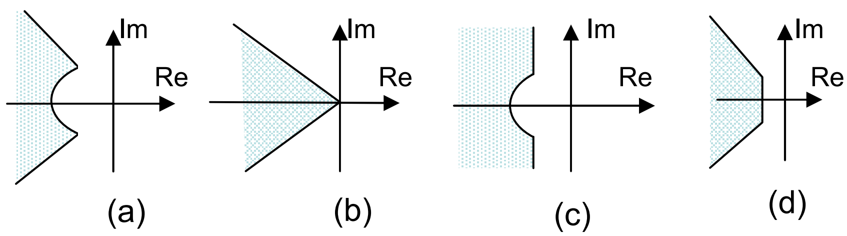
16.413 Linear Feedback Systems
Midterm Exam

Name: ____________________ ID: ____________________

Total 26 points. 20 is a full score.

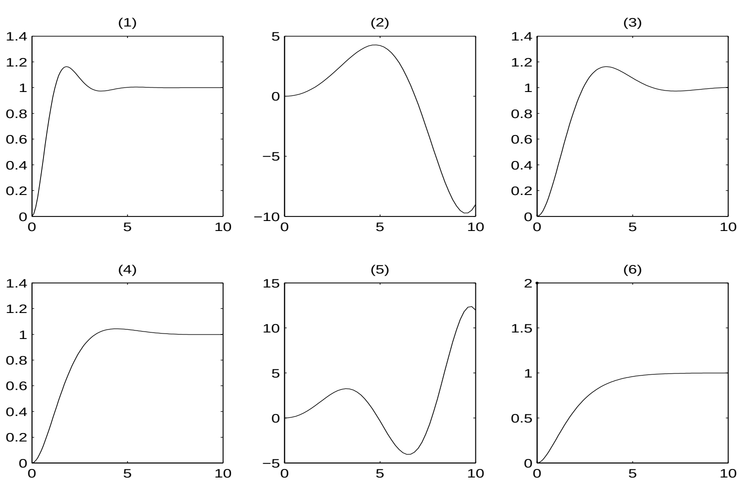
1. (2) Match time domain specifications and regions of pole locations.



(d) $M_p \le M_p^*,\ t_s \le t_s^*$ (c) $t_s \le t_s^*, t_r \le t_r^*$ ( ) $t_s \le t_s^*, M_p \le M_p^*,\ t_r \le t_r^*$

(a) $M_p \le M_p^*,\ t_r \le t_r^*$ ( ) $t_r \le t_r^*$ (b) $M_p \le M_p^*$

2. (3) Match step responses and transfer functions



(5) $H(s) = \dfrac{1}{s^2 - 0.5s + 1}$

(4) $H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

(3) $H(s) = \dfrac{1}{s^2 + s + 1}$

(6) $H(s) = \dfrac{1}{s^2 + 2s + 1}$

(2) $H(s) = \dfrac{0.5}{s^2 - 0.5s + 0.5}$

(1) $H(s) = \dfrac{1}{s^2 + 2s + 4}$

(The numerator should be 4)

3. (4) Laplace Transform:

(1) Given L[f(t)] = F(s), express L[(cos t) $\int_0^t f(\tau)d\tau$] in terms of F(s).

(2) Solve $\ddot{y} - y = 0, \quad y(0) = 0;\ \dot{y}(0) = 1.$

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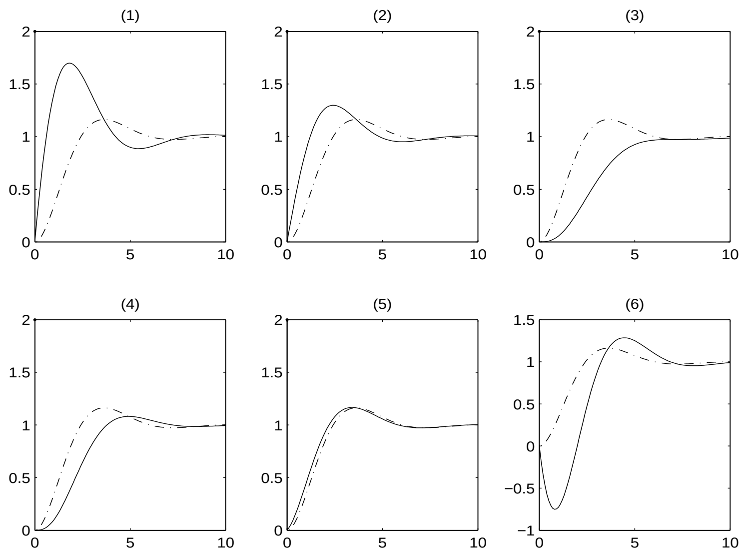
4. (3) Examine the effect of additional poles and zeroes. Match step responses and transfer functions. The dashed curves correspond to $H_0(s)$.



