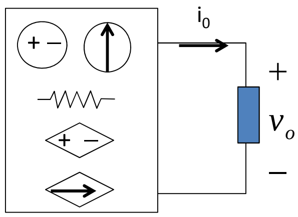
§ 4.5 Thevenin's Theorem

A common situation: Most part of the circuit is fixed, only one element is variable, e.g., a load:



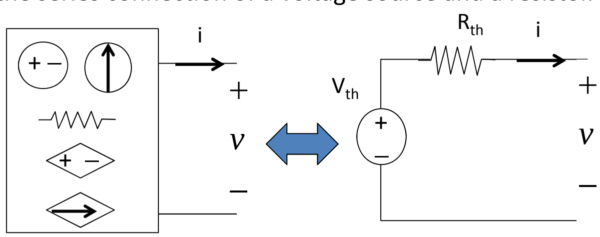
The fixed part, a two terminal circuit, can be very complex.

How the variable part affects v_o, i_o, or the power?

By Thevenin's theorem, the complex two terminal circuit can be replaced with a simple circuit. Making the analysis very easy

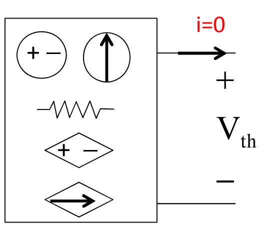
Thevenin's theorem is the most important result in Circuit Theory. It will be a main tool for Chapters 7, 8.

<u>Thevenin's theorem</u>: Every linear two terminal circuit can be made equivalent to the series connection of a voltage source and a resistor.



Same v~i relationship

V_{th}: Open circuit voltage across the two terminals



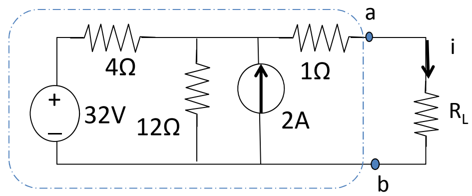
Also called V_{oc}

R_{th}: equivalent resistance with respect to the two terminals when all <u>independent sources</u> are turned off

Case 1: No dependent source. $R_{th}=R_{eq}$ (as in Chapter 2)

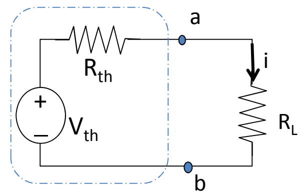
Case 2: With dependent source. Need to supply an external independent source at the two terminal. Then R_{th} is the ratio between the voltage and current.

Example: What is the relationship between R_L and i?



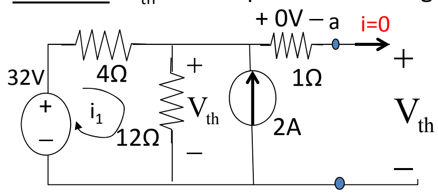
Replacing the part to the left of "a", "b" by Thevenin's equivalent

Clear relationship: $i = \frac{V_{th}}{R_{th} + R_L}$

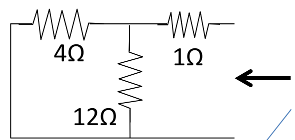


Key point: V_{th}=?, R_{th}=?

Solution: V_{th} is the open circuit voltage.



For R_{th}, turn off all independent sources:



R_{th}=1+4//12 =4Ω

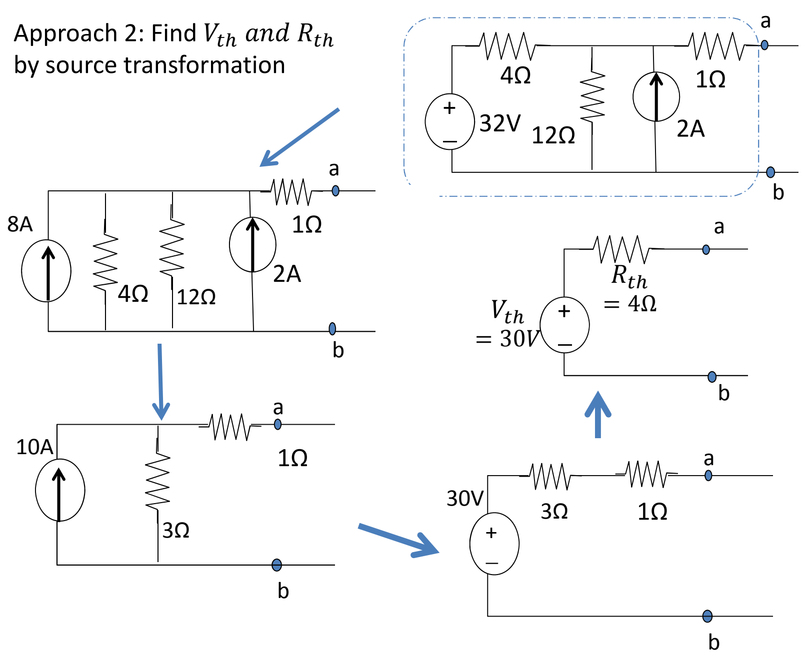
Assign mesh current i_1

KVL along the mesh: $4i_1+12(i_1+2)=32$

$i_1=0.5A$ $V_{th}=12(i_1+2)=30V$

$$i = \frac{V_{th}}{R_{th} + R_L} = \frac{30}{4 + R_L}$$

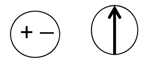
Approach 2: Find V_{th} and R_{th} by source transformation



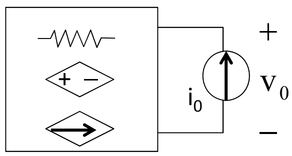
Case 2: There are dependent sources

V_{th}: The same as case 1, the open circuit voltage

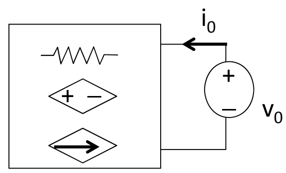
For R_{th}:

1. Turn off all independent sources. 

2. Connect the two terminals to a current/voltage source
3. Solve for the voltage/current of the source



or,



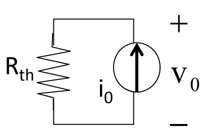
Give any number for i_0, e.g., 1A, 2A, 10A, Solve for v_0

$$R_{th} = \frac{V_0}{i_0}$$

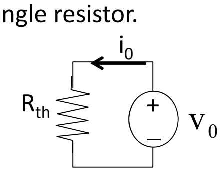
Give any number for v_0, e.g., 1V, 2V, 20V, Solve for i_0

Note: i_0 *and* v_0 assigned by active sign convention.

