s alone: turn off I_s

Set $I_s = 0 \Leftrightarrow$ replace current source with open circuit



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Key points: when you examine the effect due to one source,
turn off the other sources

- Turn off voltage source with short circuit
- Turn of current source with open circuit

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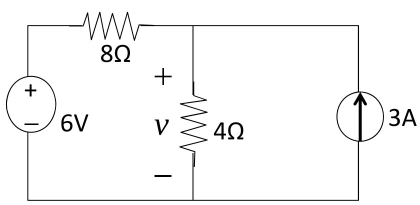
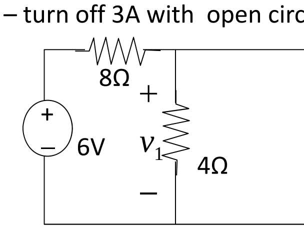
Example : Find the output v.



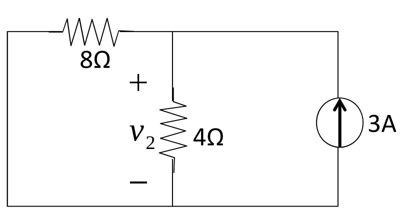
$v = v_1 + v_2 = 2+8=10V$

Step 1: Due to 6V alone
– turn off 3A with open circuit



By voltage division: $v_1 = \frac{4}{4+8} \times 6 = 2V$

Step 2: Due to 3A – turn off 6V with short circuit



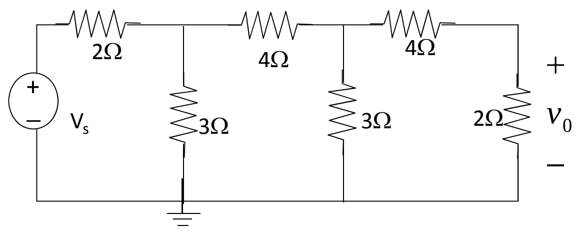
By equivalent resistance and ohm's law

$$v_2 = (8 // 4) \times 3$$
$$= \frac{4 \times 8}{4+8} \times 3 = 8V$$

Superposition breaks down a circuit into several circuits with only one independent source – typically can be solved with simple methods in Chapter 2.

R12

Practice 1: Find v_0 for V_s=27V by linearity

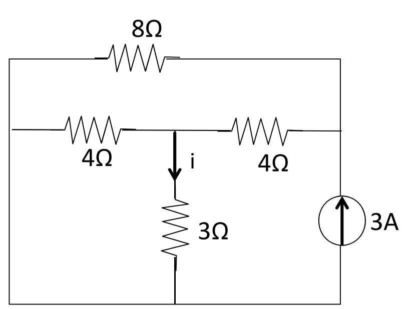


Step 1: Find k such that v_0=kV_s by assuming v_0= some value

Step 2: For V_s=27V, v_0= k V_s

R12

Practice 2: Find i using linearity



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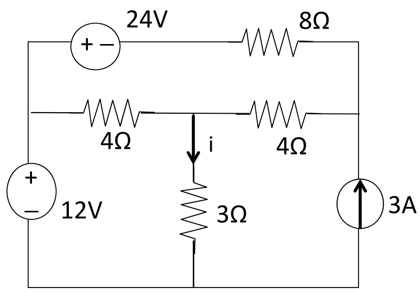
Practice 3: Find i using superposition.



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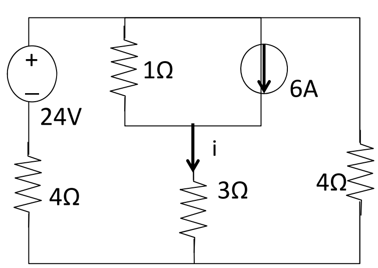
Practice 4: Find i using superposition.



Example: Find I_0 using superposition.

