For the following $L(s)$,

$$(a)\ L(s)=\frac{1}{s(s+2)(s^2+2s+2)},\qquad (b)\ L(s)=\frac{(s+1)(s+2)}{s^3},\qquad (c)\ L(s)=\frac{s}{(s+1)^2(s+3)}$$

$$(d)\ L(s)=\frac{1}{s^2(s^2+2s+2)},\qquad (e)\ L(s)=\frac{s^2+2s+2}{s^2(s+2)(s+3)}$$

(1) Determine the intervals on the real axis where root locus exist.

(2) Find the angles of asymptotes ϕ_l and the radiating out point on the real axis.

(3) Departure angles & arriving angles.

Solution:

(a) $a(s)=s(s+2)(s^2+2s+2),\ \ b(s)=1,\ \ n=4,\ m=0,$

The poles are $p_1=0,\ p_2=-2,\ p_3=-1+j,\ p_4=-1-j,$ No zeros. Thus,

(1) $[-2,\ 0]$ is the interval on the real axis where root locus exist.

(2) the angles of asymptotes

$$\phi_l=\frac{2l+1}{n-m}\pi=\frac{2l+1}{4}\pi=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$$

the radiating out point

$$\alpha=\frac{\sum p_i-\sum z_i}{n-m}=\frac{0+(-2)+(-1+j)+(-1-j)}{4}=-1$$

(3) For $p_1=0,\ \phi_2=\measuredangle(p_1-p_2)=\measuredangle 2=0,$

$$\phi_3=\measuredangle(p_1-p_3)=\measuredangle(1-j)=-\frac{\pi}{4},$$

$$\phi_4=\measuredangle(p_1-p_4)=\measuredangle(1+j)=\frac{\pi}{4},$$

Departure angle $\psi_{1,\text{dep}}=-\sum_{i\neq 1}\phi_i-(2l+1)\pi=-(0-\frac{\pi}{4}+\frac{\pi}{4})-(2l+1)\pi=\pi$

For $p_2=-2,\ \phi_1=\measuredangle(p_2-p_1)=\measuredangle-2=\pi,$

$$\phi_3=\measuredangle(p_2-p_3)=\measuredangle(-1-j)=-\frac{3\pi}{4},$$

$$\phi_4=\measuredangle(p_2-p_4)=\measuredangle(-1+j)=\frac{3\pi}{4},$$

Departure angle $\psi_{2,\text{dep}}=-\sum_{i\neq 2}\phi_i-(2l+1)\pi=-(\pi-\frac{3\pi}{4}+\frac{3\pi}{4})-(2l+1)\pi=0$

For $p_3=-1+j,\ \phi_1=\measuredangle(p_3-p_1)=\measuredangle(-1+j)=\frac{3\pi}{4},$

