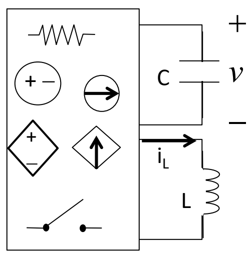
Chapter 8 Second-order circuits

A circuit with two energy storage elements:
- One inductor, one capacitor
- Two capacitors, or
- Two inductors



Such a circuit will be described with second-order differential equation:

$$\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b;$$
$$v(0) = V_0, \qquad \dot{v}(0) = V_1$$

Equivalently:

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2 v = b;$$
$$v(0) = V_0; \quad \frac{dv}{dt}(0) = V_1$$

We first learn how to solve 2nd-order diff equ.

Solving 2nd-order differential equations

Consider a second-order differential equation

$\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b$ with initial condition: $v(0) = V_0$, $\dot{v}(0) = V_1$

Need to find v(t) for all t.

In chapter 8, b=0 corresponds to source free case, b≠0 for step response.

Let $v(\infty) = b/\omega_0^2$ (Assume $\alpha > 0$. Then as $t \to \infty$, $v(t)$ goes to a constant .)

Let the roots to $s^2+2\alpha s+\omega_0^2 = 0$ (not =b) be s_1, s_2, $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

For example:

$$s^2 + 3s + 2 = 0 \Rightarrow (s+1)(s+2) = 0 \Rightarrow s_1 = -1; \; s_2 = -2$$
$$s^2 + 4s + 5 = 0 \Rightarrow (s+2+j)(s+2-j) = 0 \Rightarrow s_1, s_2 = -2 \pm j$$
$$s^2 + 10s + 25 = 0 \Rightarrow (s+5)^2 = 0 \Rightarrow s_1 = s_2 = -5$$

$j = \sqrt{-1}$

The solution will be constructed using the roots.

Three cases:
- Case 1: $\alpha > \omega_0$, two distinct real roots
- Case 2: $\alpha = \omega_0$, two identical real roots
- Case 3: $\alpha < \omega_0$, two distinct complex roots

$$\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b; \qquad (1)$$
$$v(0) = V_0,\ \dot{v}(0) = V_1 \qquad \text{(IC)}$$

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Case 1: $\alpha > \omega_0$, two distinct real roots for $s^2+2\alpha s+\omega_0^2 = 0$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The solution to (1) is not unique.
For any real numbers A_1, A_2, the following function satisfies (1),

$$v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{Where } v(\infty) = b/\omega_0^2$$

This is called the general solution. To verify,

$$\dot{v}(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t} \qquad \ddot{v}(t) = s_1^2 A_1 e^{s_1 t} + s_2^2 A_2 e^{s_2 t}$$

$$\Rightarrow \quad \ddot{v} \quad + \quad 2\alpha\dot{v} \quad + \quad \omega_0^2 v$$
$$= s_1^2 A_1 e^{s_1 t} + s_2^2 A_2 e^{s_2 t} + 2\alpha(s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}) + \omega_0^2(v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t})$$
$$= A_1 e^{s_1 t}(\underbrace{s_1^2 + 2\alpha s_1 + \omega_0^2}_{0}) + A_2 e^{s_2 t}(\underbrace{s_2^2 + 2\alpha s_2 + \omega_0^2}_{0}) + \omega_0^2 v(\infty)$$
$$= \omega_0^2 v(\infty) = \omega_0^2 \frac{b}{\omega_0^2} = b \quad \Rightarrow$$

v(t) satisfies (1) for any A_1, and A_2

The solution will be uniquely determined by the initial conditions (IC)

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$$\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b; \qquad (1)$$
$$v(0) = V_0,\ \dot{v}(0) = V_1 \qquad \text{(IC)}$$

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The general solution

$$v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{Where } v(\infty) = b/\omega_0^2$$

There is only one pair of A_1, A_2 that satisfy the IC

To find A_1, A_2, use IC to form two equations. Recall: $\dot{v}(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$

Evaluate v(t) and $\dot{v}$(t) at t=0:

$$v(0) = v(\infty) + A_1 + A_2$$
$$\dot{v}(0) = s_1 A_1 + s_2 A_2$$

To satisfy the IC, we obtain

$$\begin{aligned} v(\infty) + A_1 + A_2 &= V_0 \\ s_1 A_1 + s_2 A_2 &= V_1 \end{aligned} \quad \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} V_0 - v(\infty) \\ V_1 \end{bmatrix} \Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix}^{-1}\begin{bmatrix} V_0 - v(\infty) \\ V_1 \end{bmatrix}$$

You may use other methods to solve for A_1 and A_2.

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$\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b$ (To compare)

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Example: Solve $\ddot{v} + 4\dot{v} + 3v = 6,\quad v(0) = 1;\ \dot{v}(0) = -1$

$2\alpha = 4; \omega_0^2 = 3; b = 6 \Rightarrow \alpha = 2; \omega_0 = \sqrt{3};\qquad s^2 + 2\alpha s + \omega^2 = 0.$

$\alpha > \omega_0; \Rightarrow$ Case 1, two distinct real roots for $s^2 + 4s + 3 = 0$.

Step 1: Find s_1, s_2, $v(\infty)$

$$\boxed{s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$$

$$s_1, s_2 = -2 \pm \sqrt{2^2 - 3} = -2 \pm 1,\quad s_1 = -1, s_2 = -3$$

$v(\infty) = b/\omega_0^2$ =6/3 =2

Note:
s_1, s_2 are roots to:
$s^2 + 4s + 3 = 0$
$(s+3)(s+1) = 0$
$s_1 = -1, s_2 = -3$

The general solution: $v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$$v(t) = 2 + A_1 e^{-t} + A_2 e^{-3t}$$

Step 2: Find A_1,A_2, use initial condition

$$v(0) = 2 + A_1 + A_2 = 1 \quad (*)$$
$$\dot{v}(0) = -A_1 - 3A_2 = -1 \quad (**)$$

Note:
$\dot{v}(t) = -A_1 e^{-t} - 3A_2 e^{-3t}$

Solve (*) and (**) to obtain A_1=-2; A_2=1

Finally, $\boxed{v(t) = 2 - 2e^{-t} + e^{-3t}}$

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$$\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b; \qquad (1)$$
$$v(0) = V_0,\ \dot{v}(0) = V_1 \qquad \text{(IC)}$$

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An Example: $\ddot{v} + 4\dot{v} + 3v = 6,\quad v(0) = 1;\ \dot{v}(0) = -1$

The general solution: $v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t},\quad v(\infty) = \dfrac{b}{\omega_0^2}$

$$v(t) = 2 + A_1 e^{-t} + A_2 e^{-3t}$$

Since s_1, s_2 are negative,

$$\lim_{t\to\infty} v(t) = 2 = v(\infty)$$

Another way to see $v(\infty) = b/\omega_0^2$

$$\dot{v}(t) = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t},\quad \lim_{t\to\infty} \dot{v}(t) = 0$$
$$\ddot{v}(t) = A_1 s_1^2 e^{s_1 t} + A_2 s_2^2 e^{s_2 t},\quad \lim_{t\to\infty} \ddot{v}(t) = 0$$

From (1), $\lim_{t\to\infty} \left(\ddot{v}(t) + 2\alpha\dot{v}(t) + \omega_0^2 v(t)\right) = \omega_0^2 v(\infty) \quad = b$

$$v(\infty) = \frac{b}{\omega_0^2}$$

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Case 2: $\alpha = \omega_0$ $s^2 + 2\alpha s + \alpha^2 = (s+\alpha)^2$

Two identical roots, $s_1 = s_2 = -\alpha$ to $s^2+2\alpha s+\omega_0^2=0$

The general solution: $v(t) = v(\infty) + (A_1 + A_2 t)\, e^{-\alpha t}$ $v(\infty) = b/\omega_0^2$

Its derivative: $\dot{v}(t) = A_2 e^{-\alpha t} - \alpha(A_1 + A_2 t)e^{-\alpha t}$

Use IC to find A_1, A_2.