Explanation: as the independent sources are turned off, the two terminal circuit behaves like a single resistor.

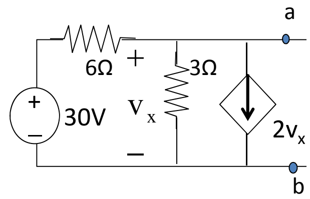


$$R_{th} = \frac{V_0}{i_0}$$

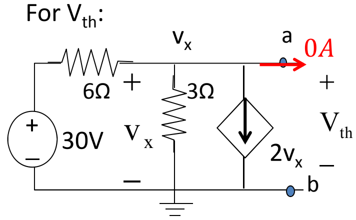


Be careful, v_0 and i_0 are assigned according to active sign convention with respect to the source (passive sign convention for R_{th}).

Example:
Find R_{th}, V_{th} for the two terminal circuit



For V_{th}:



Use nodal analysis
Pick ground. v_x is the node voltage

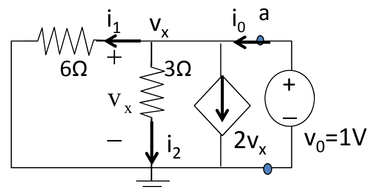
$V_{th} = v_x$

KCL at node v_x:

$$\frac{v_x - 30}{6} + \frac{v_x}{3} + 2v_x = 0 \Rightarrow v_x = 2V, \quad V_{th} = 2V$$

For R_{th}: turn off 30V with short circuit.

Approach 1: connect to voltage source v_0=1V,



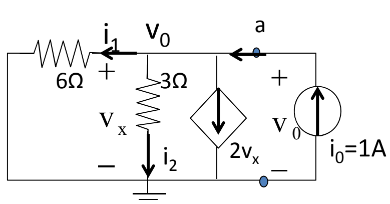
Need to find i_0

Given v_x=1V   KCL at node v_x: $i_0 = i_1 + i_2 + 2v_x$

$$i_0 = \frac{1}{6} + \frac{1}{3} + 2 = 2.5A$$

$$R_{th} = \frac{v_0}{i_0} = \frac{1}{2.5} = 0.4\Omega$$

Approach 2: connect to current source i_0=1A, $v_0 = v_x$



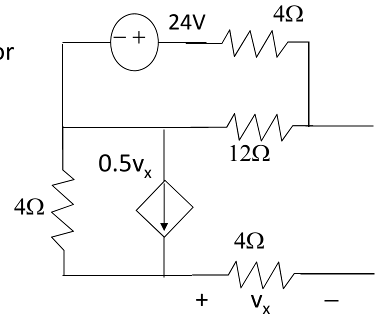
KCL at node v_0: $i_1 + i_2 + 2v_x = 1$

$$\frac{v_0}{6} + \frac{v_0}{3} + 2v_0 = 1 \qquad v_0 = 0.4$$

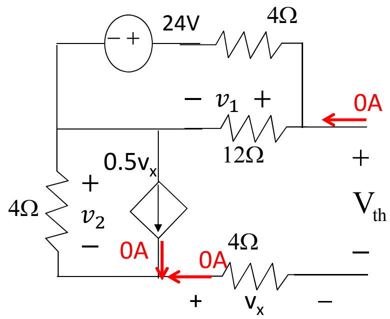
$$R_{th} = \frac{v_0}{i_0} = \frac{0.4}{1} = 0.4\Omega$$

Two approaches yield the same R_{th}.

Example: Find V_{th} and R_{th} for the two terminal circuit



For V_{th}:



By KVL, $V_{th} = v_1 + v_2 + v_x$

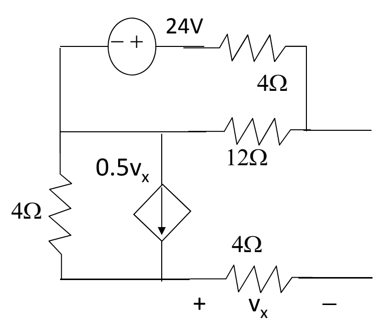
$$v_x = 0V$$

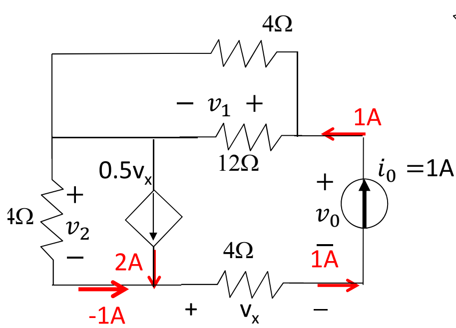
$$v_2 = 0V$$

$$v_1 = \frac{12}{12+4} \times 24 = 18V$$ By voltage div

$$V_{th} = 18 + 0 + 0 = 18V$$

For R_{th}, turn off 24V,
Supply a 1A current source.
Find the voltage v_0 across the current source