$$\phi_2=\measuredangle(p_3-p_2)=\measuredangle(1+j)=\frac{\pi}{4},$$

$$\phi_4=\measuredangle(p_3-p_4)=\measuredangle(2j)=\frac{\pi}{2},$$

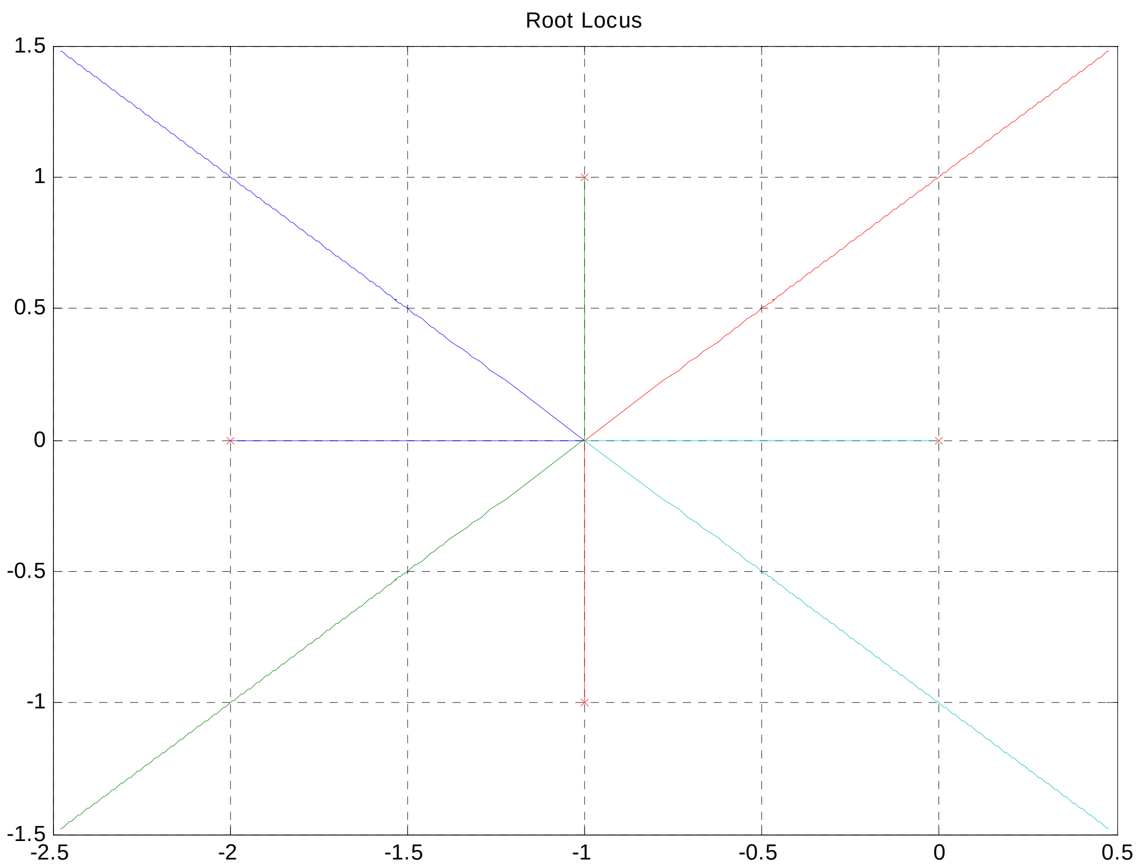
Departure angle $\psi_{3,\text{dep}} = -\sum_{i\neq 3}\phi_i - (2l+1)\pi = -(\frac{3\pi}{4}+\frac{\pi}{4}+\frac{\pi}{2}) - (2l+1)\pi = \frac{\pi}{2}$

For $p_4 = -1-j,\ \phi_1 = \measuredangle(p_4 - p_1) = \measuredangle(-1-j) = \frac{-3\pi}{4},$

$$\phi_2 = \measuredangle(p_4 - p_2) = \measuredangle(1-j) = \frac{-\pi}{4},$$

$$\phi_3 = \measuredangle(p_4 - p_3) = \measuredangle(-2j) = \frac{-\pi}{2},$$

Departure angle $\psi_{4,\text{dep}} = -\sum_{i\neq 4}\phi_i - (2l+1)\pi = -(\frac{-3\pi}{4}+\frac{-\pi}{4}+\frac{-\pi}{2}) - (2l+1)\pi = -\frac{\pi}{2}$



(b) $a(s) = s^3,\quad b(s) = (s+1)(s+2),\quad n=3,\quad m=2,$

The poles are $p_1 = 0\ (q=3)$, The zeros are $z_1 = -1,\ z_2 = -2$. Thus,

(1) $(-\infty, -2],\ [-1, 0]$ are the intervals on the real axis where root locus exist.

(2) the angles of asymptotes

$$\phi_l = \frac{2l+1}{n-m}\pi = (2l+1)\pi = \pi$$

the radiating out point

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = 0-(-1)-(-2) = 3$$

(3) For $p_1 = 0$, $\psi_1 = \measuredangle(p_1 - z_1) = \measuredangle 1 = 0$,

$$\psi_2 = \measuredangle(p_1 - z_2) = \measuredangle 2 = 0,$$

Departure angle $q\psi_{1,\text{dep}} = \sum \psi_\text{i} - \sum_{i\neq 1} \phi_\text{i} - (2l+1)\pi = -(2l+1)\pi$

$$\Rightarrow \quad \psi_{1,\text{dep}} = \frac{-(2l+1)\pi}{3} = -\frac{\pi}{3}, \pi, \frac{\pi}{3}$$

