3.2 Find the Laplace transform of the following time functions:

Solution: $(b)\ f(t) = 3 + 7t + t^2 + \delta(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\} + \mathcal{L}\{7t\} + \mathcal{L}\{t^2\} + \mathcal{L}\{\delta(t)\}$$

$$= \frac{3}{s} + \frac{7}{s^2} + \frac{2!}{s^3} + 1$$

$$= \frac{s^3 + 3s^2 + 7s + 2}{s^3}$$

3.3 Find the Laplace transform of the following time functions:

Solution: $(c)\ f(t) = t^2 + e^{-2t}\sin 3t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{e^{-2t}\sin 3t\}$$

$$= \frac{2!}{s^3} + \frac{3}{(s+2)^2 + 9}$$

$$= \frac{2}{s^3} + \frac{3}{(s+2)^2 + 9}$$

3.4 Find the Laplace transform of the following time functions:

Solution: $(c)\ f(t) = te^{-t} + 2t\cos t$

Use the following Laplace transforms and properties (Table A.1, entries 4,11, and 3),

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

$$\mathcal{L}\{tg(t)\} = -\frac{d}{ds}G(s)$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{te^{-t}\} + 2\mathcal{L}\{t\cos t\}$$

$$= \frac{1}{(s+1)^2} + 2\left(-\frac{d}{ds}\frac{s}{s^2+1}\right)$$

$$= \frac{1}{(s+1)^2} + 2\frac{s^2 - 1}{(s^2+1)^2}$$

3.5 Find the Laplace transform of the following time functions (* denotes convolution):

Solution: $(e)\ f(t) = \int_0^t \cos(t-\tau)\sin\tau\, d\tau$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\int_0^t \cos(t-\tau)\sin\tau\, d\tau\} = \mathcal{L}\{\cos t * \sin t\}$$

This is just the deTnition of the convolution theorem,

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+1}\frac{1}{s^2+1} = \frac{s}{s^4 + 2s^2 + 1}$$

3.6 Given that the Laplace transform of $f(t)$ is $F(s)$, find the Laplace transform of the following:

Solution: (*a*) $g(t) = f(t)\cos t$

First write $\cos t$ in terms of the related Euler identity (Eq. B.33),

$$g(t) = f(t)\cos t = f(t)\frac{e^{jt} + e^{-jt}}{2} = \frac{1}{2} f(t)e^{jt} + \frac{1}{2} f(t)e^{-jt}$$

Then using entry 4 of Table A.1 we have,

$$G(s) = \frac{1}{2}F(s-j) + \frac{1}{2}F(s+j) = \frac{1}{2}\left[F(s-j) + F(s+j)\right]$$

3.7 Find the time function corresponding to each of the following Laplace transforms using partial fraction expansions:

Solution: (*a*) $F(s) = \dfrac{2}{s(s+2)}$

Perform partial fraction expansion,

$$F(s) = \frac{2}{s(s+2)} = \frac{C_1}{s} + \frac{C_2}{s+2}$$

$$C_1 = \frac{2}{s+2}\Big|_{s=0} = 1, \qquad C_2 = \frac{2}{s}\Big|_{s=-2} = -1$$

$$F(s) = \frac{1}{s} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{1}{s+2}\}$$

$$f(t) = 1(t) - e^{-2t}1(t)$$

(*b*) $F(s) = \dfrac{10}{s(s+1)(s+10)}$

Perform partial fraction expansion,

$$F(s) = \frac{10}{s(s+1)(s+10)} = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+10}$$

$$C_1 = \frac{10}{(s+1)(s+10)}\Big|_{s=0} = 1, \qquad C_2 = \frac{10}{s(s+10)}\Big|_{s=-1} = -\frac{10}{9}, \qquad C_3 = \frac{10}{s(s+1)}\Big|_{s=-10} = \frac{1}{9}$$

$$F(s) = \frac{1}{s} - \frac{\frac{10}{9}}{s+1} + \frac{\frac{1}{9}}{s+10}$$

$$f(t) = \mathcal{L}^{-1}\{f(t)\} = 1(t) - \frac{10}{9}e^{-t}1(t) + \frac{1}{9}e^{-10t}1(t)$$

$$(f)\ F(s)=\frac{2(s+2)}{(s+1)(s^2+4)}$$

$$F(s)=\frac{C_1}{(s+1)}+\frac{C_2 s+C_3}{(s^2+4)}$$

$$C_1=\frac{2(s+2)}{(s^2+4)}\Big|_{s=-1}=\frac{2}{5}$$

Equate numerators and like powers of s terms:

$$(\frac{2}{5}+C_2)s^2+(C_2+C_3)s+(\frac{8}{5}+C_3)=2s+4$$

$$\frac{8}{5}+C_3=4 \qquad \Rightarrow C_3=\frac{12}{5}$$

$$C_2+C_3=2 \qquad \Rightarrow C_2=-\frac{2}{5}$$

$$\frac{2}{5}+C_2=0$$

$$F(s)=\frac{\frac{2}{5}}{(s+1)}+\frac{-\frac{2}{5}s+\frac{12}{5}}{(s^2+4)}=\frac{2}{5}\frac{1}{(s+1)}-\frac{2}{5}\frac{s}{(s^2+2^2)}+\frac{6}{5}\frac{2}{(s^2+2^2)}$$

$$f(t)=\frac{2}{5}e^{-t}-\frac{2}{5}\cos 2t+\frac{6}{5}\sin 2t$$

$$(j)\ F(s)=\frac{e^{-s}}{s^2}$$

Using entry #2 of Table A.1,

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=(t-1)1(t-1)$$

3.9 Solve the following ordinary differential equations using Laplace transforms:

Solution: $(b)\ \ddot{y}(t)-2\dot{y}(t)+4y(t)=0;$ y(0)=1, $\dot{y}(0)=2$

$$s^2Y(s)-sy(0)-\dot{y}(0)-2sY(s)+2y(0)+4Y(s)=0$$

$$Y(s)=\frac{sy(0)+\dot{y}(0)-2y(0)}{s^2-2s+4}$$

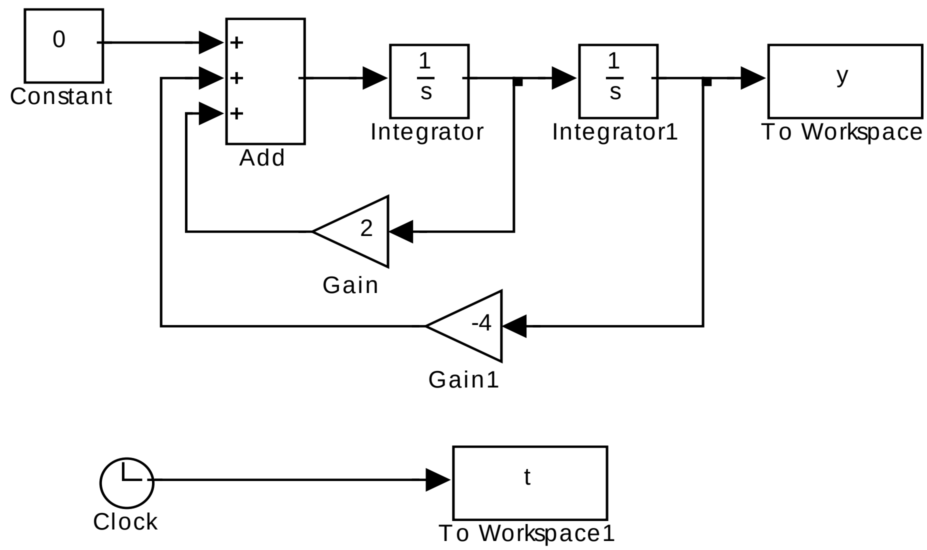
$$=\frac{s}{s^2-2s+4}$$

$$=\frac{s}{(s-1)^2+3}$$

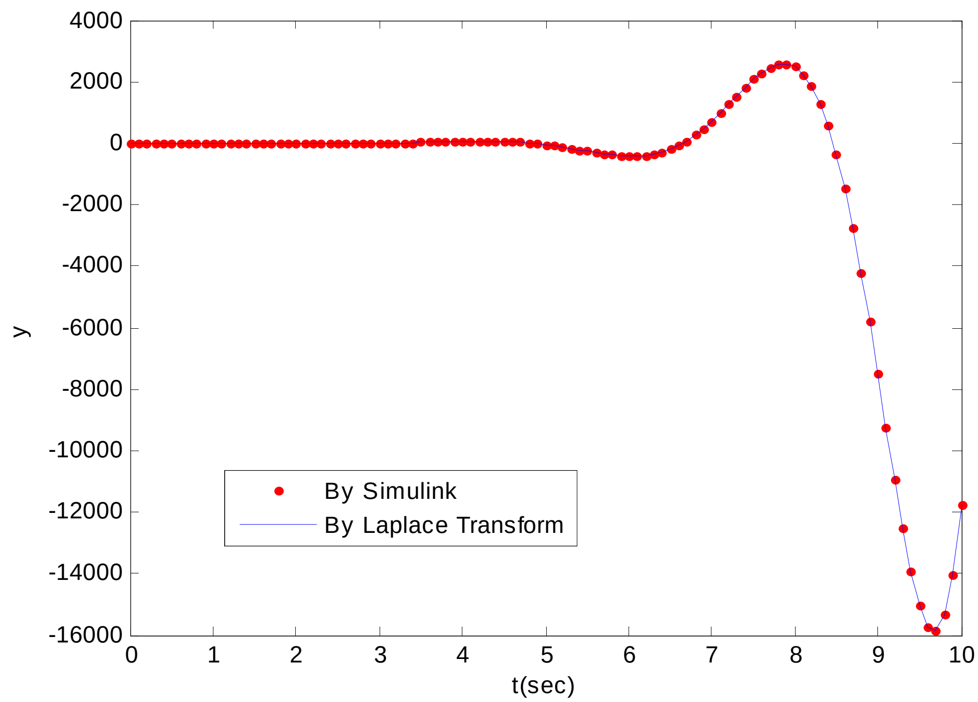
$$=\frac{s-1}{(s-1)^2+\sqrt{3}^2}+\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2+\sqrt{3}^2}$$