Then $R_{th} = \frac{v_0}{i_0}$

By KVL, $v_0 = v_1 + v_2 + v_x$

$$v_x = 4V$$

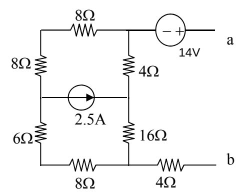
$$v_2 = 4 \times (1-2) = -4V$$

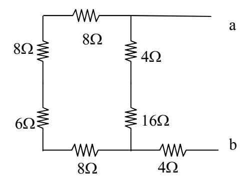
$$v_1 = (4//12) \times 1 = 3V$$

$$v_0 = 3 - 4 + 4 = 3V$$

$$R_{th} = \frac{v_0}{i_0} = 3\Omega$$

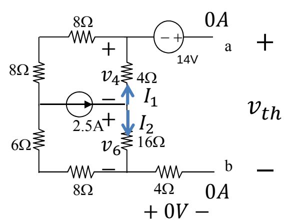
Example: Find Thevenin's equivalent with respect to terminals "a" and "b".





$R_{th} = 4 + (4 + 16)//(8+8+6+8)$=16Ω

v_{th}: open circuit voltage



$v_{th} = 14 + v_4 + v_6$

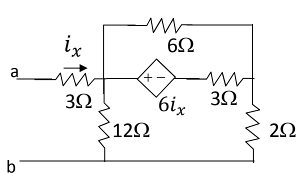
By current division

$$I_1 = \frac{30}{20 + 30} \times 2.5 = 1.5A$$
$$I_2 = 2.5 - 1.5 = 1A$$

$$v_4 = -1.5 \times 4 = -6V$$
$$v_6 = 1 \times 16 = 16V$$

$$v_{th} = 14 - 6 + 16 = 24V$$

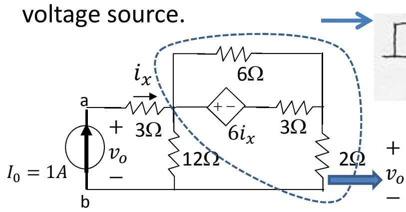
Example: Find Thevenin's equivalent circuit with respect to terminals "a" and "b"

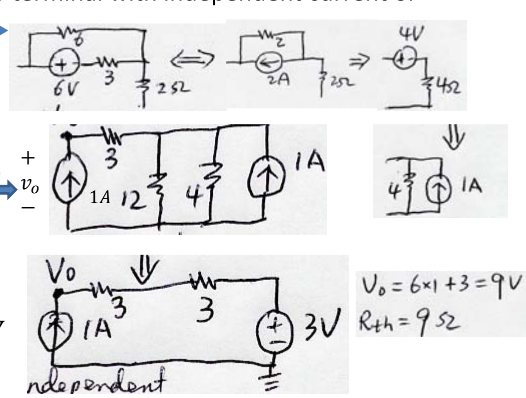


For V_{th}, the open circuit voltage,
since there is no independent source ,
all voltages and currents are 0,

$$V_{th} = 0.$$

For R_{th}, need to supply the two terminal with independent current or voltage source.





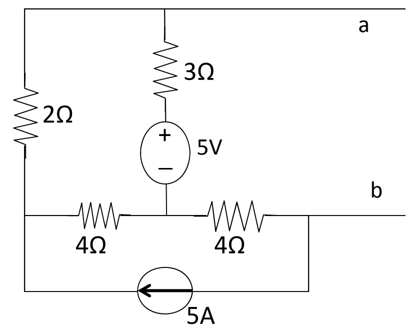
Set $I_0 = 1A$, need to find v_o.
Then $R_{th} = \frac{v_o}{I_o}$

$$i_x = I_o = 1A, \quad 6i_x = 6V$$

Use source transformation

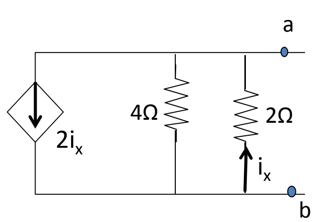
R14

Practice 1: Find R_{th}, V_{th} for the two terminal circuit



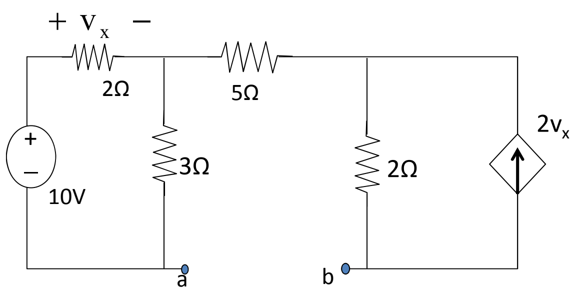
R14

Practice 2: Find R_{th}, V_{th} for the two terminal circuit



R14

Practice 3: Find R_{th}, V_{th} for the two terminal circuit



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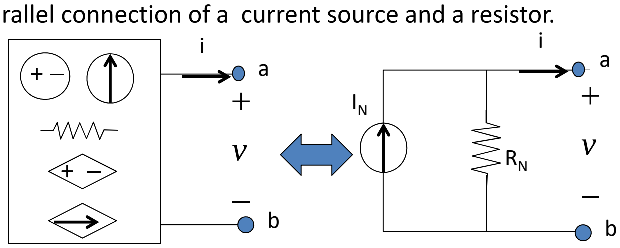
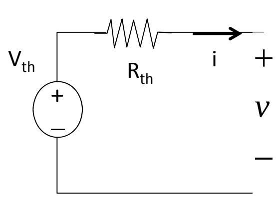
§4.6 Norton's Theorem

Norton's theorem: Every linear two terminal circuit can be made equivalent to the parallel connection of a current source and a resistor.