For $z_1 = -1$, $\phi_1 = \measuredangle(z_1 - p_1) = \measuredangle -1 = \pi$,

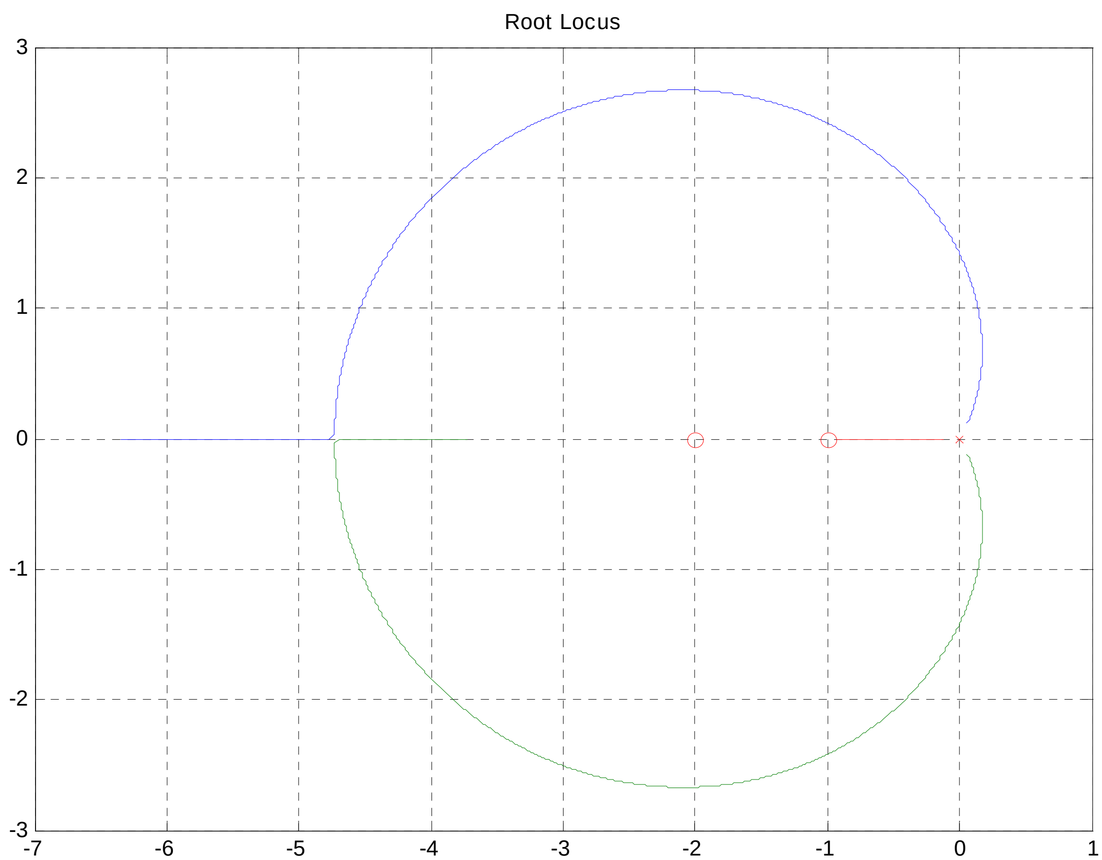
$$\psi_2 = \measuredangle(z_1 - z_2) = \measuredangle 1 = 0,$$

Arriving angle $\psi_{1,\text{arr}} = \sum \phi_\text{i} - \sum_{i\neq 1} \psi_\text{i} + (2l+1)\pi = 3\pi - 0 + (2l+1)\pi = 0$

For $z_2 = -2$, $\phi_1 = \measuredangle(z_2 - p_1) = \measuredangle -2 = \pi$,

$$\psi_1 = \measuredangle(z_2 - z_1) = \measuredangle -1 = \pi,$$

Arriving angle $\psi_{2,\text{arr}} = \sum \phi_\text{i} - \sum_{i\neq 2} \psi_\text{i} + (2l+1)\pi = 3\pi - \pi + (2l+1)\pi = \pi$



(c) $a(s)=(s+1)^2(s+3),\quad b(s)=s,\quad n=3,\quad m=1,$

The poles are $p_1=-1\ (q=2),\ p_2=-3,$ The zeros are $z_1=0.$ Thus,

(1) $[-3,0]$ is the interval on the real axis where root locus exist.

