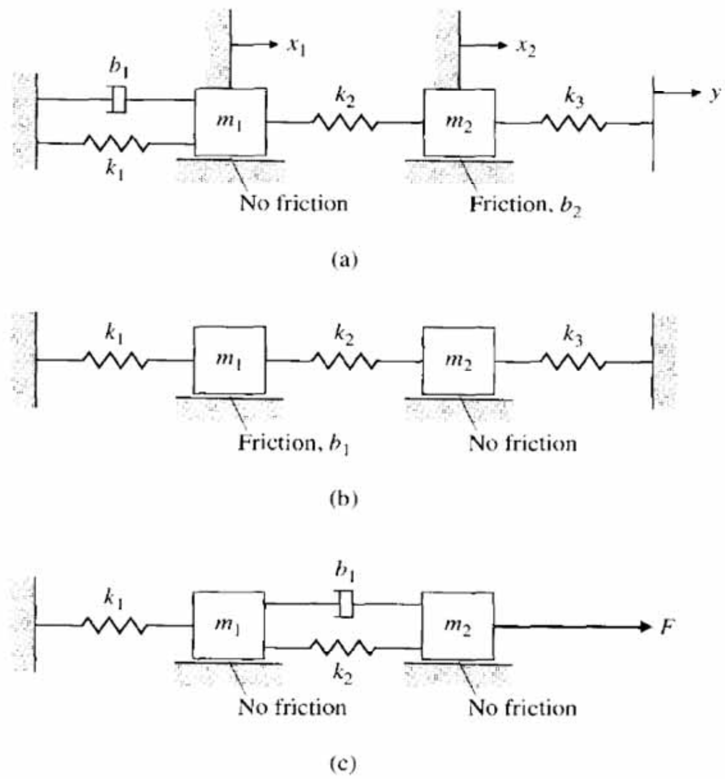
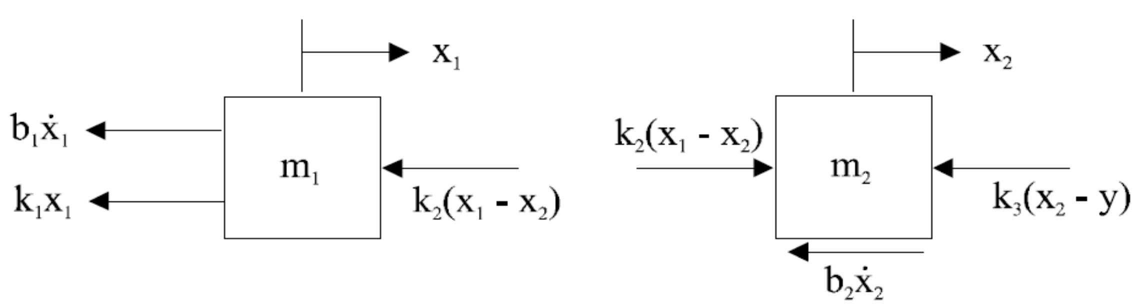
1. Write the differential equations for the mechanical systems shown in Fig. 2.38.

Figure 2.38
Mechanical systems



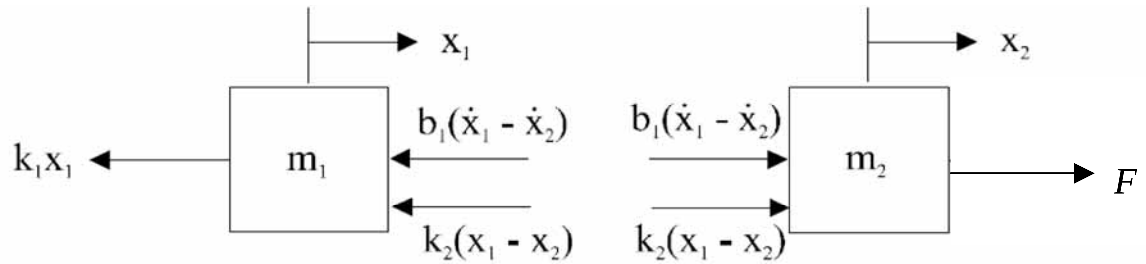
Solution:

The key is to draw the Free Body Diagram (FBD) in order to keep the signs right. For (a), to identify the direction of the spring forces on the object, let $x_2 = 0$ and fixed and increase x_1 from 0. Then the k_1 spring will be stretched producing its spring force to the left and the k_2 spring will be compressed producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.