At t = 0, $v(0) = v(\infty) + A_1 = V_0$ $\Rightarrow$ $A_1 = V_0 - v(\infty)$

$\dot{v}(0) = A_2 - \alpha A_1 = V_1$ $A_2 = V_1 + \alpha A_1$

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Case 3: $\alpha < \omega_0$

Two complex roots to $s^2+2\alpha s+\omega_0^2=0$

Let $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ $j = \sqrt{-1}$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\,\omega_d$$

$$s_1, s_2 = -\alpha \pm j\,\omega_d$$

The general solution:

$$v(t) = v(\infty) + e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \qquad v(\infty) = b/\omega_0^2$$

Its derivative:

$$\dot{v}(t) = -\alpha e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) + e^{-\alpha t}(-\omega_d B_1 \sin\omega_d t + \omega_d B_2 \cos\omega_d t)$$

Use IC to find B_1, B_2.

$v(0) = v(\infty) + B_1 = V_0$ $\Rightarrow$ $B_1 = V_0 - v(\infty)$

$\dot{v}(0) = -\alpha B_1 + \omega_d B_2 = V_1$ $B_2 = (V_1 + \alpha B_1)/\omega_d$

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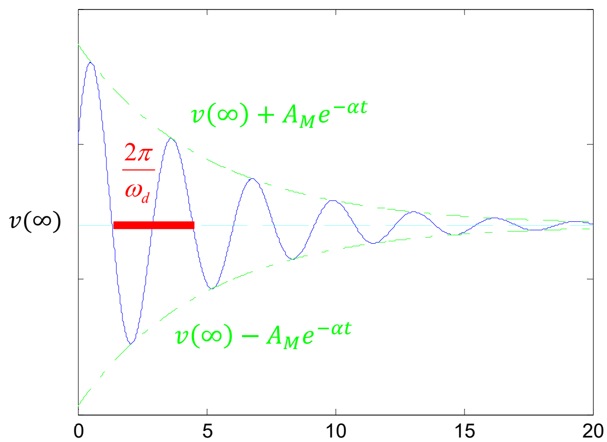
$$v(t) = v(\infty) + e^{-\alpha t}(B_1 \cos\omega_d t \ + B_2 \sin\omega_d t)$$



Case 3: $\alpha < \omega_0$

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Example: Solve $\ddot{v} + 8\dot{v} + 25v = 200, \quad \dot{v}(0) = 0, \ v(0) = 12$

Compare with $\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b$

Solution: $\alpha = ?\, \omega_0 = ?, b = ? \ \ which\ case?$

$\alpha = 4, \omega_0 = 5, b = 200, \quad \alpha < \omega_0 \rightarrow$ case 3, two complex roots : $-\alpha \pm j\omega_d$

Step 1: Form general solution using roots and $v(\infty)$.

$v(t) = v(\infty) + e^{-\alpha t}(B_1 \cos\omega_d t \ + B_2 \sin\omega_d t\,),\ v(\infty) = ? \ \ \omega_d = ?$

Imaginary part of the roots: $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{25 - 16} = 3$

Complex roots: $-\alpha \pm j\omega_d^2 = -4 \pm j3$

$v(\infty) = ?\, v(\infty) = \frac{b}{\omega_0^2} = \frac{200}{25} = 8 \qquad \cancel{\ddot{v}(\infty)}^{0} + 8\cancel{\dot{v}(\infty)}^{0} + 25v(\infty) = 200$

$v(t) = 8 + e^{-4t}(B_1 \cos 3t \ + B_2 \sin 3t\,),$ ← General solution

Step 2: Find B_1, B_2, using initial conditions, $\quad \dot{v}(0) = 0, \ v(0) = 12$

$\dot{v}(t) = -4e^{-4t}\,(B_1 \cos 3t + B_2 \sin 3t) + e^{-4t}\,(-3B_1 \sin 3t + 3B_2 \cos 3t\,)$

$\dot{v}\,(0) = -4B_1 + 3B_2 = 0$

$v(0) = 8 + B_1 = 12$ ⇨ $B_1 = 4; B_2 = \frac{16}{3}$

Final solution: $v(t) = 8 + e^{-4t}\,(4\cos 3t + \frac{16}{3}\sin 3t\,)\ V$

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Practice 1: Solve

$$\ddot{v} + 5\,\dot{v} + 6v = 18, \qquad v(0) = 0; \quad \dot{v}(0) = 4$$

Practice 2: Solve

$$\ddot{v} + 4\,\dot{v} + 4v = 8, \qquad v(0) = 1; \quad \dot{v}(0) = -1$$

Practice 3: Solve

$$\ddot{v} + 4\,\dot{v} + 13v = 39, \qquad v(0) = 1; \quad \dot{v}(0) = -1$$

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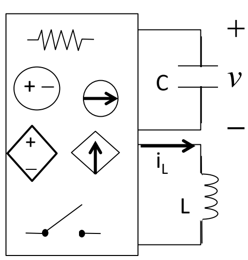
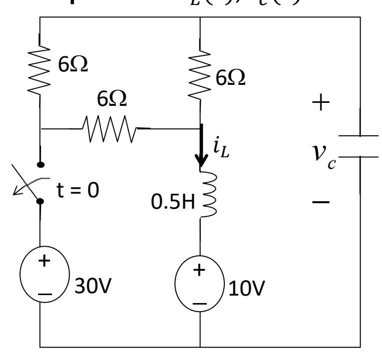
Second-order RLC circuits

General circuit:



Example : Find $i_L(t), v_c(t)$ for t > 0. L21



As always, $i_L(t), v_c(t)$ are continuous function of time.
Use the circuit before switch to find $i_L(0), v_c(0)$.

For t>0, $v_c(t)$ satisfies

$$\frac{d^2 v_c}{dt^2} + 2\alpha\frac{dv_c}{dt} + \omega_0^2 v_c = b;$$

$$v_c(0) = V_0; \quad \frac{dv_c}{dt}(0) = V_1$$

We also need $\frac{dv_c}{dt}(0)$.

Generally, $\frac{dv_c}{dt}(t)$ is not continuous at t=0.

How to find $\frac{dv_c}{dt}(0)$?

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Converting a circuit problem to a math problem L21

To solve a second order circuit, we need to

1. Derive the differential equation: $\ddot{v} + 2\alpha\dot{v} + \omega_0^2 v = b;$
2. Find the initial condition $v(0) = V_0,\ \dot{v}(0) = V_1$

§ 8.2. Finding initial values

We need to find $v_c(0^+), i_L(0^+), \dfrac{dv_C(0^+)}{dt}, \dfrac{di_L(0^+)}{dt}$

Key points:

1. Capacitor voltage and inductor current don't jump

   $v_C(0^+) = v_C(0^-), i_L(0^+) = i_L(0^-)$

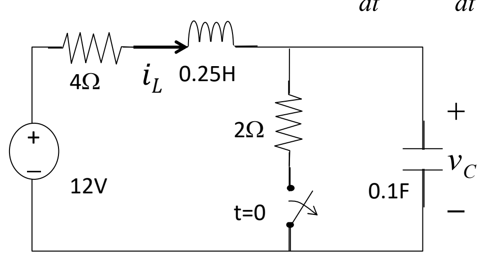
   Use circuit before switch (t<0) to find these two.