(2) the angles of asymptotes

$$\phi_l=\frac{2l+1}{n-m}\pi=\frac{(2l+1)\pi}{2}=\frac{\pi}{2},\frac{3\pi}{2}$$

the radiating out point

$$\alpha=\frac{\sum p_i-\sum z_i}{n-m}=\frac{-1-1-3-0}{2}=-2.5$$

(3) For $p_1=-1,\ \phi_2=\measuredangle(p_1-p_2)=\measuredangle 2=0,$

$$\psi_1=\measuredangle(p_1-z_1)=\measuredangle -1=\pi,$$

Departure angle $q\psi_{1,\text{dep}}=\sum\psi_{\text{i}}-\sum_{i\neq 1}\phi_{\text{i}}-(2l+1)\pi=\pi-0-(2l+1)\pi=-2l\pi$

$$\Rightarrow\quad \psi_{1,\text{dep}}=\frac{-2l\pi}{q}=-l\pi=0,\pi$$

For $p_2=-3,\ \phi_1=\measuredangle(p_2-p_1)=\measuredangle -2=\pi,$

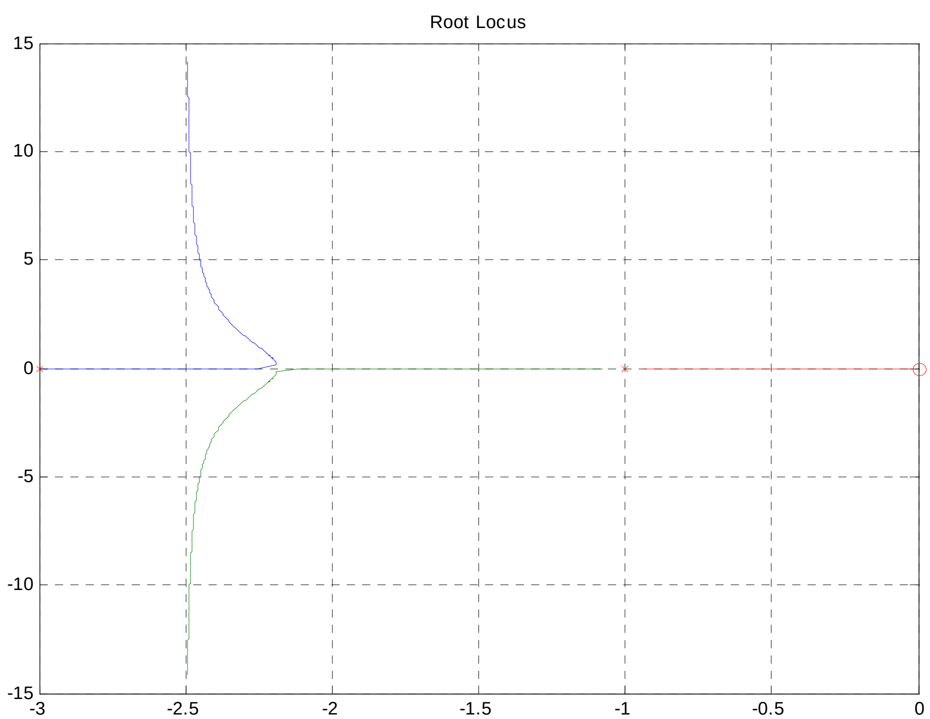
$$\psi_1=\measuredangle(p_2-z_1)=\measuredangle -3=\pi,$$

Departure angle $\psi_{2,\text{dep}}=\sum\psi_{\text{i}}-\sum_{i\neq 2}\phi_{\text{i}}-(2l+1)\pi=\pi-2\pi-(2l+1)\pi=0$

For $z_1=0,\ \phi_1=\measuredangle(z_1-p_1)=\measuredangle 1=0,$

$$\phi_2=\measuredangle(z_1-p_2)=\measuredangle 3=0,$$

Arriving angle $\psi_{1,\text{arr}}=\sum\phi_{\text{i}}-\sum_{i\neq 1}\psi_{\text{i}}+(2l+1)\pi=0+0+(2l+1)\pi=\pi$



$(d)\ a(s)=s^2(s^2+2s+2),\quad b(s)=1,\quad n=4,\quad m=0,$

The poles are $p_1=0\ (q=2),\ p_2=-1+j,\ p_3=-1-j,$ No zeros. Thus,

(1) There is NO interval on the real axis where root locus exist.