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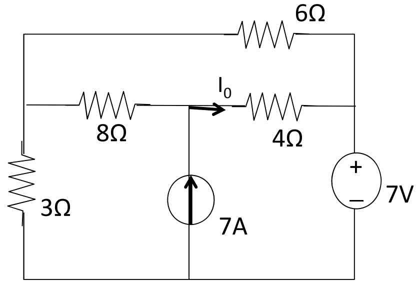
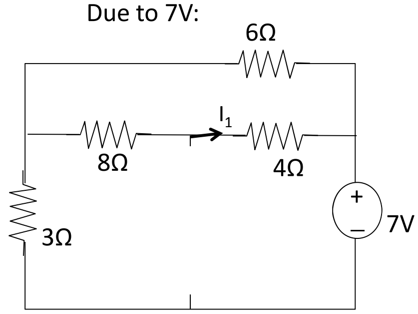
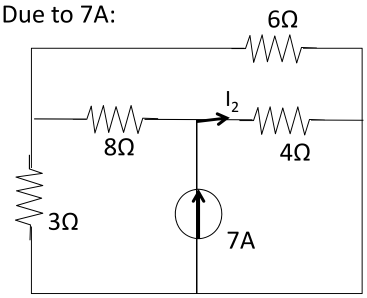


Due to 7V:



Due to 7A:



Due to 7V: replace 7A with open circuit
Due to 7A: replace 7V with short circuit

$$I_0 = I_1 + I_2$$

Due to 7A:

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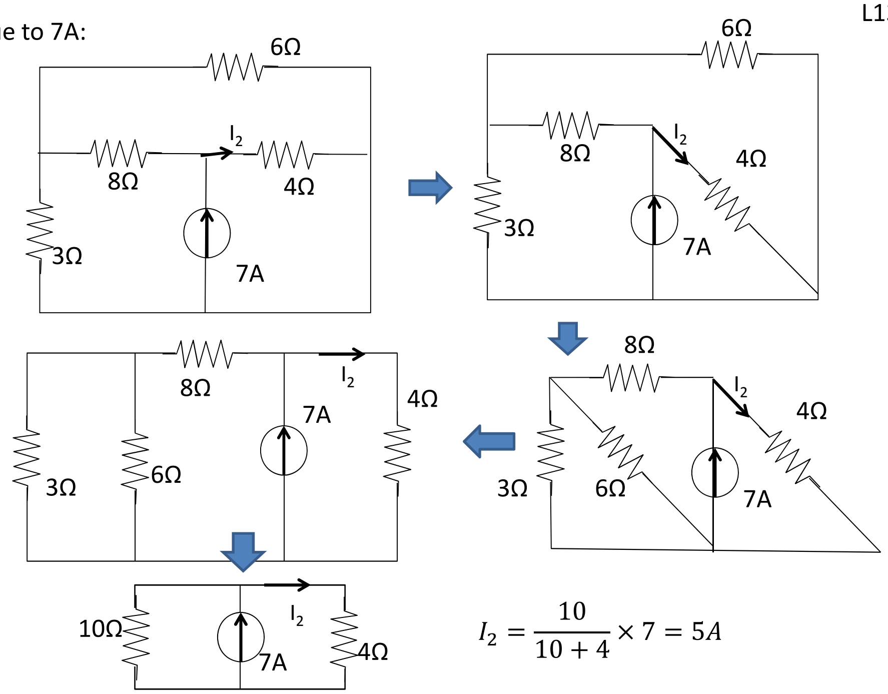

$$I_2 = \frac{10}{10+4} \times 7 = 5A$$

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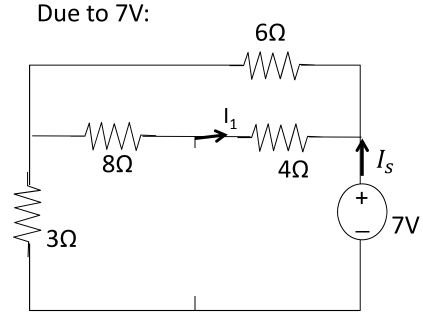
Due to 7V:



R_{eq} with respect to $7A$:

$R_{eq} = 3 + 6//(8+4) = 7\Omega$

$I_s = 1A$

By current division:

$$I_1 = -\frac{6}{8+4+6} I_s = -0.333A$$

Going back to the original circuit: $I_0 = I_1 + I_2 = 5 - 0.333 = 4.667A$

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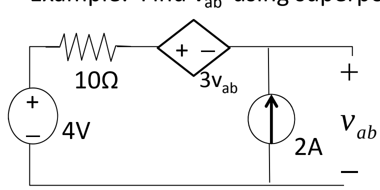
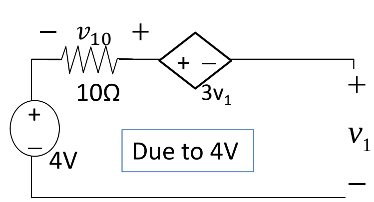
Superposition with dependent sources:

For circuit with dependent sources, keep the dependent sources.
They cannot be turned off . They are not input