I_N: Short circuit current from "a" to "b", R_N: same as R_{th}

Since the same circuit is also equivalent to Thevenin's



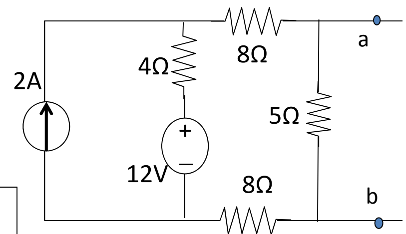
By source transformation,

$R_N = R_{th}$, $V_{th} = R_{th} I_N$,

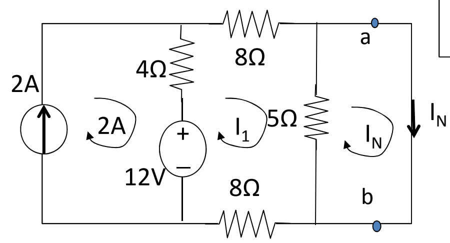
$I_N = V_{th}/R_{th}$

You can get Thevenin's equivalent first, then use source transformation to get Norton's equivalent, or vice vesa

Example:
Find the Norton's equivalent



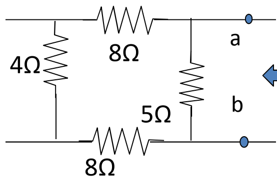
For I_N, short circuit "a", "b"



What is the relationship between I_1 and I_N?
Be careful: a resistor in parallel with short circuit does not take any current. It can be discarded.
$I_1 = I_N$, voltage across 5Ω is 0.

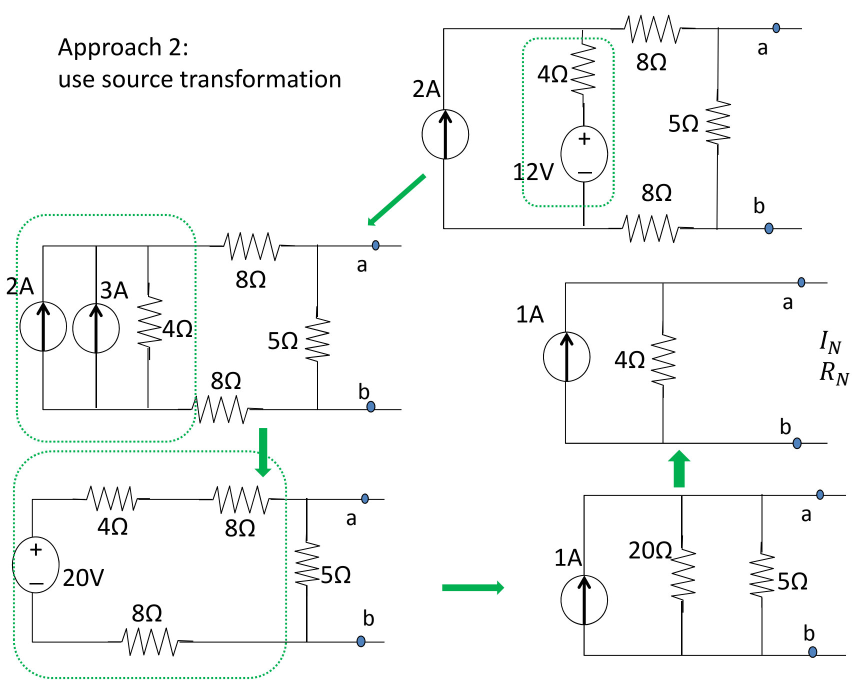
KVL along center mesh: $8I_N+8I_N-12+4(I_N-2)=0$ ⇒ $I_N=1A$

For R_N, turn off 2A and 12V sources:



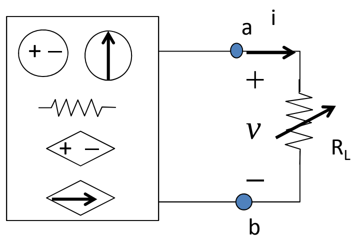
⇐ $R_N=5//(8+8+4)=4\Omega$

Approach 2:
use source transformation



$I_N = 1A$
$R_N = 4\Omega$

§4.8 Maximum power transfer



Linear circuit connected to a variable load R_L

Power consumed by load:

$$p = R_L i^2$$

Question: for what value of R_L is p maximized?

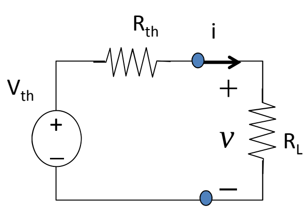
Note that the current i also depends on R_L.

Key point: find the exact relationship between i and R_L.

Main tool: Thevenin's and Norton's theorem

Applications examples:
Radio transmitter - maximize power delivered to the antenna or transmission line
Grid tied inverter - maximize power delivered to the grid
Electric vehicle - maximize power delivered to drive motor

By Thevenin's theorem, the two terminal circuit can be replaced with a voltage source in series with a resistor:



Clear relationship:

$$i = \frac{V_{th}}{R_{th} + R_L}$$

$$p = R_L i^2 = R_L \frac{V_{th}^2}{(R_{th} + R_L)^2}$$

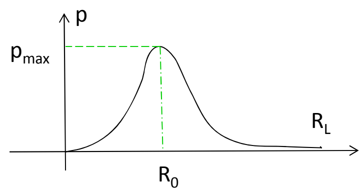
$$p = V_{th}^2 \frac{R_L}{(R_{th} + R_L)^2}$$

$For\ \ R_L = 0, \qquad p = 0.$

As $R_L \to \infty, \qquad p \to 0$

L15

$$p = V_{th}^2 \frac{R_L}{(R_{th}+R_L)^2}$$



A pure math problem: find R_L so that p is maximized.