(2) the angles of asymptotes

$$\phi_l=\frac{2l+1}{n-m}\pi=\frac{(2l+1)\pi}{4}=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$$

the radiating out point

$$\alpha=\frac{\sum p_i-\sum z_i}{n-m}=\frac{0+(-1+j)+(-1-j)}{4}=-0.5$$

(3) For $p_1=0,\ \phi_2=\measuredangle(p_1-p_2)=\measuredangle(1-j)=-\frac{\pi}{4},$

$$\phi_3=\measuredangle(p_1-p_3)=\measuredangle(1+j)=\frac{\pi}{4},$$

Departure angle $q\psi_{1,\text{dep}}=\sum\psi_{\text{i}}-\sum_{i\neq 1}\phi_{\text{i}}-(2l+1)\pi=\frac{\pi}{4}-\frac{\pi}{4}-(2l+1)\pi=-(2l+1)\pi$

$$\Rightarrow\quad \psi_{1,\text{dep}}=\frac{-(2l+1)\pi}{q}=\frac{\pi}{2},\frac{3\pi}{2}$$

For $p_2=-1+j,\ \phi_1=\measuredangle(p_2-p_1)=\measuredangle(-1+j)=\frac{3\pi}{4},$

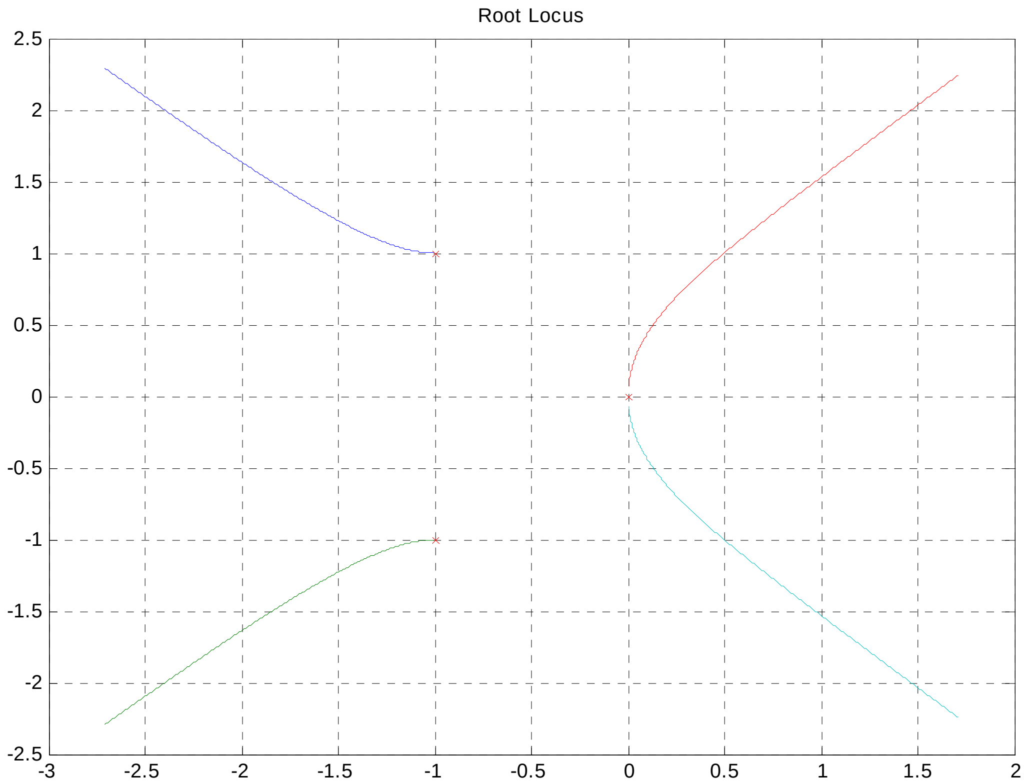
$$\phi_3=\measuredangle(p_2-p_3)=\measuredangle 2j=\frac{\pi}{2},$$

Departure angle $\psi_{2,\text{dep}}=\sum\psi_{\text{i}}-\sum_{i\neq 2}\phi_{\text{i}}-(2l+1)\pi=-2\frac{3\pi}{4}-\frac{\pi}{2}-(2l+1)\pi=\pi$