2. For $\dfrac{dv_C(0^+)}{dt}, \dfrac{di_L(0^+)}{dt}$ Recall: $v_L = L\frac{di_L}{dt},\ \ i_c = C\frac{dv_c}{dt}$

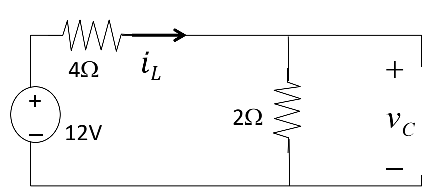
   Consider the circuit right after switch, i.e., at t = 0^+

- Apply KVL to find $v_L(0^+)$, then $\dfrac{di_L(0^+)}{dt} = \dfrac{1}{L}v_L(0^+),$
- Apply KCL to find $i_C(0^+)$, then $\dfrac{dv_C(0^+)}{dt} = \dfrac{1}{C}i_C(0^+),$

Example: Find $v_C(0^+), i_L(0^+), \dfrac{dv_C(0^+)}{dt}, \dfrac{di_L(0^+)}{dt}$ L21



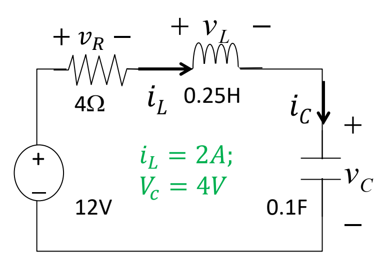
Use circuit before switch to find $v_C(0^+), i_L(0^+),$
At t = 0^-



$i_L(0^-) = 2A;\ v_c(0^-) = 2 \times 2 = 4V$

By continuity, $i_L(0^+) = 2A,\ \ v_C(0^+) = 4V$

To find $\dfrac{dv_c(0^+)}{dt}, \dfrac{di_L(0^+)}{dt}$



Consider t = 0^+

$$v_L(0^+) = 12 - 4\times 2 - 4 = 0V$$

$$\frac{di_L(0^+)}{dt} = \frac{1}{L}v_L(0^+) = 0$$

$$i_c(0^+) = i_L(0^+) = 2\text{A}$$

$$\frac{dv_c(0^+)}{dt} = \frac{1}{C}i_c(0^+) = \frac{2}{0.1} = 20\ V/s,$$

Chapter 8 circuits

1). Source free series RLC circuits
2). Source free parallel RLC circuits
3). Step response of series RLC circuits
4). Step response of parallel RLC circuits
5). Other second order circuits

We will focus on 1) and 3).
We study 3) first.
1) is a special case of 3).

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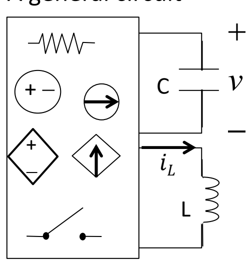
§ 8.5 Step response of series RLC circuits

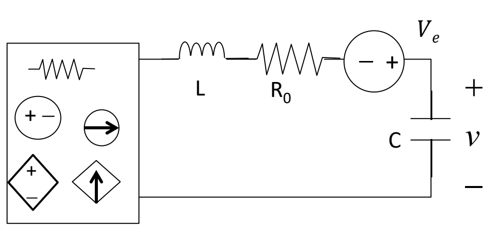
A general circuit



Before switch (t<0),
L and C may not be in series.

Determine $v_c(0), i_L(0)$
from circuit before switch
by considering L as short,
C as open.

After switch (t>0), L and C in series



By Thevenin's theorem,
the two terminal circuit
can be replaced by a
voltage source and a resistor

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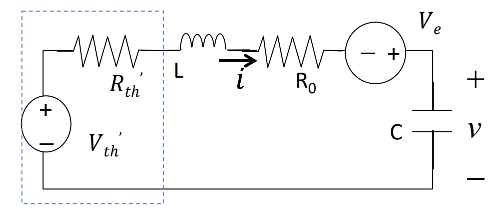
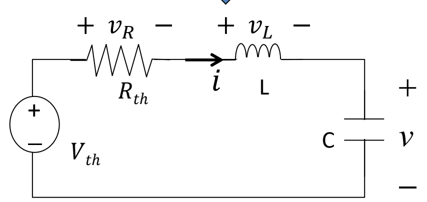
By Thevenin's theorem, the two terminal circuit can be replaced by a voltage source and a resistor



$V_{th} = V_{th}' + V_e$
$R_{th} = R_{th}' + R_0$

Rearranging the elements does not change loop current i and capacitor v



We call (V_{th}, R_{th}) Thevenin's equivalent w.r.t LC

Given $v(0^+) = V_0,$ $\dfrac{dv(0^+)}{dt} = V_1$

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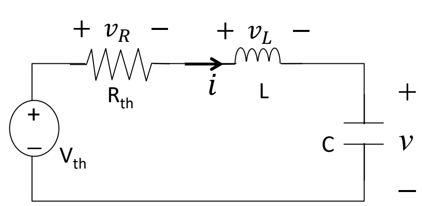
The equivalent circuit:



(V_{th}, R_{th}) :
Thevenin's equivalent w.r.t LC

Given $v(0^+) = V_0,$ $\dfrac{dv(0^+)}{dt} = V_1$

Choose v as key variable. We derive diff. equation for v

Express other variables in terms of v:

$$i = C\frac{dv}{dt}; \quad v_R = R_{th} i = R_{th} C\frac{dv}{dt}; \qquad v_L = L\frac{di}{dt} = LC\frac{d^2v}{dt^2}$$

By KVL,

$$v_R + v_L + v = V_{th} \qquad R_{th}C\frac{dv}{dt} + LC\frac{d^2v}{dt^2} + v = V_{th};$$

Normalize: $$\frac{d^2v}{dt^2} + \frac{R_{th}}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{V_{th}}{LC};$$

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Step response of series RLC circuit:

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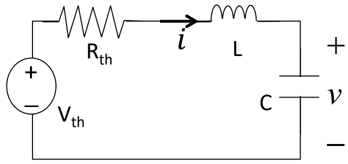

$$\frac{d^2v}{dt^2}+\frac{R_{th}}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{V_{th}}{LC};\ v(0^+)=V_0,\ \frac{dv(0^+)}{dt}=V_1$$

$$\frac{d^2v}{dt^2}+2\alpha\frac{dv}{dt}+\omega_0^2 v=b;$$ As compared with general form

$$\alpha=\frac{R_{th}}{2L},\quad \omega_0=\frac{1}{\sqrt{LC}},\quad b=\frac{V_{th}}{LC},\quad v(\infty)=\frac{b}{\omega_0^2}=V_{th}$$

If $V_{th}=0$,
source free RLC,
$\Rightarrow$ A special case

The solution:

Case 1: $\alpha>\omega_0;\ s_1,s_2=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$ $\qquad v(t)=v(\infty)+A_1e^{s_1t}+A_2e^{s_2t}$

Case 2: $\alpha=\omega_0;\ s_1=s_2=-\alpha$ $\qquad v(t)=v(\infty)+(A_1+A_2t)e^{-\alpha t}$

Case 3: $\alpha<\omega_0;\ \omega_d=\sqrt{\omega_0^2-\alpha^2},\ s_1,s_2=-\alpha\pm j\omega_d$

$$v(t)=v(\infty)+e^{-\alpha t}(B_1\cos\omega_d t+B_2\sin\omega_d t)$$

A_1,A_2,or B_1,B_2 will be determined by initial conditions

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Case 1: $\alpha>\omega_0;\ s_1,s_2=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$ $\qquad v(t)=v(\infty)+A_1e^{s_1t}+A_2e^{s_2t}$

L21

Find A_1,A_2 from
$$v(0)=v(\infty)+A_1+A_2$$
$$\frac{dv(0^+)}{dt}=s_1A_1+s_2A_2$$

Case 2: $\alpha=\omega_0;\ s_1=s_2=-\alpha$ $\qquad v(t)=v(\infty)+(A_1+A_2t)e^{-\alpha t}$

Find A_1,A_2 from
$$v(0)=v(\infty)+A_1$$
$$\frac{dv(0^+)}{dt}=-\alpha A_1+A_2$$

Case 3: $\alpha<\omega_0;\ \omega_d=\sqrt{\omega_0^2-\alpha^2},\ s_1,s_2=-\alpha\pm j\omega_d$

$$v(t)=v(\infty)+e^{-\alpha t}(B_1\cos\omega_d t+B_2\sin\omega_d t)$$

Find B_1,B_2 from
$$v(0)=v(\infty)+B_1$$
$$\frac{dv(0^+)}{dt}=-\alpha B_1+\omega_d B_2$$