(a)

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - b_1\dot{x}_1 - k_2(x_1 - x_2) \\
m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - k_3(x_2 - y) - b_2\dot{x}_2
\end{aligned}
$$



(c)

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) \\
m_2\ddot{x}_2 &= F - k_2(x_2 - x_1) - b_1(\dot{x}_2 - \dot{x}_1)
\end{aligned}
$$

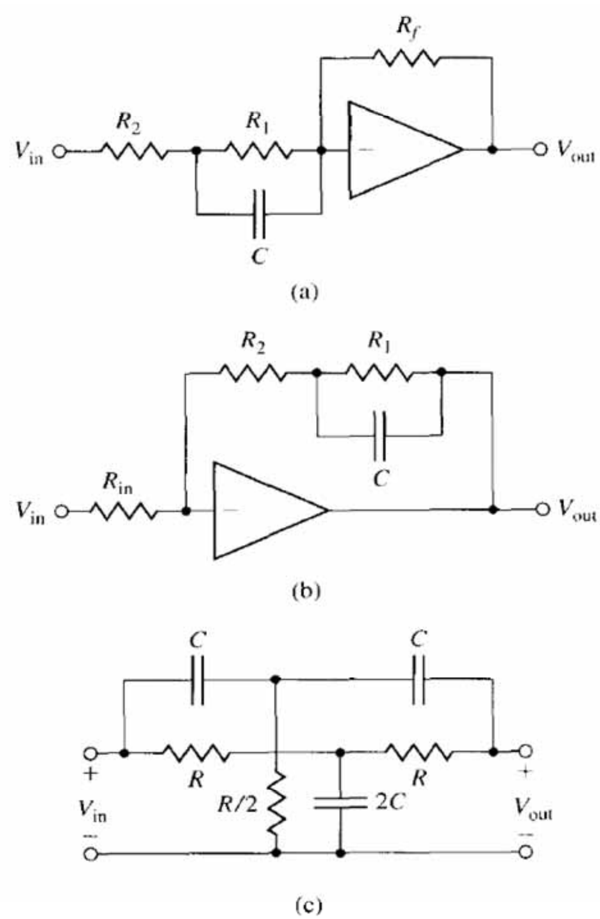
16. Write the dynamic equations and find the transfer functions for the circuits shown in Fig. 2.46.

    (a) lead circuit

    (b) lag circuit.

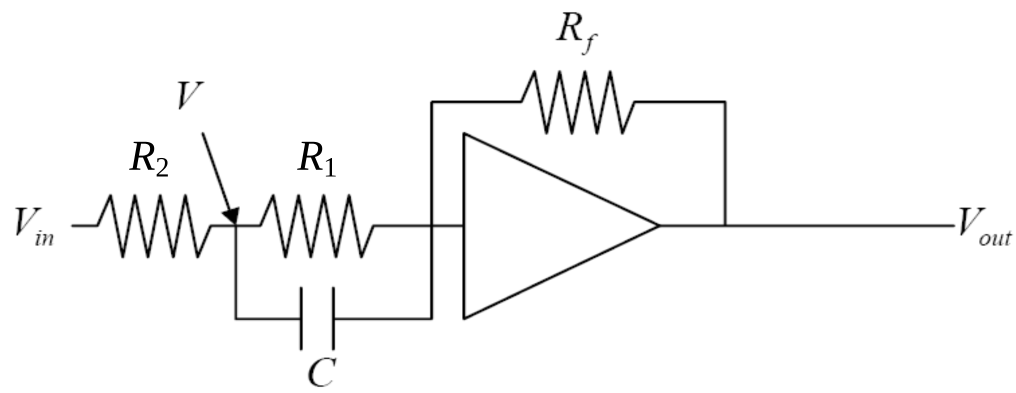
    (c) notch circuit

Figure 2.46
Lead (a), lag (b), notch (c) circuits



Solution:

(a) lead circuit



$$\frac{V_{in}-V}{R_2}+\frac{0-V}{R_1}+C\frac{d}{dt}(0-V)=0 \qquad (1)$$

$$\frac{V_{in}-V}{R_2}=\frac{0-V_{out}}{R_f} \qquad (2)$$

We need to eliminate V. From Eq. (2),

$$V=V_{in}+\frac{R_2}{R_f}V_{out}$$

Substitute V's in Eq. (1),

$$\frac{1}{R_2}\left(V_{in}-V_{in}-\frac{R_2}{R_f}V_{out}\right)-\frac{1}{R_1}\left(V_{in}+\frac{R_2}{R_f}V_{out}\right)-C\left(\dot{V}_{in}+\frac{R_2}{R_f}\dot{V}_{out}\right)=0$$

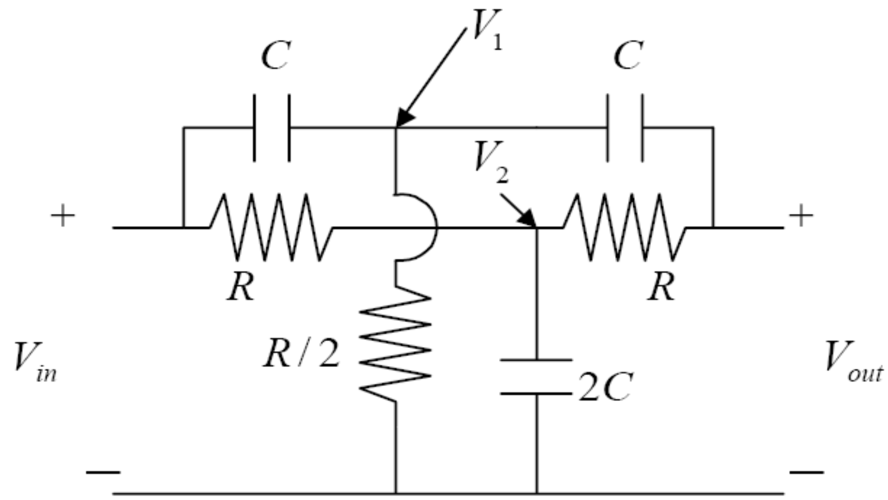
$$\frac{1}{R_1}V_{in}+C\dot{V}_{in}=-\frac{1}{R_f}\left[\left(1+\frac{R_2}{R_1}\right)V_{out}+R_2C\dot{V}_{out}\right]$$

Laplace Transform

$$\begin{aligned}\frac{V_{out}}{V_{in}} &= \frac{Cs+\frac{1}{R_1}}{-\frac{1}{R_f}\left(R_2Cs+1+\frac{R_2}{R_1}\right)} \\ &= -\frac{R_f}{R_2}\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}}\end{aligned}$$

We can see that the pole is at the left side of the zero, which means a lead compensator.

(c) notch circuit



$$
\begin{aligned}
C\frac{d}{dt}\left(V_{in}-V_1\right)+\frac{0-V_1}{R/2}+C\frac{d}{dt}\left(V_{out}-V_1\right) &= 0\\
\frac{V_{in}-V_2}{R}+2C\frac{d}{dt}\left(0-V_2\right)+\frac{V_{out}-V_2}{R} &= 0\\
C\frac{d}{dt}\left(V_1-V_{out}\right)+\frac{V_2-V_{out}}{R} &= 0
\end{aligned}
$$