For $p_3=-1-j,\ \phi_1=\measuredangle(p_3-p_1)=\measuredangle(-1-j)=-\frac{3\pi}{4},$

$$\phi_2=\measuredangle(p_3-p_2)=\measuredangle-2j=-\frac{\pi}{2},$$

Departure angle $\psi_{3,\text{dep}}=\sum\psi_{\text{i}}-\sum_{i\neq 3}\phi_{\text{i}}-(2l+1)\pi=2\frac{3\pi}{4}+\frac{\pi}{2}-(2l+1)\pi=\pi$



(e) $a(s) = s^2(s+2)(s+3),\quad b(s) = s^2 + 2s + 2,\quad n = 4,\quad m = 2,$

The poles are $p_1 = 0\ (q = 2),\ p_2 = -2,\ p_3 = -3,$ The zeros are $z_1 = -1 + j,\ z_1 = -1 - j.$ Thus,

(1) $[-3, -2]$ is the interval on the real axis where root locus exist.

(2) the angles of asymptotes

$$\phi_l = \frac{2l+1}{n-m}\pi = \frac{(2l+1)\pi}{2} = \frac{\pi}{2}, \frac{3\pi}{2}$$

the radiating out point

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{-2-3-(-1+j)-(-1-j)}{2} = -1.5$$

(3) For $p_1 = 0,\ \phi_2 = \measuredangle(p_1 - p_2) = \measuredangle 2 = 0,$

$$\phi_3 = \measuredangle(p_1 - p_3) = \measuredangle 3 = 0,$$

$$\psi_1 = \measuredangle(p_1 - z_1) = \measuredangle(1 - j) = -\frac{\pi}{4},$$

$$\psi_2 = \measuredangle(p_1 - z_2) = \measuredangle(1 + j) = \frac{\pi}{4},$$

Departure angle $q\psi_{1,\text{dep}} = \sum \psi_i - \sum_{i \neq 1} \phi_i - (2l+1)\pi = -\frac{\pi}{4} + \frac{\pi}{4} + 0 - (2l+1)\pi = -(2l+1)\pi$

$$\Rightarrow \quad \psi_{1,\text{dep}} = \frac{-(2l+1)\pi}{q} = \frac{\pi}{2}, \frac{3\pi}{2}$$

For $p_2 = -2,\ \phi_1 = \measuredangle(p_2 - p_1) = \measuredangle -2 = \pi,$

$$\phi_3 = \measuredangle(p_2 - p_3) = \measuredangle 1 = 0,$$

$$\psi_1 = \measuredangle(p_2 - z_1) = \measuredangle(-1 - j) = -\frac{3\pi}{4},$$

$$\psi_2 = \measuredangle(p_2 - z_2) = \measuredangle(-1 + j) = \frac{3\pi}{4},$$

Departure angle $\psi_{2,\text{dep}} = \sum \psi_i - \sum_{i \neq 2} \phi_i - (2l+1)\pi = -\frac{3\pi}{4} + \frac{3\pi}{4}\pi - 2\pi - 0 - (2l+1)\pi = \pi$

For $p_3 = -3$, $\phi_1 = \measuredangle(p_3 - p_1) = \measuredangle -3 = \pi,$

$$\phi_2 = \measuredangle(p_3 - p_2) = \measuredangle -1 = \pi,$$

$$\psi_1 = \measuredangle(p_3 - z_1) = \measuredangle(-2 - j) = \tan^{-1} 0.5 - \pi,$$

$$\psi_2 = \measuredangle(p_3 - z_2) = \measuredangle(-2 + j) = \pi - \tan^{-1} 0.5,$$

Departure angle $\psi_{3,\text{dep}} = \sum \psi_i - \sum_{i \neq 3} \phi_i - (2l+1)\pi = \tan^{-1} 0.5 - \pi + \pi - \tan^{-1} 0.5 - 2\pi - \pi - (2l+1)\pi = 0$

For $z_1 = -1 + j$, $\phi_1 = \measuredangle(z_1 - p_1) = \measuredangle(-1 + j) = \frac{3\pi}{4},$

$$\phi_2 = \measuredangle(z_1 - p_2) = \measuredangle(1 + j) = \frac{\pi}{4},$$

$$\phi_3 = \measuredangle(z_1 - p_3) = \measuredangle(2 + j) = \tan^{-1} 0.5,$$

$$\psi_2 = \measuredangle(z_1 - z_2) = \measuredangle 2j = \frac{\pi}{2},$$

Arriving angle $\psi_{1,\text{arr}} = \sum \phi_i - \sum_{i \neq 1} \psi_i + (2l+1)\pi = 2\frac{3\pi}{4} + \frac{\pi}{4} + \tan^{-1} 0.5 - \frac{\pi}{2} + (2l+1)\pi = \frac{\pi}{4} + \tan^{-1} 0.5$