Example: Find v_{ab} using superposition:



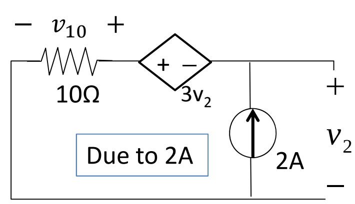
$v_{ab} = v_1 + v_2 = 1 + 5 = 6V$



By KVL: $3v_1 + v_1 = 4 + v_{10}$,

$v_{10} = 0,$

$v_1 = 1V$

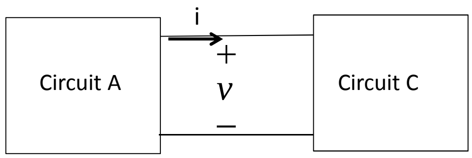


$3v_2 + v_2 = v_{10} = 2 \times 10 = 20V$

$v_2 = 5V$

§ 4.4: Source transformation

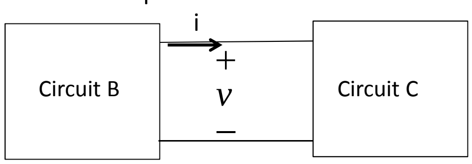
A general situation: you can separate a circuit as the connection of two parts.



You may need to make change to one part,

But want to ensure this change does not affect the other part

For circuit A, you can obtain a function v = f (i)
If there is another circuit B, with the same v~i relationship,
You can replace Circuit A with Circuit B without affecting Circuit C

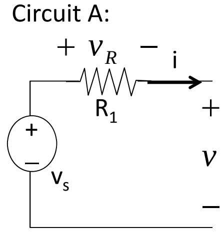


All variables in Circuit C are unchanged,
But the variables in Circuit A may not exist in Circuit B.

Sometimes you need to replace a complex circuit with a simpler one; as with Thevenin's equivalent
Sometimes with one which is more convenient for analysis as with source transformation

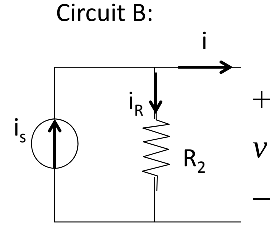
Compare the two connections:

Circuit A:



What is the v~i relationship for each circuit? How to express v in terms of i?

Circuit B:



By KVL,
$v = v_s - v_R = v_s - R_1 i$

$v = v_s - R_1 i$

By KCL, $i_R = i_s - I$
$v = R_2 i_R = R_2 i_s - R_2 i$

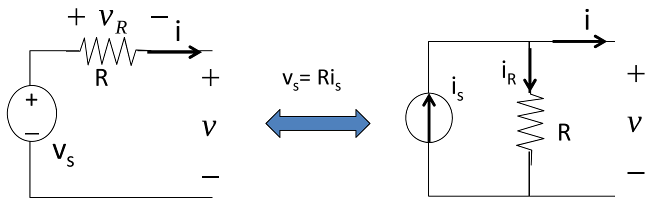
$v = R_2 i_s - R_2 i$

To have the same v~i relationship: $R_1 = R_2$, $V_s = R_2 i_s$

L13

Conclusion:

The following two connections are equivalent



If you replace one with another, the rest of the circuit will not be affected.
However, be careful:
The voltages across resistor R in the two circuits are different;
The currents through R in the two circuits also different.

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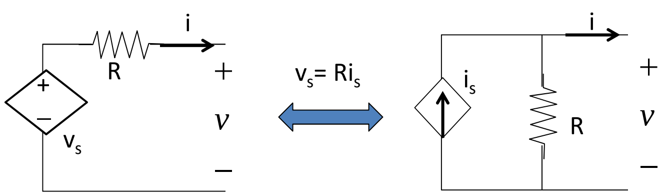
Extension to dependent sources:

The following two connections are equivalent



For example, if $i_s = kv_x$, then $v_s = Ri_s = Rkv_x$
If $v_s = ki_0$, then $i_s = ki_0/R$

The rule: if you put the two sources side by side, the arrow of the current source points from – to + of the voltage source