$$H_0(s) = \frac{1}{s^2 + s + 1}$$

(1) $H(s) = H_0(s)\dfrac{s+0.5}{0.5}$

( ) $H(s) = H_0(s)\dfrac{1}{s-1}$

(6) $H(s) = -H_0(s)\dfrac{s-0.5}{0.5}$

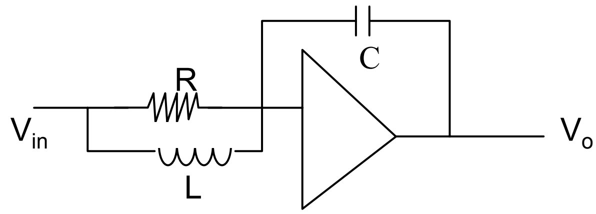
(3) $H(s) = H_0(s)\dfrac{0.5}{s+0.5}$

(5) $H(s) = H_0(s)\dfrac{s+5}{5}$

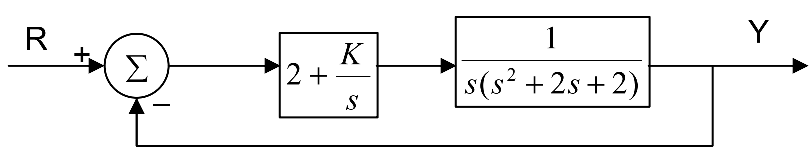
(2) $H(s) = H_0(s)(s+1)$

(4) $H(s) = H_0(s)\dfrac{1}{s+1}$

5. (3) Derive the dynamical equation and transfer function V_o/V_{in} for the following circuit:



6. (4) Determine the range of K for the following feedback system to be stable:



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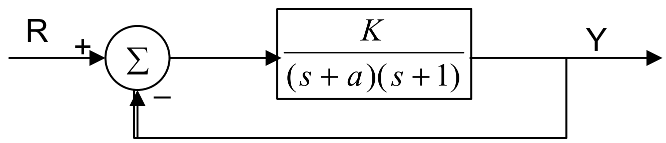
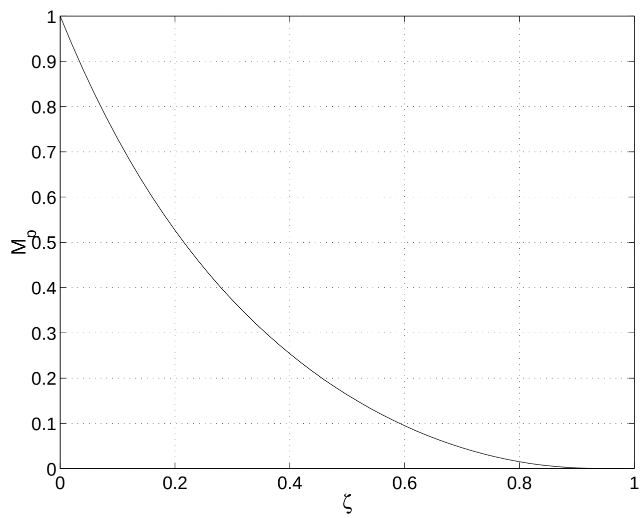
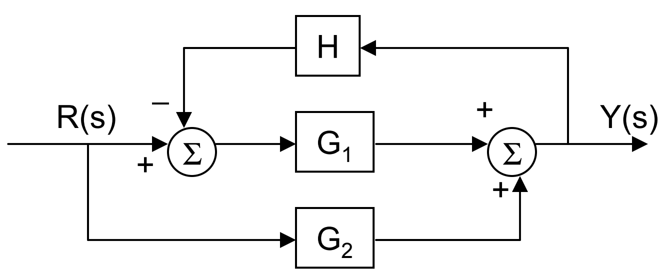
7. (5) For the closed-loop system,



Find a and K so that the overshoot is less than 10% and the settling time is less than 1 second. You may use the $M_p - \zeta$ graph for your design.



8. (2) Find the transfer function from R(s) to Y(s):



3. (4) Laplace Transform:

(1) Given L[f(t)] = F(s), express L[(cos t) $\int_0^t f(\tau)d\tau$] in terms of F(s).

(2) Solve $\ddot{y} - y = 0, \quad y(0) = 0; \ \dot{y}(0) = 1$.

Solution:

(1) Given $\mathcal{L}[f(t)] = F(s)$,

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

$$\mathcal{L}\left[(\cos t)\int_0^t f(\tau)d\tau\right] = \mathcal{L}\left[\frac{e^{jt}+e^{-jt}}{2}\int_0^t f(\tau)d\tau\right]$$

$$= \frac{1}{2}\mathcal{L}\left[e^{jt}\int_0^t f(\tau)d\tau\right] + \frac{1}{2}\mathcal{L}\left[e^{-jt}\int_0^t f(\tau)d\tau\right]$$

$$= \frac{1}{2}\frac{1}{s-j}F(s-j) + \frac{1}{2}\frac{1}{s+j}F(s+j)$$

(2) Denote $\mathcal{L}[y(t)] = Y(s)$, then the Laplace transform of the equation $\ddot{y} - y = 0$ is,

