Given $G(s)=\dfrac{1}{s^2-s+1}$. Consider the following types of controllers

$(a)\ C(s)=k_p,\quad (b)\ C(s)=k_p+k_I/s,\quad (c)\ C(s)=k_D s+k_p,\quad (d)\ C(s)=k_D s+k_p+k_I/s$

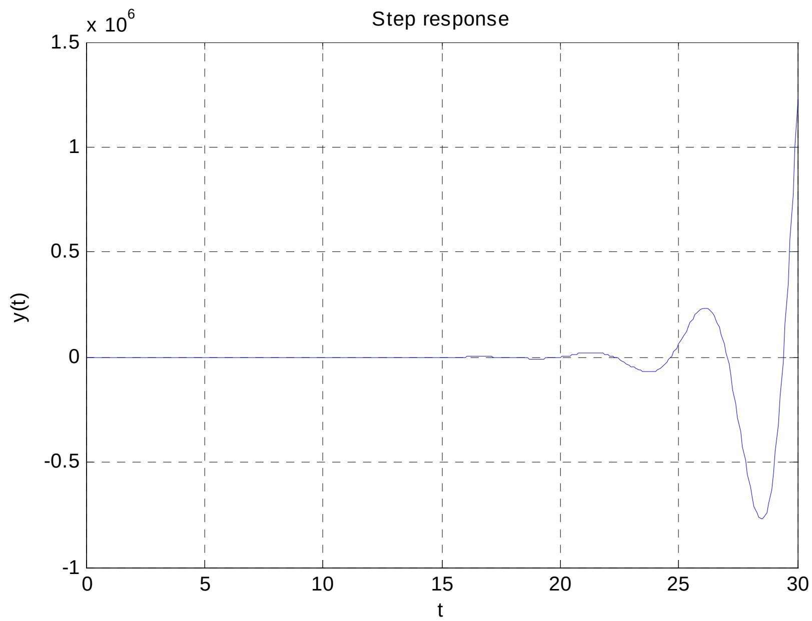
For each type, is it possible to stabilize the closed-loop system by choosing suitable parameters? If it is possible, pick the parameters so that the closed-loop poles are at $-2\pm j\,2$ and -6 (if a third order system is obtained). In each case, what is the steady state error of the step response? Plot the step response with Matlab.

Solution: The closed-loop transfer function is $\dfrac{Y(s)}{R(s)}=\dfrac{G(s)C(s)}{1+G(s)C(s)}=\dfrac{C(s)}{s^2-s+1+C(s)}$

$(a)\ C(s)=k_p,\qquad \dfrac{Y(s)}{R(s)}=\dfrac{k_p}{s^2-s+1+k_p},\qquad a(s)=s^2+a_1 s+a_0,$

where $a_1=-1,\ a_0=1+k_p$.

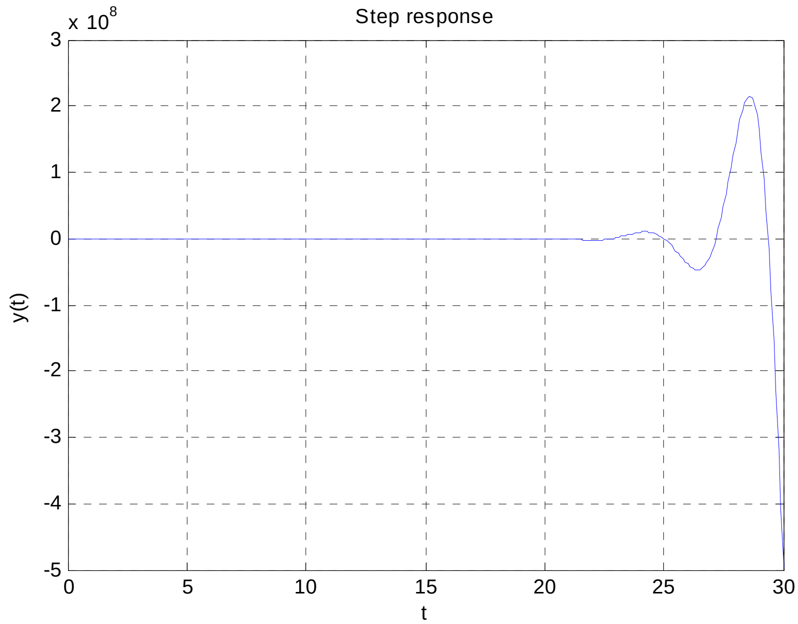
Since $\sum_{i=1}^{2} s_i=-a_1=1$ (s_i is the i$^{\text{th}}$ root of $a(s)$), we can't have all the poles in LHP to stabilize the closed-loop system. Choose $k_p=1,$ the step response is,