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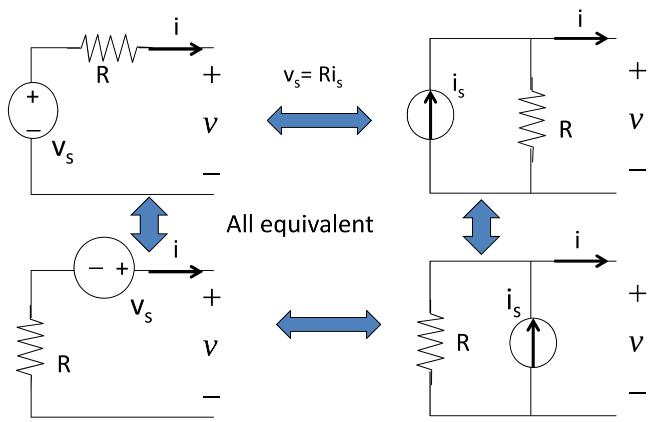

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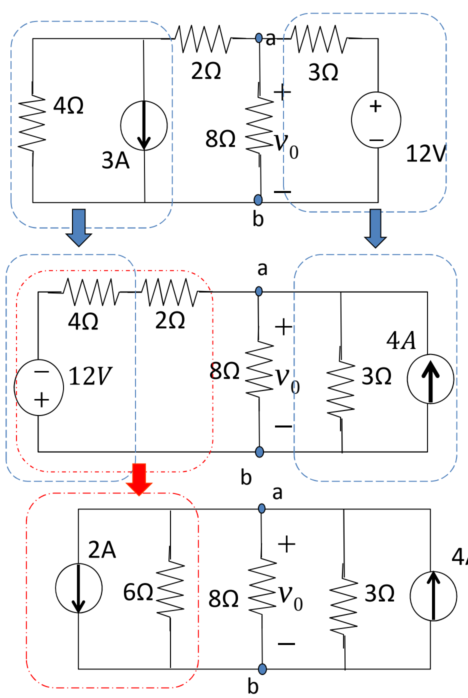

Example: Find the voltage v_0 using source transformation

Caution:
Need to keep the two nodes “a” and “b” so that v_0 is kept after every change

After these transformations, 4Ω and 2Ω can be combined, 3Ω and 8Ω can be combined. Reducing the number of elements

When everything in parallel, you can rearrange the elements without changing the voltage and branch current.

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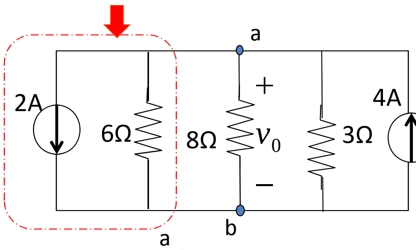
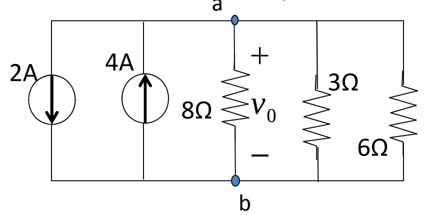
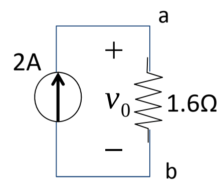


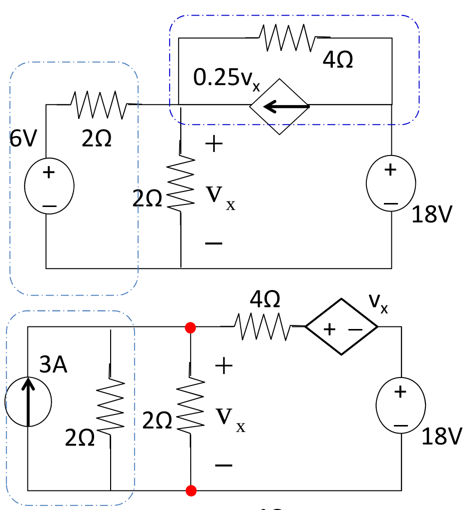
When everything in parallel, you can rearrange the elements without changing the voltage and branch current.





v_0=3.2V

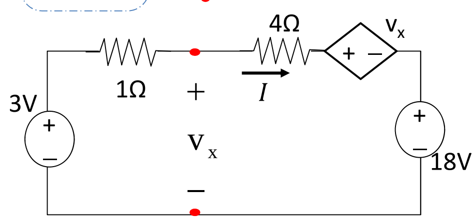
L13



Example: Find v_x

Approach 1:

Convert to a circuit with one loop.