At the maximal point, $\left.\frac{dp}{dR_L}\right|_{R_L=R_0} = 0$

$$\frac{dp}{dR_L} = V_{th}^2 \frac{(R_L+R_{th})^2 - 2R_L(R_L+R_{th})}{(R_L+R_{th})^4}$$

$$= V_{th}^2 \frac{R_{th}-R_L}{(R_L+R_{th})^3}$$

For $dp/dR_L = 0$, we must have $R_L=R_{th}$

At $R_L=R_{th}$, $$\boxed{p_{max} = \frac{V_{th}^2}{4R_{th}}}$$

L15

<u>Conclusion</u>: Maximal power is transferred to the load when $R_L=R_{th}$, and the maximal power is

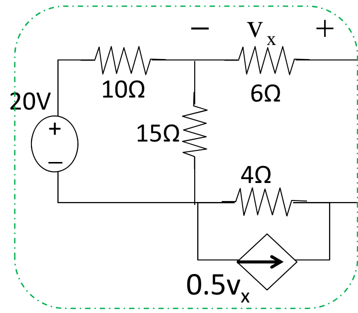
$$p_{max} = \frac{V_{th}^2}{4R_{th}}$$

When the Norton's equivalent is used, similar result:

<u>Conclusion</u>: Maximal power is transferred to the load when $R_L=R_N$, and the maximal power is

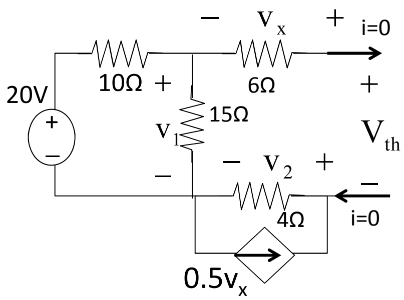
$$p_{max} = \frac{R_N I_N^2}{4}$$

Example:
Determine R so that maximal power is delivered.
Also find the maximal power.



Solution: Need to find V_{th} and R_{th} with respect two the two terminals of R.

For V_{th}, disconnect R



Since i=0, v_x= 0. 10Ω and 15Ω in series
By voltage division

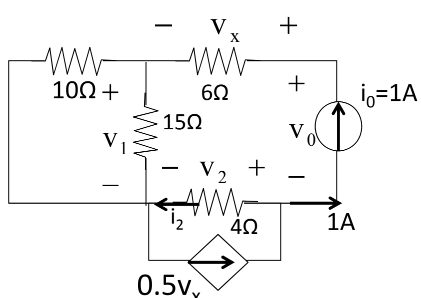
$$v_1 = \frac{15}{15+10} \times 20 = 12V$$

Also since i=0, $0.5v_x$=0 flows through 4Ω.
Thus v_2= 0V

By KVL,

$V_{th}=v_x+v_1-v_2=0+12+0=12V$

For R_{th}, turn off 20V, but keep $0.5v_x$ dependent source.
Because of the dependent source, supply an external current source i_0= 1A. (you may also try a voltage source and compare)



Need to find v_0 across the 1A current source (Active sign convention)

$v_x=6i_0=6V$

$$v_1 = (15//10) \times i_0 = 6 \times 1 = 6V$$

Assign current i_2 for 4Ω
By KCL, $i_2=0.5v_x-1$. Thus

$v_2= 4(0.5v_x-1)$
$= 4(3-1)=8V$

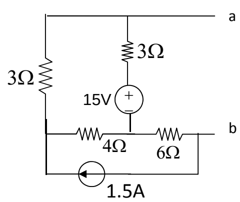
$$v_0 = v_x + v_1 - v_2 = 4V$$

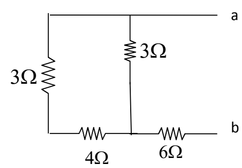
$$R_{th} = \frac{v_0}{i_0} = \frac{4}{1} = 4\Omega$$

Finally, The maximal power is delivered when $R=R_{th}=4\Omega$.
The maximal power is

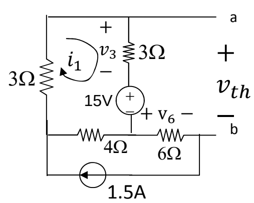
$$p_{max} = \frac{V_{th}^2}{4R_{th}} = \frac{12^2}{4 \times 4} = 9W$$

Example: Find Thevenin's equivalent with respect to terminals "a" and "b".





$R_{th} = 6 + 3//7 = 8.1\Omega$



KVL along mesh 1:

$$3i_1 + 15 + 4(i_1 - 1.5) + 3i_1 = 0$$
$$10i_1 = -9$$
$$i_1 = -0.9$$

$$V_{th} = v_3 + 15 + v_6 = -2.7 + 15 + 9 = 21.3V$$

Question: Any relationship between the following circuits?

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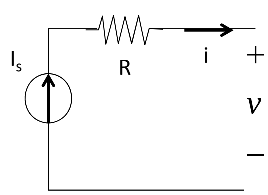
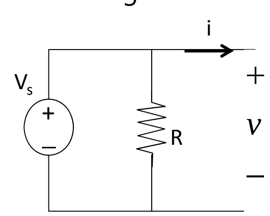




Illegal to open circuit

Norton's equivalent:

$I_N = I_s$

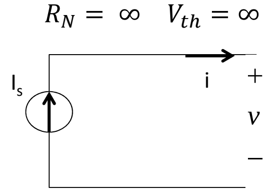
$R_N = \infty \quad V_{th} = \infty$

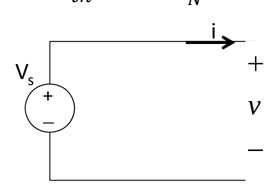
Illegal to short circuit

Thevenin's equivalent:

$V_{th} = V_s$

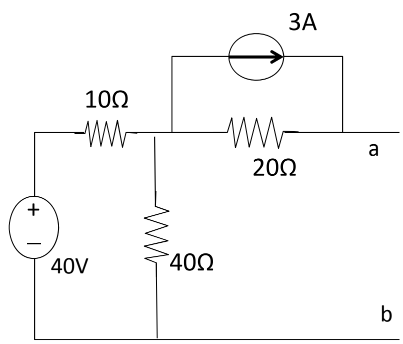
$R_{th} = 0 \quad I_N = \infty$



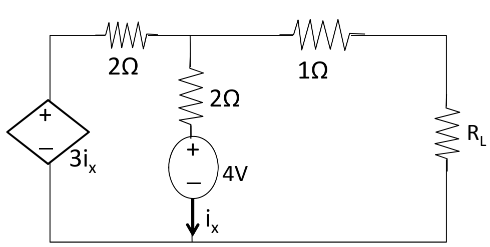


Any resistor in series with current source can be discarded
Any resistor in parallel with voltage source can be discarded

Practice 4: Find the Norton's equivalent

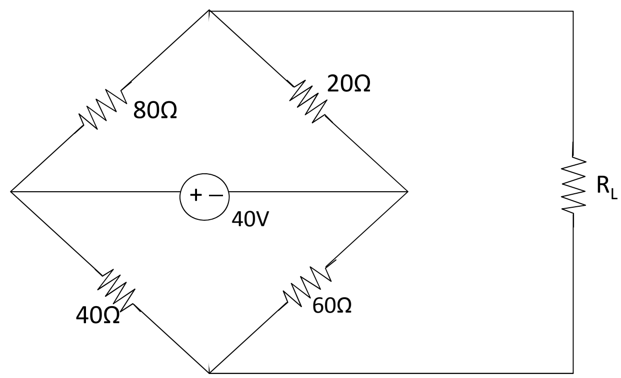


Practice 5: Find the value of R_L so that maximum power is delivered. Also find the maximum power.



R15

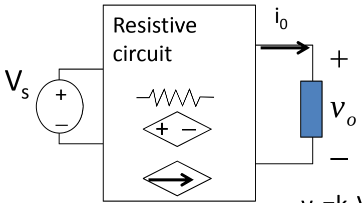
Practice 6: Find the unknown R_L so that maximum power is delivered. Also find the maximum power.



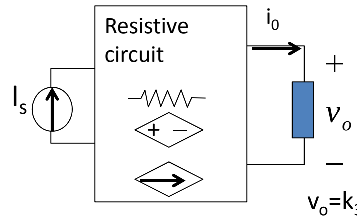
Chapter 4 Review

- Linearity
- Source transformation
- Superposition
- Thevenin's Theorem
- Norton's Theorem
- Maximum power Transfer

<u>Linearity:</u> the input-output relationship of a resistive circuit is linear.

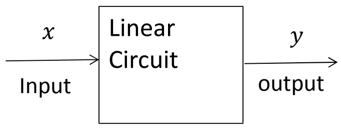


$v_o = k_1 V_s$
$i_0 = k_2 V_s$



$v_o = k_3 I_s$
$i_0 = k_4 I_s$

In general,



$y = kx$ for a certain k

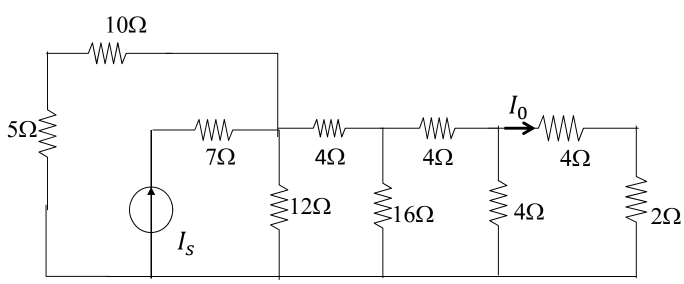
How to find k?

If you know any pair of input and output,
$(x, y) = (x_0, y_0)$

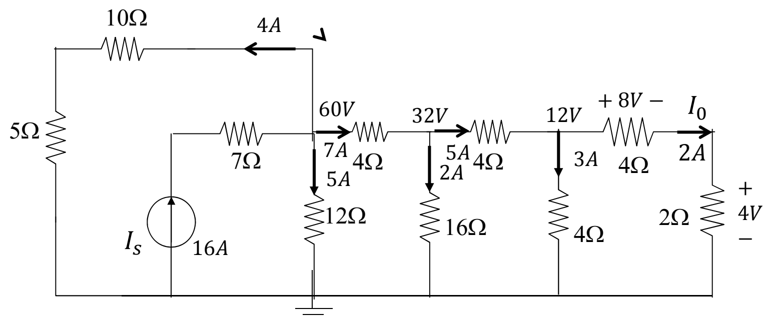
Then $y_0 = kx_0$ ⇒ $k = \frac{y_0}{x_0}$

Backward approach:
Assume $y = y_0$, find $x = x_0$

Example 1: Find I_s for $I_0 = 2A$. Compute I_0 for $I_s = 10, 24A$.



Solution: Find I_s for $I_0 = 2A$. Set the bottom node as ground



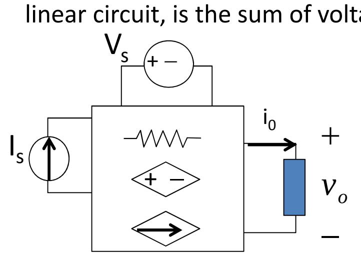
For $I_0 = 2A$; $I_s = 16A$; $k = \frac{I_0}{I_s} = \frac{2}{16} = \frac{1}{8}$, $I_0 = \frac{1}{8} I_s$

If $I_s = 10A$, $I_0 = \frac{1}{8} I_s = \frac{10}{8} = 1.25A$

If $I_s = 24A$, $I_0 = \frac{1}{8} I_s = \frac{24}{8} = 3A$

Superposition

Superposition principle: the voltage across (or current through) an element in a linear circuit, is the sum of voltage/current due to each <u>independent</u> source alone.