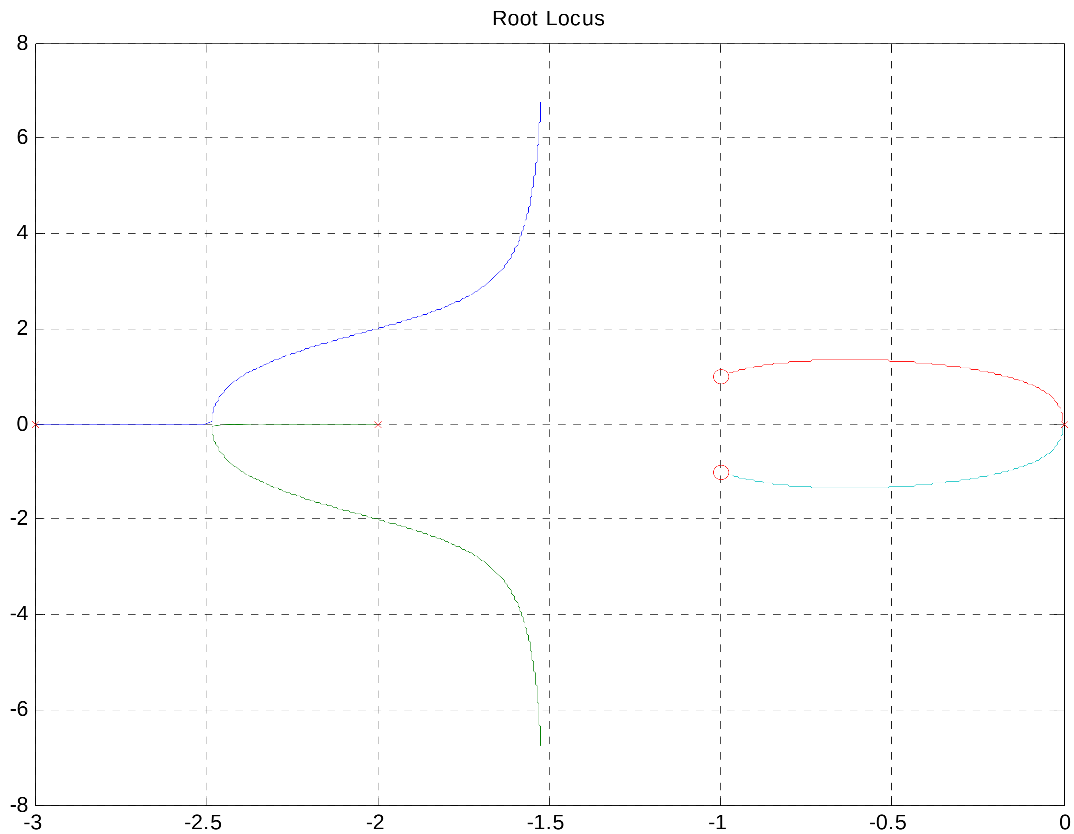
For $z_2 = -1 - j$, $\phi_1 = \measuredangle(z_2 - p_1) = \measuredangle(-1 - j) = -\frac{3\pi}{4},$

$$\phi_2 = \measuredangle(z_2 - p_2) = \measuredangle(1 - j) = -\frac{\pi}{4},$$

$$\phi_3 = \measuredangle(z_2 - p_3) = \measuredangle(2 - j) = -\tan^{-1} 0.5,$$

$$\psi_1 = \measuredangle(z_2 - z_1) = \measuredangle -2j = -\frac{\pi}{2},$$

Arriving angle $\psi_{2,\text{arr}} = \sum \phi_i - \sum_{i \neq 2} \psi_i + (2l+1)\pi = -2\frac{3\pi}{4} - \frac{\pi}{4} - \tan^{-1} 0.5 + \frac{\pi}{2} + (2l+1)\pi = -\frac{\pi}{4} - \tan^{-1} 0.5$


Root Locus
8
6
4
2
0
-2
-4
-6
-8
-3
-2.5
-2
-1.5
-1
-0.5
0