$$f(t)=(e^t\cos\sqrt{3}t+\frac{1}{\sqrt{3}}e^t\sin\sqrt{3}t)1(t)$$

Below is the Simulink Model and Running result,




```
>>plot(t,y,'r.')
>> hold on
>> plot(t,exp(t).*(cos(sqrt(3)*t)+sin(sqrt(3)*t)/sqrt(3)))
>> xlabel('t(sec)');ylabel('y');
>> legend('By Simulink','By Laplace Transform')
```



$$(d)\ \ddot{y}(t)+3y(t)=\sin t;\ \text{y(0)=1},\ \dot{y}(0)=2$$

$$s^2Y(s)-sy(0)-\dot{y}(0)+3Y(s)=\frac{1}{s^2+1}$$

$$Y(s)=\frac{sy(0)+\dot{y}(0)+\frac{1}{s^2+1}}{s^2+3}$$

$$=\frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)}$$

$$=\frac{C_1s+C_2}{s^2+3}+\frac{C_3s+C_4}{s^2+1}$$

$$\frac{(C_1s+C_2)(s^2+1)+(C_3s+C_4)(s^2+3)}{(s^2+3)(s^2+1)}=\frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)}$$

Match coefficients of like powers of s:

$$s^3(C_1+C_3)+s^2(C_2+C_4)+s(C_1+3C_3)+(C_2+3C_4)=s^3+2s^2+s+3$$

$$C_1+C_3=1 \quad \Rightarrow\ C_1=1-C_3$$

$$C_2+C_4=2 \quad \Rightarrow\ C_2=2-C_4$$

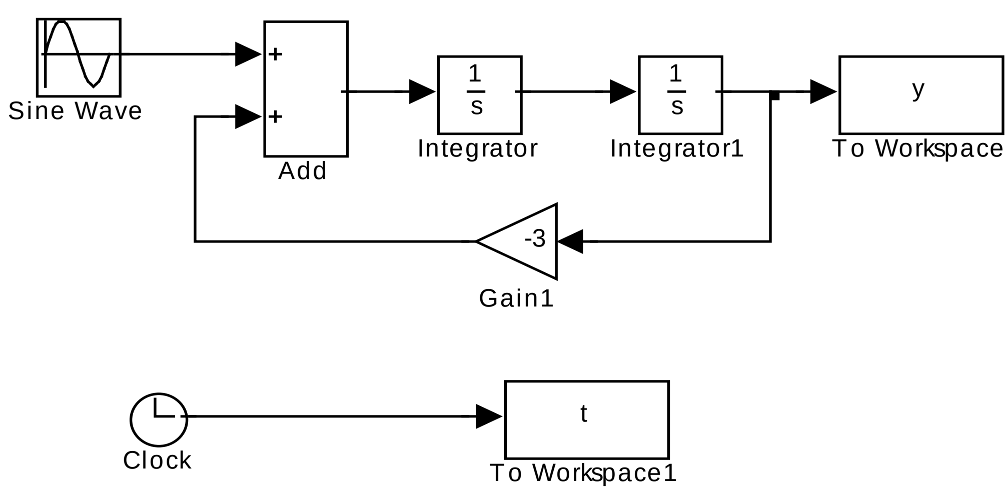
$$C_1+3C_3=1 \quad \Rightarrow\ 1-C_3+3C_3=1 \quad \Rightarrow\ C_3=0,\ C_1=1$$

$$C_2+3C_4=3 \quad \Rightarrow\ 2-C_4+3C_4=3 \quad \Rightarrow\ C_4=\frac{1}{2},\ C_2=\frac{3}{2}$$

$$Y(s)=\frac{s+\frac{3}{2}}{s^2+3}+\frac{\frac{1}{2}}{s^2+1}=\frac{s}{s^2+\sqrt{3}^2}+\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{s^2+\sqrt{3}^2}+\frac{1}{2}\frac{1}{s^2+1}$$

$$f(t)=\cos\sqrt{3}t+\frac{\sqrt{3}}{2}\sin\sqrt{3}t+\frac{1}{2}\sin t$$

Below is the Simulink Model and Running result,



```
>> plot(t,y,'r.')
>> hold on
>> plot(t,cos(sqrt(3)*t)+sin(sqrt(3)*t)*sqrt(3)/2+sin(t)/2)
>> xlabel('t(sec)');ylabel('y');
>> legend('By Simulink','By Laplace Transform')
```