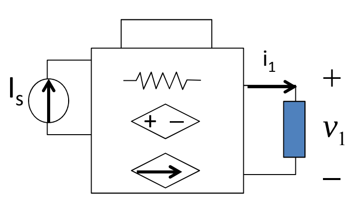


Key points:

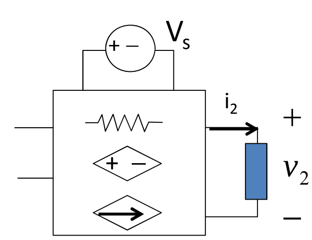
- Turn off voltage source with short circuit
- Turn of current source with open circuit

$v_0 = v_1 + v_2$
$i_0 = i_1 + i_2$

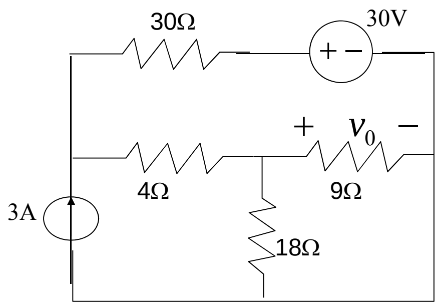
Due to I_s alone: turn off V_s
Set $V_s = 0 \Leftrightarrow$ replace voltage source with short circuit



Due to V_s alone: turn off I_s
Set $I_s = 0 \Leftrightarrow$ replace current source with open circuit

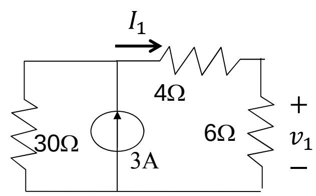


Example 3: Use **superposition** to compute v_0

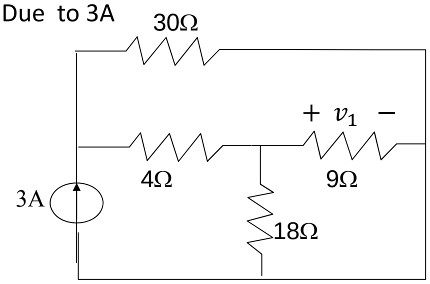


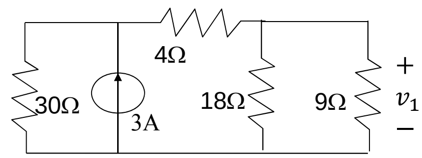
$$I_1 = \frac{30}{10+30} \times 3 = 2.25A$$

$$v_1 = 6I_1 = 13.5V$$

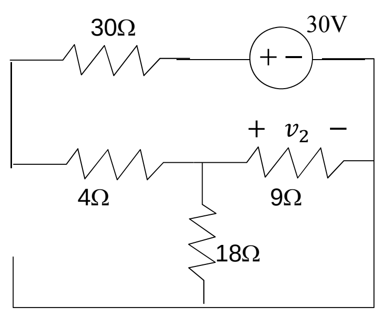


Due to 3A





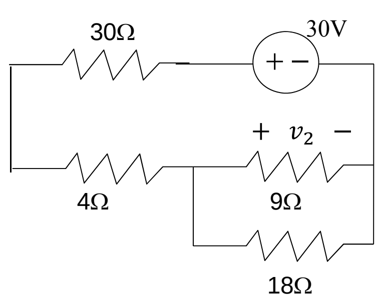
Due to 30V:

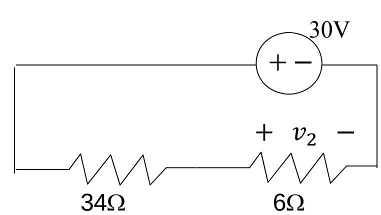


$$v_0 = v_1 + v_2 = 13.5 + 4.5 = 18V$$

By voltage division:

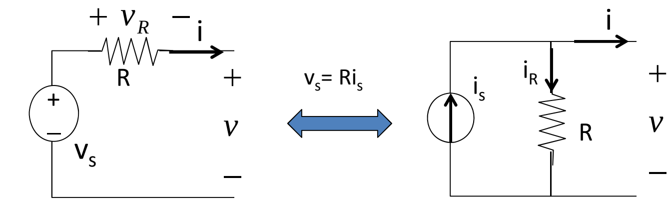
$$v_2 = \frac{6}{6+34} \times 30 = 4.5V$$



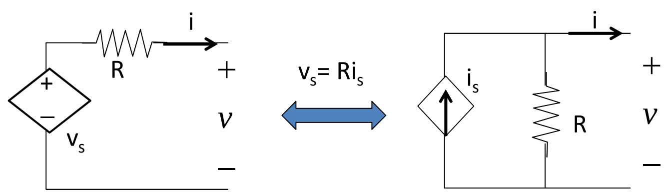


Source Transformation:

The following two connections are equivalent



The following two connections are equivalent



<u>The rule:</u> if you put the two sources side by side,
the arrow of the current source points from – to + of the voltage source

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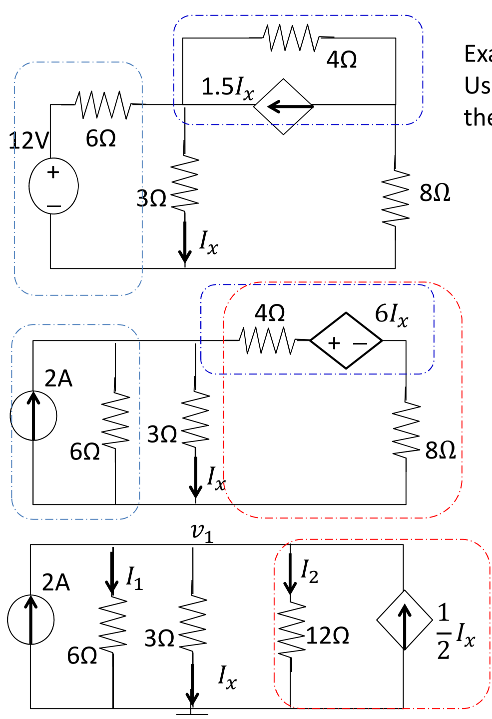


Example : Compute I_x:
Use **source Transformation** to convert the circuit to one with only two nodes.