We need to eliminat V_1, V_2 from three equations and find the relation between V_{in} and V_{out}

$$
\begin{aligned}
V_1 &= \frac{Cs}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right)\\
V_2 &= \frac{\frac{1}{R}}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right)
\end{aligned}
$$

$$
\begin{aligned}
& CsV_1-CsV_{out}+\frac{1}{R}V_2-\frac{1}{R}V_{out}\\
= \ & Cs\frac{Cs}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right)+\frac{1}{R}\frac{\frac{1}{R}}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right)-\left(Cs+\frac{1}{R}\right)V_{out}\\
= \ & 0
\end{aligned}
$$

$$\frac{C^2s^2 + \frac{1}{R^2}}{2\left(Cs + \frac{1}{R}\right)} V_{in} = \left[\left(Cs + \frac{1}{R}\right) - \frac{C^2s^2 + \frac{1}{R^2}}{2\left(Cs + \frac{1}{R}\right)}\right] V_{out}$$

$$\frac{V_{out}}{V_{in}} = \frac{\frac{C^2s^2 + \frac{1}{R^2}}{2\left(Cs + \frac{1}{R}\right)}}{\left(Cs + \frac{1}{R}\right) - \frac{C^2s^2 + \frac{1}{R^2}}{2\left(Cs + \frac{1}{R}\right)}}$$

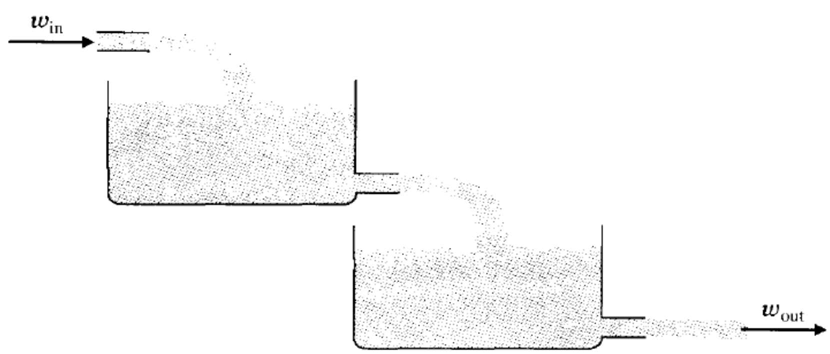
$$= \frac{\left(C^2s^2 + \frac{1}{R^2}\right)}{2\left(Cs + \frac{1}{R}\right)^2 - \left(C^2s^2 + \frac{1}{R^2}\right)}$$

$$= \frac{C^2\left(s^2 + \frac{1}{R^2C^2}\right)}{C^2s^2 + 4\frac{Cs}{R} + \frac{1}{R^2}}$$

$$= \frac{s^2 + \frac{1}{R^2C^2}}{s^2 + \frac{4}{RC}s + \frac{1}{R^2C^2}}$$

26. For the two-tank fluid-flow system shown in Fig. 2.53, find the differential equations relating the flow into the first tank to the flow out of the second tank.

Figure 2.53
Two-tank fluid-flow system for Problem 2.26



Solution:

This is a variation on the problem solved in Example 2.20 and the definitions of terms is taken from that. From the relation between the height of the water and mass flow rate, the continuity equations are

$$\begin{aligned} \dot{m}_1 &= \rho A_1 \dot{h}_1 = w_{in} - w \\ \dot{m}_2 &= \rho A_2 \dot{h}_2 = w - w_{out} \end{aligned}$$

Also from the relation between the pressure and outgoing mass flow rate,

$$\begin{aligned} w &= \frac{1}{R_1} (\rho g h_1)^{\frac{1}{2}} \\ w_{out} &= \frac{1}{R_2} (\rho g h_2)^{\frac{1}{2}} \end{aligned}$$

Finally,

$$\begin{aligned} \dot{h}_1 &= -\frac{1}{\rho A_1 R_1} (\rho g h_1)^{\frac{1}{2}} + \frac{1}{\rho A_1} w_{in} \\ \dot{h}_2 &= \frac{1}{\rho A_2 R_1} (\rho g h_1)^{\frac{1}{2}} - \frac{1}{\rho A_2 R_2} (\rho g h_2)^{\frac{1}{2}} . \end{aligned}$$