$(b)\ C(s)=k_p+k_I/s,\qquad \dfrac{Y(s)}{R(s)}=\dfrac{k_p+k_I/s}{s^2-s+1+k_p+k_I/s}=\dfrac{k_p s+k_I}{s^3-s^2+(1+k_p)s+k_I},$

$a(s)=s^3+a_2 s^2+a_1 s+a_0,$ where $a_2=-1,\ a_1=1+k_p,\ a_0=k_I$

Since $\sum_{i=1}^{3} s_i=-a_1=1$ (s_i is the i$^{\text{th}}$ root of $a(s)$), we can't have all the poles in LHP to stabilize the closed-loop system. Choose $k_p=k_I=1,$ the step response is,



(c) $C(s) = k_D s + k_p$, $\quad \dfrac{Y(s)}{R(s)} = \dfrac{k_D s + k_p}{s^2 - s + 1 + k_D s + k_p} = \dfrac{k_D s + k_p}{s^2 + (k_D - 1)s + 1 + k_p}$

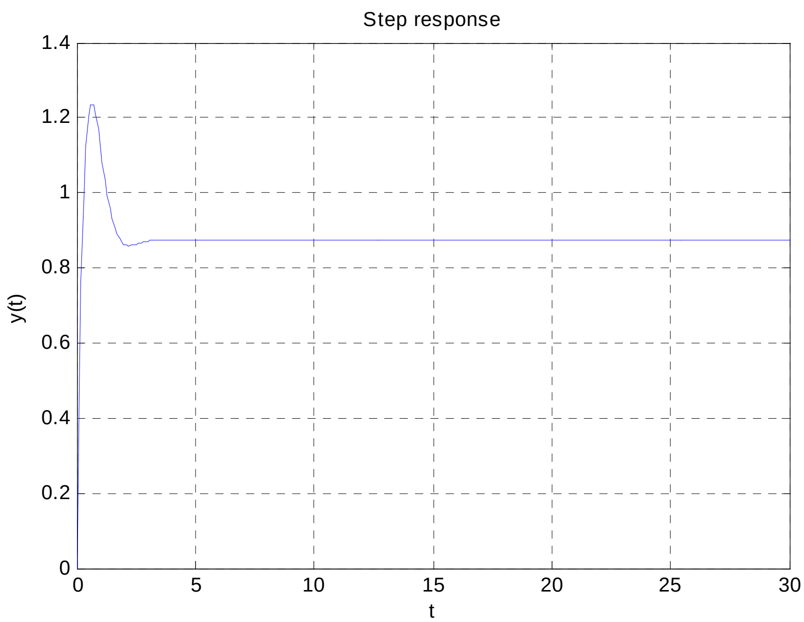
$a(s) = s^2 + a_1 s + a_0$, where $a_1 = k_D - 1$, $a_0 = 1 + k_p$

Since $s_1 + s_2 = -a_1 = 1 - k_D$, $s_1 s_2 = a_0 = 1 + k_p$, it is possible to stabilize the closed-loop system.

For $s_{1,2} = -2 \pm j2$, $1 - k_D = s_1 + s_2 = -4$, $1 + k_p = s_1 s_2 = 8$, $\quad \Rightarrow \quad k_D = 5$, $k_p = 7$

Thus, the transfer function is $\dfrac{Y(s)}{R(s)} = \dfrac{5s + 7}{s^2 + 4s + 8}$, the steady state error is

$1 - \lim_{s \to 0} \dfrac{Y(s)}{R(s)} = 1 - \dfrac{7}{8} = \dfrac{1}{8}$, and the step response is,



(d) $C(s) = k_D s + k_p + k_I / s$, $\dfrac{Y(s)}{R(s)} = \dfrac{k_D s + k_p + k_I / s}{s^2 - s + 1 + k_D s + k_p + k_I / s} = \dfrac{k_D s^2 + k_p s + k_I}{s^3 + (k_D - 1)s^2 + (1 + k_p)s + k_I}$,

$a(s) = s^3 + a_2 s^2 + a_1 s + a_0$, where $a_2 = k_D - 1$, $a_1 = 1 + k_p$, $a_0 = k_I$

It is possible to stabilize the closed-loop system.

Since $a(s) = (s - s_1)(s - s_2)(s - s_3) = (s + 2 - j2)(s + 2 + j2)(s + 6) = s^3 + 10s^2 + 32s + 48$

$k_D - 1 = 10,\ \ 1 + k_p = 32,\ \ k_I = 48 \quad \Rightarrow \quad k_D = 11,\ \ k_p = 31,\ \ k_I = 48$

Thus, the transfer function is $\dfrac{Y(s)}{R(s)} = \dfrac{11s^2 + 31s + 48}{s^3 + 10s^2 + 32s + 48}$, the steady state error is

$1 - \lim\limits_{s \to 0} \dfrac{Y(s)}{R(s)} = 1 - 1 = 0$, and the step response is,