Use Nodal analysis method:

$$I_1 = \frac{v_1}{6};\ \ I_2 = \frac{v_1}{12};\ \ I_x = \frac{v_1}{3}$$

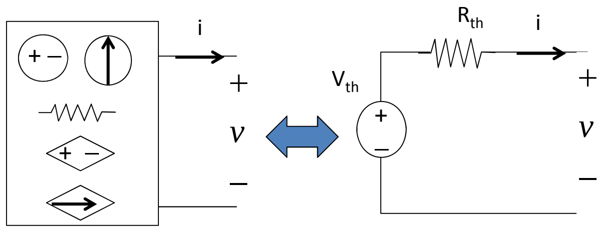
KCL at v_1:

$$I_1 + I_x + I_2 - \frac{1}{2}I_x = 2$$

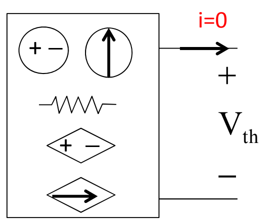
$$\frac{v_1}{6} + \frac{v_1}{3} + \frac{v_1}{12} - \frac{v_1}{6} = 2$$

$$v_1 = 4.8V;\ \ I_x = \frac{4.8}{3} = 1.6A$$

<u>Thevenin's theorem</u>: Every linear two terminal circuit can be made equivalent to the series connection of a voltage source and a resistor.



V_{th}: Open circuit voltage across the two terminals



Also called V_{oc}

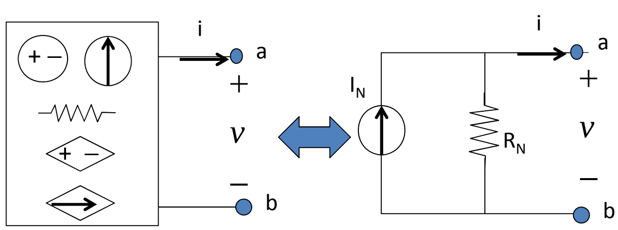
R_{th}: equivalent resistance with respect to the two terminals when all independent sources are turned off

Case 1: No dependent source. R_{th}=R_{eq} (as in Chapter 2)

Case 2: With dependent source. Need to supply an external independent source at the two terminal. Then R_{th} is the ratio between the voltage and current.

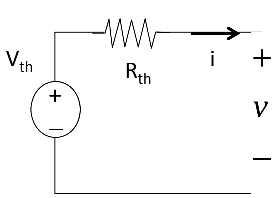
Norton's Theorem

Every linear two terminal circuit can be made equivalent to the parallel connection of a current source and a resistor.



I_N: Short circuit current from "a" to "b", R_N: same as R_{th}

Since the same circuit is also equivalent to Thevenin's



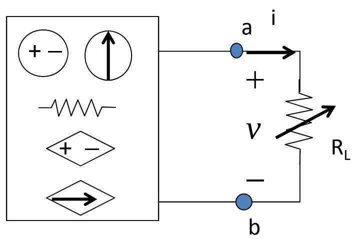
By source transformation,

R_N=R_{th}, V_{th}=$R_{th}I_N$,

I_N=V_{th}/R_{th}

You can get Thevenin's equivalent first, then use source transformation to get Norton's equivalent, or vice vesa

Maximum power transfer



Linear circuit connected to a variable load R_L

Power consumed by load:

$$p = R_L i^2$$

<u>Conclusion</u>: Maximal power is transferred to the load when R_L=R_{th}, and the maximal power is

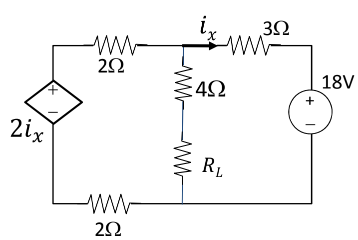
$$p_{max} = \frac{V_{th}^2}{4R_{th}}$$

When the Norton's equivalent is used, similar result:

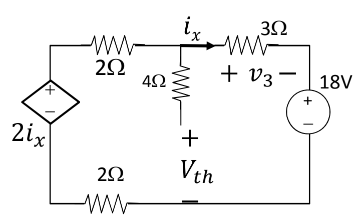
<u>Conclusion</u>: Maximal power is transferred to the load when R_L=R_N, and the maximal power is

$$p_{max} = \frac{R_N I_N^2}{4}$$

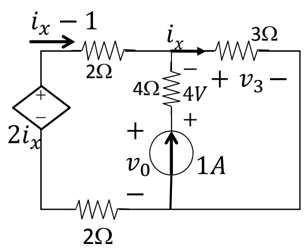
Example 4: Find the unknown resistance R_L so that maximum power is delivered to it. Also find the maximum power.



For V_{th}, disconnect R_L



For R_{th}, turn off 18V, supply current source



$By\ KVL,$

$$2i_x + 3i_x + 18 + 2i_x - 2i_x = 0$$

$$5i_x = -18; \quad i_x = -3.6A$$

$$V_{th} = 18 + 3i_x = 7.2V$$

KVL along outer loop:

$$-2i_x + 2(i_x - 1) + 3i_x + 2(i_x - 1) = 0$$

i_x =0.8, $v_0 = 4 + 3i_x = 6.4V$, $R_{th} = \frac{v_0}{1} = 6.4\Omega$

When $R_L = 6.4\Omega$; $p_{max} = \frac{V_{th}^2}{4R_{th}} = \frac{7.2^2}{4\times 6.4}$=2.025W