Express v_x: $v_x = 3 - I$

KVL around loop:

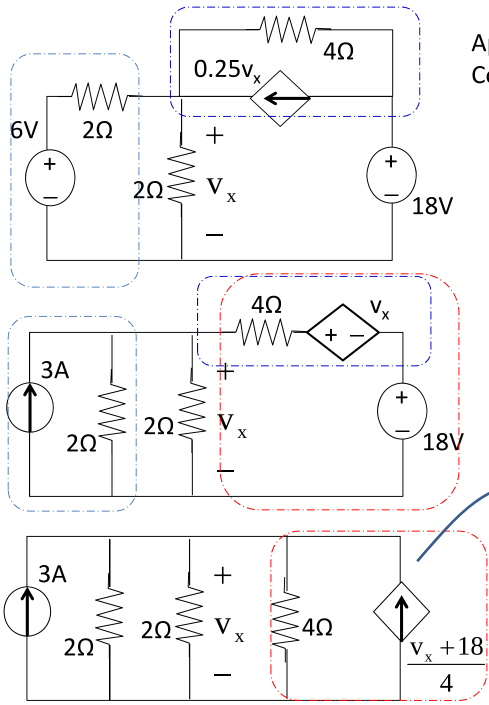
$I + 4I + v_x + 18 - 3 = 0$

Plug in $v_x = 3 - I$

$I + 4I + 3 - I + 18 - 3 = 0$

$4I = -18 \quad I = -4.5A$

$v_x = 3 - I = 3 - (-4.5) = 7.5V$



L13

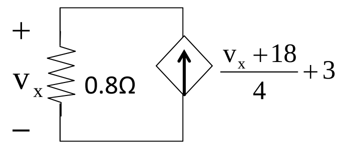
Approach 2:
Convert to a circuit with two nodes.

Combine parallel resistors

4Ω//2Ω//2Ω=0.8Ω

Combine parallel current sources

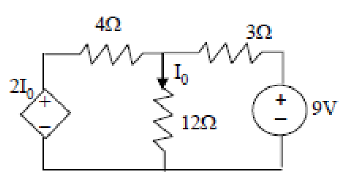
$$\frac{v_x + 18}{4} + 3A$$

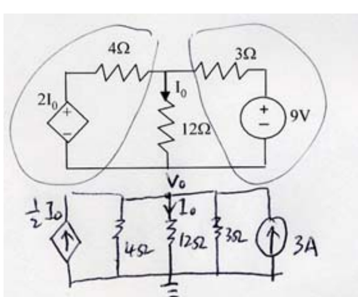


$$0.8\left(\frac{v_x + 18}{4} + 3\right) = v_x$$

v_x=7.5V

2. (4pts) Use source transformation to determine I_0





KCL at V_0 $\quad \frac{V_0}{4} + \frac{V_0}{12} + \frac{V_0}{3} = 3 + I_0/2$
$= 3 + V_0/24$

$I_0 = \frac{V_0}{12}$

multiply both sides with 24.

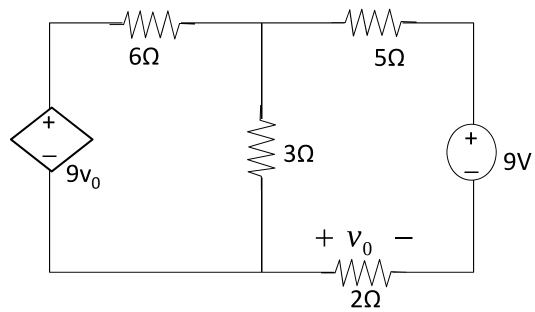
$6V_0 + 2V_0 + 8V_0 = 3\times24 + V_0$

$15V_0 = 3\times24$, $\quad V_0 = \frac{3\times24}{15}$

$I_0 = \frac{V_0}{12} = \frac{3\times24}{15\times12} = \frac{1}{5}\cdot\frac{2}{1} = 0.4A$.

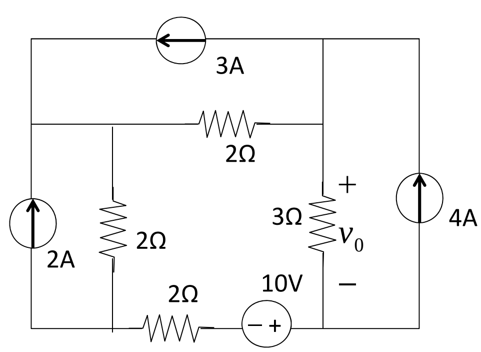
R13

Practice 5: Find v_0 using source transformation



R13

Practice 6: Find v_0 using source transformation



R13

Practice 7: Find I_0 using source transformation