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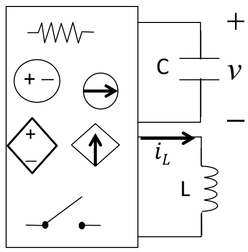
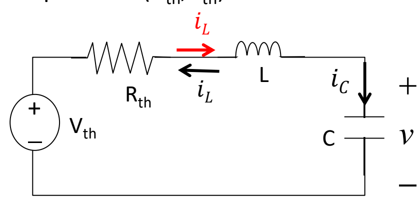
Key points for step resp. of series RLC circuits:



Obtain $v(0)$, $i_L(0)$ from circuit before switch

For circuit after switch, obtain Thevenin's equivalent (V_{th},R_{th}) w.r.t LC



Assign i_C by passive sign convention

$$\frac{d^2v}{dt^2}+\frac{R_{th}}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{V_{th}}{LC};$$
$$\alpha=\frac{R_{th}}{2L},\quad \omega_0=\frac{1}{\sqrt{LC}},\quad v(\infty)=V_{th}$$

$$i_C(0^+)=\pm i_L(0)$$

Depending on how i_L is assigned. Then

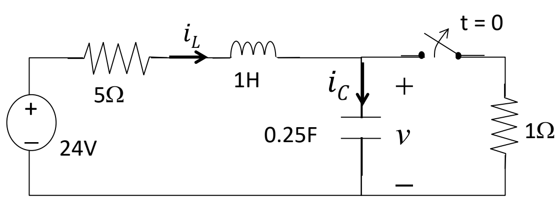
$$\frac{dv(0^+)}{dt}=\frac{i_C(0^+)}{C}$$

Then follow straightforward math computation on previous slide

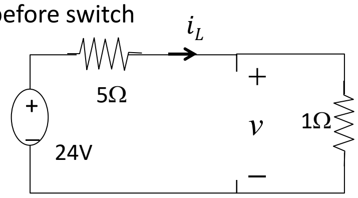
Key parameters to obtain:

- $v(0), i_L(0)$ from circuit before switch
- V_{th}, R_{th} from circuit after switch.

Example 1: Find v(t) for t > 0.



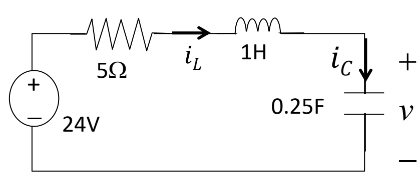
Step 1: find v(0), i_L(0) from circuit before switch



i_L(0)=24/6=4A

v(0) = 4V

Step 2: Find V_{th}, R_{th}, w.r.t LC from circuit after switch



V_{th}=24V, R_{th}=5Ω v(∞) = 24V

Since $i_C = i_L$, $\frac{dv_C(0^+)}{dt}=\frac{i_L(0)}{C}=\frac{4}{0.25}=16V/s$

Step 3: Math computation

L21

$$\alpha = \frac{R_{th}}{2L} = \frac{5}{2} = 2.5,\ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.25}} = 2 \qquad \alpha > \omega_0,\ \text{Case 1}$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2.5 \pm \sqrt{6.25 - 4} = -2.5 \pm 1.5 \qquad s_1 = -1, s_2 = -4$$

The general solution: $v(t) = 24 + A_1 e^{-t} + A_2 e^{-4t}$

Note:
$v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

Find A_1, A_2 from initial condition: v(0) = 4V; $dv(0^+)/dt=16$

$$v(0) = 24 + A_1 + A_2 = 4$$
$$\frac{dv(0^+)}{dt} = -A_1 - 4A_2 = 16$$
$$\Rightarrow A_1 = -64/3,\ A_2 = 4/3$$

Final solution: $v(t) = 24 - \frac{64}{3} e^{-t} + \frac{4}{3} e^{-4t}\ V$

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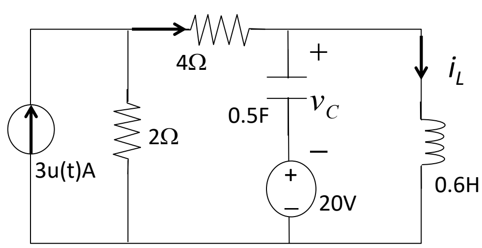
Practice 4: Find $v_C(0^+), i_L(0^+), \frac{dv_C(0^+)}{dt}, \frac{di_L(0^+)}{dt}$

R21



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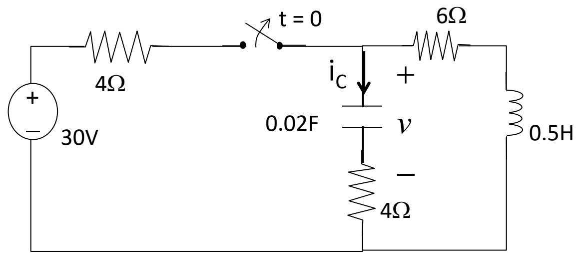
R21

Practice 5: Find v(t) for t > 0.



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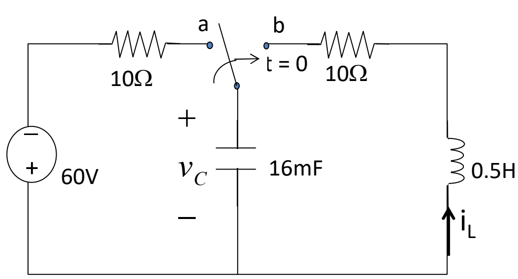
R21

Practice 6: The switch is at position a for a long time before swinging to position b at t = 0. Find $i_L(t)$, $v_C(t)$ for t > 0.



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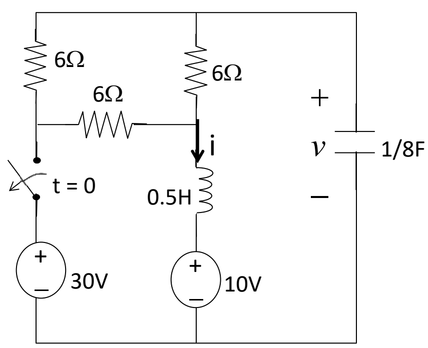
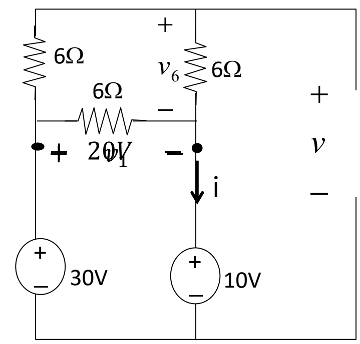
Example 2: Find i(t), v(t) for t > 0.



Step 1: find v(0), i(0) from circuit before switch

L22



A simple circuit in Chapter 2.

$$i(0) = \frac{(30-10)}{6//(6+6)} = \frac{20}{4} = 5A$$

By voltage division,

$$v_6 = \frac{6}{6+6} \times (30-10) = 10V$$

By KVL, v(0)=20V

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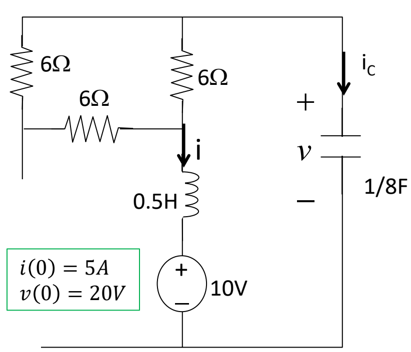
Step 2: After switch



$i(0) = 5A$, $\dfrac{dv(0^+)}{dt} = ?$

Since $i_C = -i$, $i_C(0^+) = -5A$

$\dfrac{dv(0^+)}{dt} = -5/(1/8) = -40V/s$

R_{th}=?, V_{th}=?

R_{th}=(6+6)//6=4Ω, V_{th}=10V = v(∞)

Step 3: math

L22

$\alpha = ?\ \omega_0 = ?$ Which case?

$$\alpha = \frac{R_{th}}{2L} = \frac{4}{2\times 0.5} = 4; \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5/8}} = 4$$

➡ Case 2.

General solution: $v(t) = 10+(A_1+A_2t)e^{-4t}$

Find A_1,A_2 using v(0), $\dfrac{dv(0^+)}{dt}$:

$$v(0) = 10 + A_1 = 20$$

A_1=10, A_2=0

$$\frac{dv(0^+)}{dt} = -4A_1 + A_2 = -40$$

Final solution: $v(t) = 10 + 10\,e^{-4t}$V

Find inductor current,

$$i(t) = -i_c(t) = -C\frac{dv}{dt}$$

$i(t) = -(1/8)*10(-4)e^{-4t} = 5\,e^{-4t}$A

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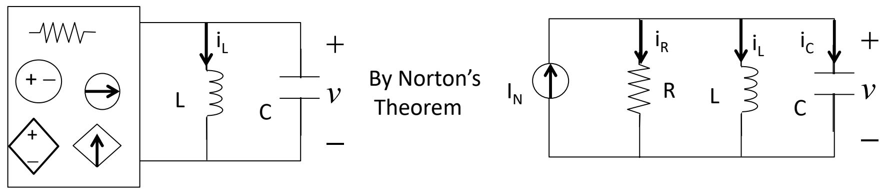
For parallel RLC circuit



Choose i_L as key variable. Derive diff equ. with KCL

$$v = L\frac{di_L}{dt}, \quad i_R = \frac{L}{R}\frac{di_L}{dt}, \quad i_C = LC\frac{d^2 i_L}{dt^2}$$

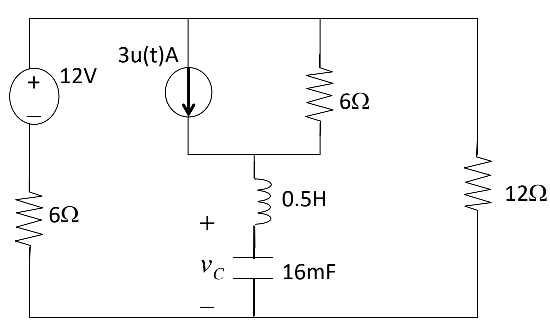
$$i_R + i_C + i_L = I_N \qquad \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} + i_L = I_N$$

Normalize: $\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{I_N}{LC} \qquad \alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad b = \frac{I_N}{LC}$

Same math problem.

No parallel RLC circuits in homework or Final Exam.

Example 3: Find $v_C(t)$ for $t > 0$.



Recall:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Thus for t<0, 3u(t)A=0A

The current source is turned off.

Replace it with open circuit for t<0

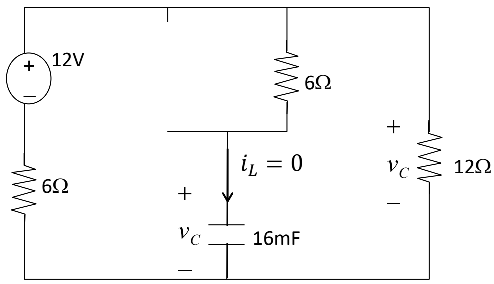
Key parameters to obtain:
- $v_C(0)$, $i_L(0)$ from circuit before switch (t<0)
- V_{th}, R_{th} w.r.t. LC from circuit after switch (t>0) .

Step 1: Find $v_c(0), i_L(0)$ from circuit for t<0

Step 1: Find $v_c(0), i_L(0)$ from circuit for t<0

L22

For t < 0, 3u(t)A=0, current source open, C open , L short



Since i_L=0, 12V, 6Ω *and* 12Ω in the outer loop are in series

By voltage division:

$$\boxed{v_c(0) = v_R = \frac{12}{6+12} \times 12 = 8V,}$$

$$\boxed{i_L(0) = 0, \quad \frac{dv_c(0)}{dt} = 0}$$

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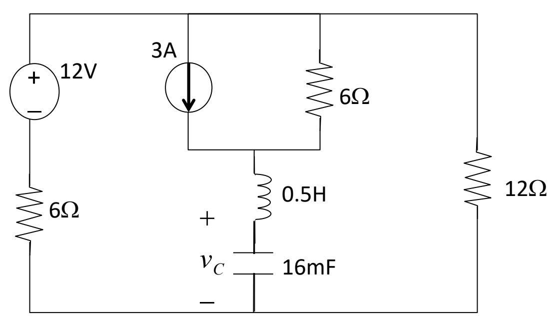
Step 2: For t>0, 3u(t)A=3A. Need to find $V_{th} = v_c(\infty), R_{th}$ w.r.t LC

L22



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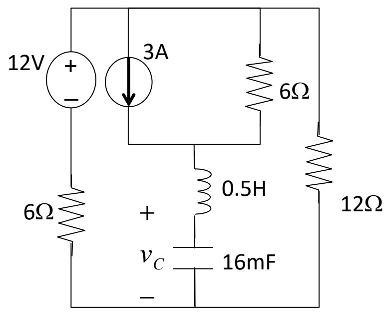
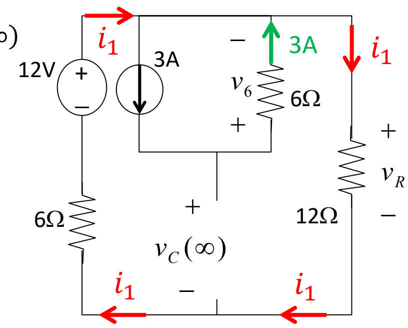
Step 2: Fot t>0, 3u(t)A=3A. Need to find $V_{th} = v_c(\infty), R_{th}$ w.r.t LC

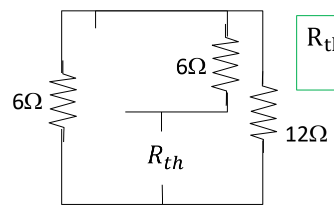
L22



For $v(\infty)$



For R_{th}, turn off 12V with short,
turn off 3A with open



$$\mathrm{R_{th}} = 6 + 6//12$$
$$=10\Omega$$

By KVL, $v_c(\infty) = v_6 + v_R$

$v_6 = 3 \times 6 = 18V$, by ohms law

$i_1 = \frac{12}{18}A = \frac{2}{3}A,$

$v_R = 12i_1 = 12 \times \frac{2}{3} = 8V$

$$\boxed{v_c(\infty) = 18 + 8 = 26V}$$

Step 3: Math

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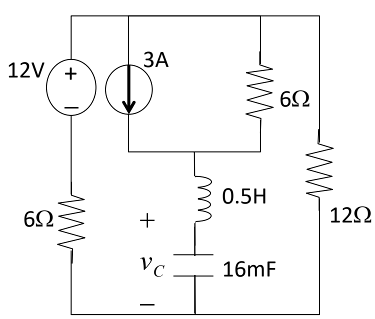
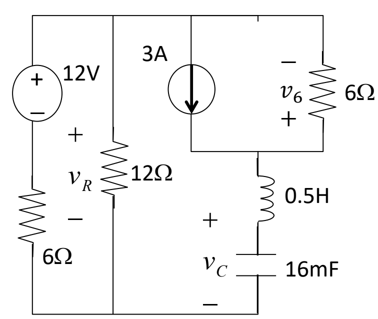
Step 2: Another way to look at the circuit for t>0

L22



To see connection better, move 12 Ω to the left



$R_{th} = 6 + 12//6 = 10\Omega$

Under DC condition,

$v_R = 8V, v_6 = 18V,$

$v_c(\infty) = V_{th} = 8 + 18 = 26V$

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Step 3: math From step 1, step 2, found

$R_{th} = 10\Omega,\ \ v_c(\infty) = V_{th} = 8 + 18 = 26V,\ \ v_c(0) = 8V, \frac{dv_c(0^+)}{dt} = 0$

Also note: $L = 0.5H;\ \ C = 16mF = 0.016F \quad \alpha = ?, \omega_0 = ?\ \ which\ case?$

$$\alpha = \frac{R_{th}}{2L};\quad \omega_0 = \frac{1}{\sqrt{LC}};$$

$$\alpha = \frac{R_{th}}{2L} = \frac{10}{2 \times 0.5} = 10;\quad \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{1}{0.5 \times 0.016}} = \sqrt{125},$$

$\alpha < \omega_0,\ case\ 3, \quad \omega_d = \sqrt{125 - 100} = 5$

The general solution:

$\mathrm{v_c(t) = 26 + e^{-10t}(B_1 \cos 5t + B_2 \sin 5t)}$

$v(0) = v(\infty) + B_1$

$\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2$

To satisfy initial condition:

$v_c(0) = 26 + B_1 = 8$

$\dot{v}_c(0) = -10B_1 + 5B_2 = 0$

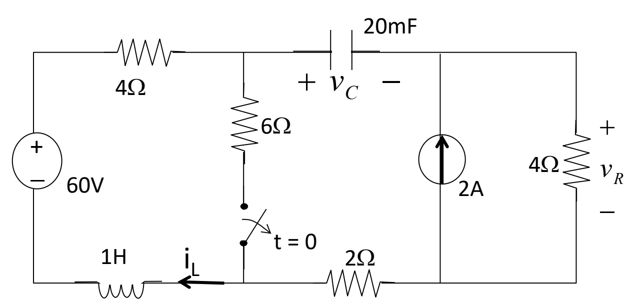
Solving equations to get $B_1 = -18; B_2 = -36$

Step response:

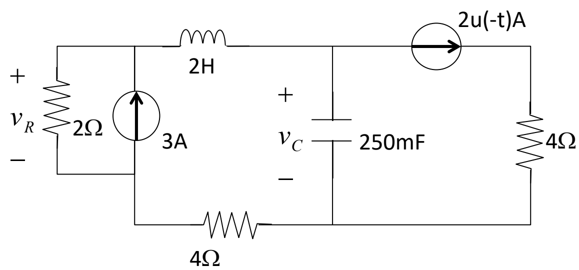
$\mathrm{v_c(t)} = 26 + \mathrm{e}^{-10t}(-18\cos 5t - 36 \sin 5t)V$

Practice 7: Find $i_L(t)$, $v_C(t)$, $v_R(t)$ for t > 0.

R22



Practice 8: Find $v_C(t)$ and $v_R(t)$ for $t > 0$. R22



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Chapter 7, Chapter 8 Review

Final exam: 12/16/19 (Monday),
8-11am,
BL210
One page one-sided note allowed,
Calculators allowed
Don't include the solution to any circuit problem in the note!

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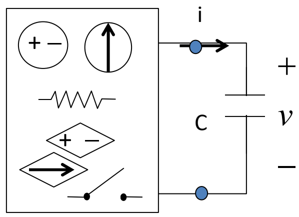
Step response of general RC circuit



One or more switches changes structure of circuit at t = 0.

$For\ \ t > 0,$ step response:

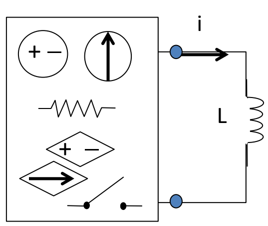
$$v(t) = v(\infty) + (v(0) - v(\infty))\, e^{-\frac{t}{R_{th}C}}$$

Three key parameters:

1. Initial condition $v(0)=V_0$,
   Established from circuit before switch. Obtained by solving a DC circuit, t < 0 (capacitor = open circuit)
2. Final condition $v(\infty)$
   Established from circuit after switch. Obtained by solving another DC circuit, t > 0 (capacitor = open circuit)
3. Equivalent resistance R_{th}, or R_{eq}
   with respect to capacitor for circuit after switch

Since $v(\infty) = V_{th}$, (V_{th}, R_{th}) can be together considered as the Thevenin's equivalent, with respect to the capacitor, from circuit after switch.

Step response of RL circuits



Step response – For t > 0,

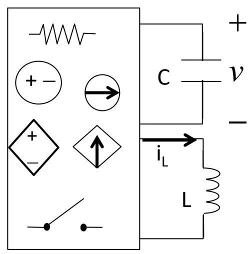
$$i(t) = i(\infty) + (i(0) - i(\infty))e^{-\frac{R_N}{L}t}$$

Three key parameters:

1. i(0), initial condition, from the DC circuit for t < 0. (inductor =short)
2. $i(\infty)=I_N$, final value, from DC circuit for t > 0. (inductor =short)
3. $R_N=R_{th}$, equivalent resistance w.r.t inductor from circuit for t > 0

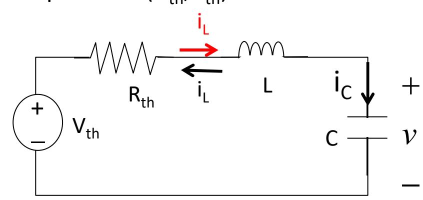
(I_N, R_N) together as Norton's equivalent, w.r.t inductor, for t > 0.

Key points for step resp. of series RLC circuits:



Obtain v(0), $i_L(0)$ from circuit before switch

For circuit after switch, obtain Thevenin's equivalent (V_{th},R_{th}) w.r.t LC



Assign i_C by passive sign convention

$$\frac{d^2v}{dt^2}+\frac{R_{th}}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{V_{th}}{LC};$$

$$\alpha=\frac{R_{th}}{2L},\quad \omega_0=\frac{1}{\sqrt{LC}},\quad v(\infty)=V_{th}$$

$$i_C(0^+)=\pm i_L(0)$$

Depending on how i_L is assigned. Then

$$\frac{dv(0^+)}{dt}=\frac{i_C(0^+)}{C}$$

Then follow straightforward math computation on next slide

Key parameters to obtain:

- v(0), $i_L(0)$ from circuit before switch
- $V_{th,}$ R_{th} from circuit after switch.

R_{th} is the total equivalent resistance in the loop

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Case 1: $\alpha>\omega_0;\, s_1,s_2=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$ $\quad v(t)=v(\infty)+A_1e^{s_1t}+A_2e^{s_2t}$

Find A_1,A_2 from
$$\begin{cases} v(0)=v(\infty)+A_1+A_2 \\ \dfrac{dv(0^+)}{dt}=s_1A_1+s_2A_2 \end{cases}$$

Case 2: $\alpha=\omega_0;\, s_1=s_2=-\alpha$ $\quad v(t)=v(\infty)+(A_1+A_2t)e^{-\alpha t}$

Find A_1,A_2 from
$$\begin{cases} v(0)=v(\infty)+A_1 \\ \dfrac{dv(0^+)}{dt}=-\alpha A_1+A_2 \end{cases}$$

Case 3: $\alpha<\omega_0;\, \omega_d=\sqrt{\omega_0^2-\alpha^2},\, s_1,s_2=-\alpha\pm j\omega_d$

$$v(t)=v(\infty)+e^{-\alpha t}(B_1\cos\omega_dt+B_2\sin\omega_dt)$$

Find B_1,B_2 from
$$\begin{cases} v(0)=v(\infty)+B_1 \\ \dfrac{dv(0^+)}{dt}=-\alpha B_1+\omega_dB_2 \end{cases}$$

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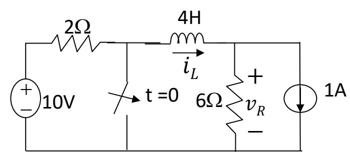
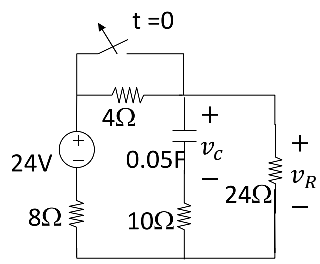
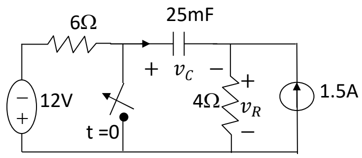
Problem 1: The switch in the circuit has been open for a long time before closed at t = 0. Find $i_L(t)$ *and* $v_R(t)$ for t ≥ 0. ($i_L(1) = 1.2231A,\ v_R(1) = 1.3388V$)



Problem 2: The switch in the circuit has been closed for a long time before opening at t = 0. Find $v_c(t), v_R(t)$ for t ≥ 0.



Problem 3: The switch in the circuit has been closed for a long time before opening at t = 0. Find $v_R(t)$ and $v_C(t)$ for t ≥ 0. $(v_c(1) = -17.78V,\ v_R(1) = 5.91V)$



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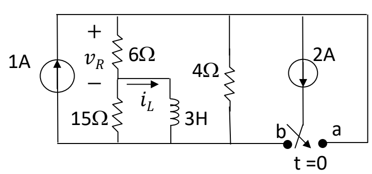
Problem 4: The switch has been in position “**b”** for a long time before moving to position “**a**” at t = 0. Find $i_L(t)$ and $v_R(t)$ for t ≥ 0. $(i_L(1) = 0.2917A,\ v_R(1) = 2.0102V)$



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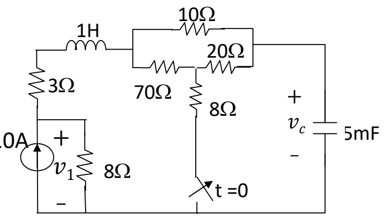
Problem 5: The switch has been closed for a long time before open at t = 0. Find v_c(t) and v_1(t) for t > 0. ($v_c(0.1) = 74.0827V,\ v_1(0.1) = 72.8574V$)



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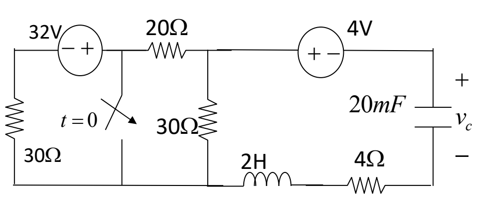
Problem 6: The switch is open for a long time before closed at t = 0. Find $v_c(t)$ for t > 0. ($v_c(1) = -4.1762V$)



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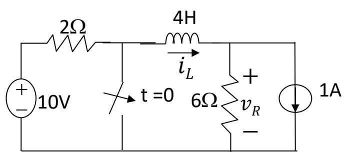
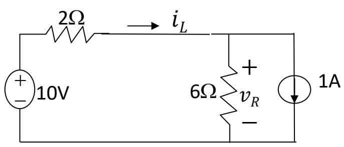
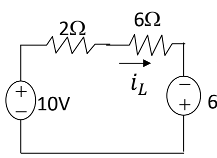
Problem 1: The switch in the circuit has been open for a long time before closed at t = 0.
Find $i_L(t)$ and $v_R(t)$ for t ≥ 0. $(i_L(1) = 1.2231A,\ v_R(1) = 1.3388V)$



Step 1) Find $i_L(0)$ from circuit before switch

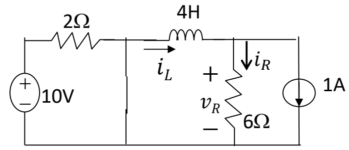


Use source transformation:



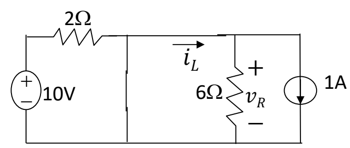
$i_L(0) = \frac{16}{8} = 2A$

Step 2): For t > 0 (After switch):



$i_L(\infty)$ =?

At t = ∞



6Ω is short circuited. $v_R = 0$.
No current through 6Ω. $i_L = 1A$.
($10V$ and 2Ω have no effect)

$i_L(\infty) = 1A$

R_{th} w.r.t inductor? $R_{th} = 6\Omega$, $\frac{R_{th}}{L} = \frac{6}{4} = 1.5$

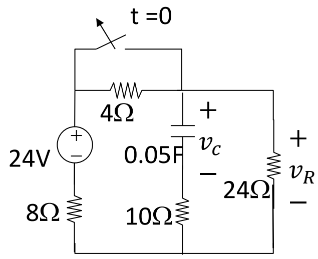
$$i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\frac{R_{th}}{L}t}$$

$$i_L(t) = 1 + e^{-1.5t}\ A$$

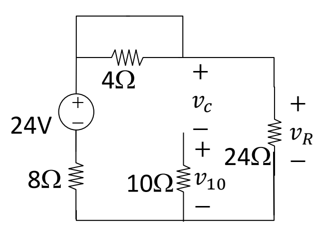
$$v_R(t) = 6i_R = 6(i_L(t) - 1) = 6e^{-1.5t}V$$

Also, $v_R(t) = -L\frac{di_L}{dt}$

Problem 2: The switch in the circuit has been closed for a long time before opening at t = 0. Find $v_c(t), v_R(t)$ for t ≥ 0.



Step 1: $v_c(0)$ from circuit for t < 0:

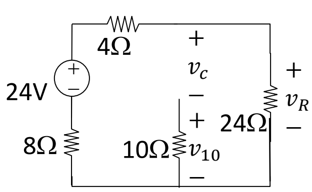


$v_{10} = 0$,
By voltage division:

$$v_R = \frac{24}{24+8} \times 24 = 18V$$

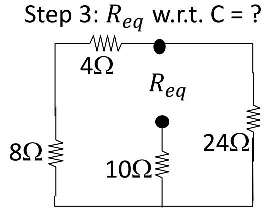
$$v_c(0) = v_R = 18V$$

Step 2: At $t = \infty$, $v_c(\infty)$ =?



$$v_c(\infty) = v_R = \frac{24}{12+24} \times 24 = 16V$$

Step 3: R_{eq} w.r.t. C = ?



$R_{eq} = 10 + (8+4)//24$
$= 18\Omega$

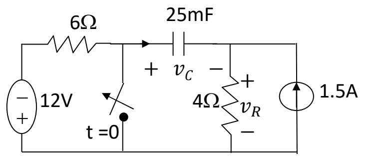
$R_{eq}C = 18 \times 0.05 = 0.9s$

$$v_c(t) = 16 + (18-16)e^{-\frac{1}{0.9}t} = 16 + 2e^{-1.11t}\ \text{V}$$

$$v_R(t) = v_c(t) + 10i_c(t) = v_c(t) + 10C\frac{dv_c}{dt}$$

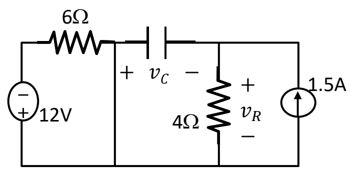
$$= 16 + 0.89e^{-1.11t}\text{V}$$

Problem 3: The switch in the circuit has been closed for a long time before opening at t = 0. Find $v_R(t)$ *and* $v_c(t)$ for t ≥ 0. $(v_c(1) = -17.78V,\ v_R(1) = 5.91V)$



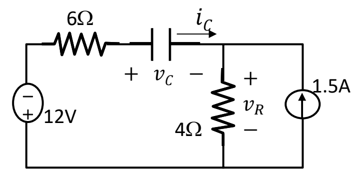
Always find $v_c(t)$ first, then derive other variables from $v_c(t)$ using basic laws and property of capacitor

Step 1) Find $v_c(0)$ from circuit before switch



$v_c(0) = -v_R = 4 \times 1.5 = -6V$

Step 2: For t > 0, switch is open



$v_c(\infty) = ?$

$v_c(\infty) = -12 - v_R(\infty) = -12 - 6 = -18V$

R_{th} w.r.t. Capacitor?

$R_{th} = 10\Omega \qquad \frac{1}{RC} = \frac{1}{10 \times 0.025} = 4$

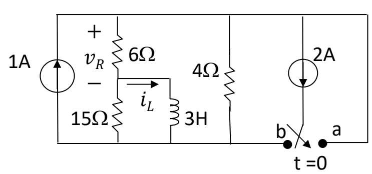
$v_c(t) = v_c(\infty) + (v_c(0) - v_c(\infty))e^{-\frac{1}{R_{th}C}t}$

$v_c(t) = -18 + 12e^{-4t}V$

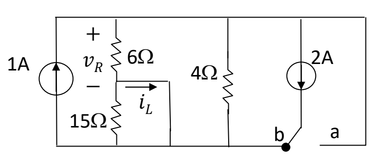
$v_R(t) = 4(i_c + 1.5) = 4(C\frac{dv_c}{dt} + 1.5)$

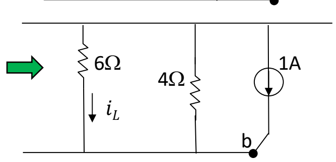
$v_R(t) = 4(0.025 \times 12 \times (-4)e^{-4t} + 1.5)$
$= -4.8e^{-4t} + 6\ V$

Problem 4: The switch has been in position "**b**" for a long time before moving to position "**a**" at t = 0. Find $i_L(t)$ *and* $v_R(t)$ for t ≥ 0. $(i_L(1) = 0.2917A,\ v_R(1) = 2.0102V)$



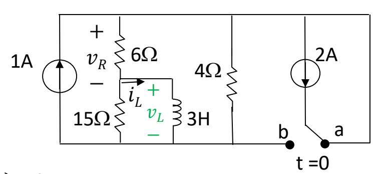
Step 1: Find $i_L(0)$ from circuit before switch



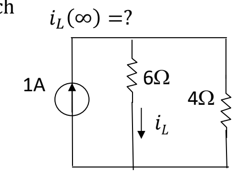


$i_L(0) = -\frac{4}{4+6} \times 1 = -0.4A$

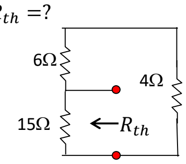
Step 2: For t > 0



$i_L(\infty) = ?$



$R_{th} = ?$



$i_L(\infty) = \frac{4}{4+6} \times 1 = 0.4A \qquad R_{th} = 15//(6+4) = 6\Omega$

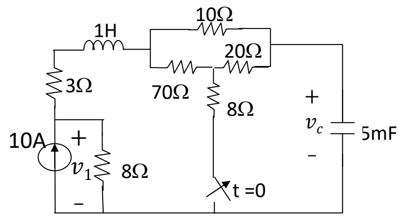
$i_L(t) = i_L(\infty) + \left(i_L(0) - i_L(\infty)\right)e^{-\frac{R_{th}}{L}t}, \qquad \frac{R_{th}}{L} = \frac{6}{3} = 2$

$i_L(t) = 0.4 - 0.8e^{-2t}A$

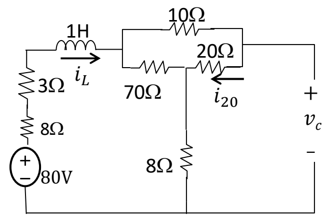
$v_R(t) = 6\left(i_L + \frac{v_L}{15}\right) = 6(i_L + \frac{L}{15}\frac{di_L}{dt})$

$v_R(t) = 2.4 - 2.88e^{-2t}V$

Problem 5: The switch has been closed for a long time before open at t = 0.
Find $v_c(t)$ and $v_1(t)$ for t > 0. $(v_c(0.1) = 74.0827V,\ v_1(0.1) = 72.8574V)$



Step 1: Find $i_L(0), v_c(0)$. Consider t < 0:

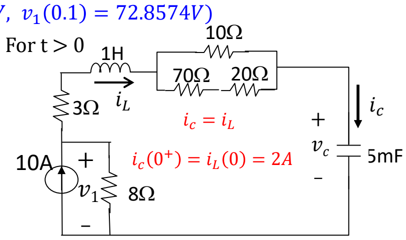


$$i_L(0) = \frac{80}{11 + 70//30 + 8} = 2A$$

$$i_{20} = \frac{70}{100} \times 2 = 1.4A$$

$$v_c(0) = 8i_L + 20i_{20} = 16 + 28 = 44V$$

Step 2: For t > 0



$v_c(\infty) = ?\ R_{th} = ?\quad \alpha = ?\ \omega_0 = ?\ \ Which\ case$

$v(\infty) = 80V, R_{th} = 11 + 10//90 = 20\Omega$

$\alpha = \frac{R_{th}}{2L} = 10,\ \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{200}$, Case 3

$\omega_d = \sqrt{200 - 100} = 10$

General solution:

$$v_c(t) = 80 + e^{-10t}(B_1 \cos 10t\ + B_2 \sin 10\,t\,)$$

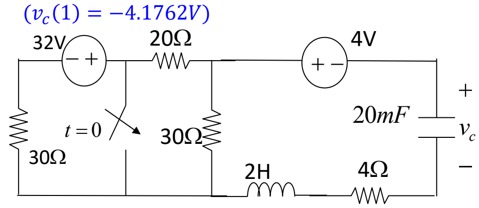
$$\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{2}{0.005} = 400V/s$$

$44 = 80 + B_1$
$400 = -10B_1 + 10B_2$ ➡ $B_1 = -36, B_2 = 4$

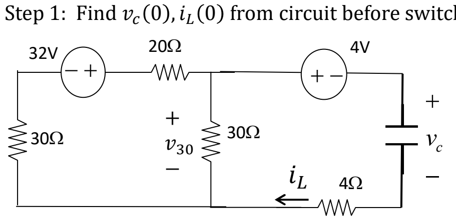
$$v_c(t) = 80 + e^{-10t}(-36 \cos 10t\ + 4 \sin 10t)\ V$$

$$v_1(t) = 8\left(10 - C\frac{dv_c}{dt}\right) = 80 - e^{-10t}(16 \cos 10t + 12.8 \sin 10t\,)\ V$$

Problem 6: The switch is open for a long time before closed at t = 0. Find $v_c(t)$ for t > 0.
$(v_c(1) = -4.1762V)$



Step 1: Find $v_c(0), i_L(0)$ from circuit before switch



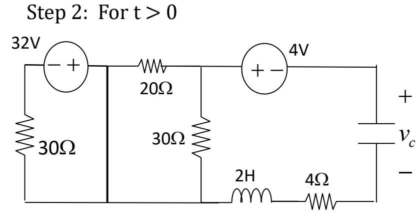
$v_c(0) = v_{30} - 4,\qquad i_L(0) = i_c(0) = 0$

Since $i_L = 0$, $20\Omega, 30\Omega, 30\Omega$ are in series.
By voltage division,

$$v_{30} = \frac{30}{30 + 20 + 30} \times 32 = 12V$$

$v_c(0) = 12 - 4 = 8V\qquad \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = 0$

Step 2: For t > 0



$v_{th} = v_c(\infty) = ?\quad R_{th} = ?$

Since 32V and 30Ω are short circuited, no effect.

$v_c(\infty) = -4V,\ R_{th} = 4 + 20//30 = 16\Omega$

$\alpha = ?\ \omega_0 = ?\ \ Which\ case?$

$\alpha = \frac{R_{th}}{2L} = \frac{16}{4} = 4,\ \ \omega_0 = \frac{1}{\sqrt{LC}} = 5$, Case 3

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 3$

The general solution?

$$v_c(t) = -4 + e^{-4t}(B_1 \cos 3t + B_2 \sin 3t\,)$$

Use initial conditions to find coefficients

$8 = -4 + B_1$
$0 = -4B_1 + 3B_2$ $\qquad B_1 = 12,\qquad B_2 = 16$

$$v_c(t) = -4 + e^{-4t}(12 \cos 3t\ + 16 \sin 3t\,)V$$