$$s^2Y(s) - sy(0) - \dot{y}(0) - Y(s) = 0$$

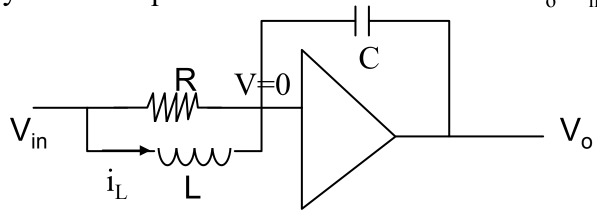
Given the initial condition $y(0) = 0; \ \dot{y}(0) = 1$,

$$s^2Y(s) - 1 - Y(s) = 0 \qquad \Rightarrow \qquad Y(s) = \frac{1}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

thus,

$$y(t) = \frac{1}{2}\left(e^t - e^{-t}\right) = \sinh t$$

5. (3) Derive the dynamical equation and transfer function V_o/V_{in} for the following circuit:



Solution: Consider the node, which voltage $V = 0$, from Kirchhoff's current law,

$$\frac{V_{in}}{R} + i_L + C\dot{V}_o = 0 \qquad (1)$$

Where i_L denotes the current through L, and

$$L\dot{i}_L = V_{in} \qquad (2)$$

From (1), (2), eliminate i_L, we can obtain the dynamic equation,

$$L\dot{V}_{in} + RV_{in} + LRC\ddot{V}_o = 0$$

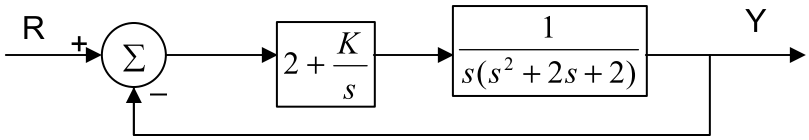
The laplace transform is,

$$sLV_{in}(s) + RV_{in}(s) + s^2 LRCV_o(s) = 0$$

Thus the transfer function is,

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{sL + R}{s^2 LRC}$$

6. (4) Determine the range of K for the following feedback system to be stable:



Solution: The transfer function is,

$$T(s)=\frac{(2+K/s)\dfrac{1}{s(s^2+2s+2)}}{1+(2+K/s)\dfrac{1}{s(s^2+2s+2)}}=\frac{2s+K}{s^4+2s^3+2s^2+2s+K},$$

$$a(s)=s^4+2s^3+2s^2+2s+K$$

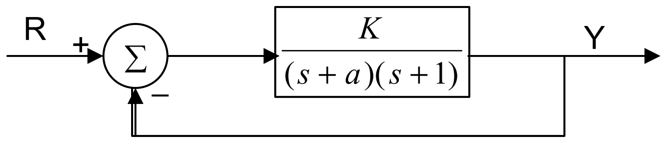
The Routh Array is

$$\begin{array}{llll} s^4: & 1 & 2 & K \\ s^3: & 2 & 2 & \\ s^2: & 1 & K & \\ s^1: & 2-2K & & \\ s^0: & K & & \end{array}$$

In order for the system to be stable, all the roots of $a(s)$ should be in LHP, thus,

$$K>0 \text{ and } 2-2K>0 \qquad \Rightarrow \qquad 0<K<1$$

7. (5) For the closed-loop system,



Find a and K so that the overshoot is less than 10% and the settling time is less than 1 second. You may use the $M_p - \zeta$ graph for your design.

Solution: The transfer function is,

$$T(s) = \frac{\dfrac{K}{(s+a)(s+1)}}{1+\dfrac{K}{(s+a)(s+1)}} = \frac{K}{s^2+(a+1)s+a+K} = \frac{K}{a+K}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},$$

Where

$$\omega_n^2 = a+K \qquad (1)$$

$$2\zeta\omega_n = a+1 \qquad (2)$$

From (2),

$$a = 2\zeta\omega_n - 1 = 2\sigma - 1 \qquad (3)$$

Given $r_s < 1\,\text{sec}$, or $\sigma > \dfrac{4.6}{1\,\text{sec}} = 4.6$,

$$a = 2\sigma - 1 > 8.2$$

Given $M_p < 10\%$, from $M_p - \zeta$ graph or $\zeta = \sqrt{\dfrac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}}$,

$$\zeta > 0.591$$

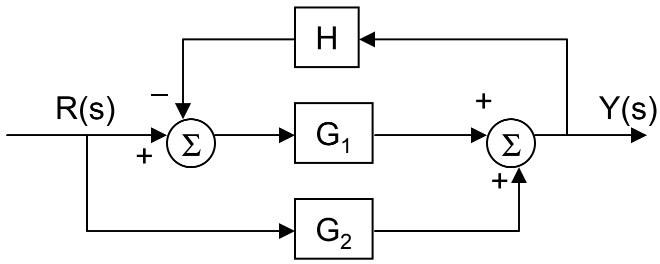
From (1),

$$K = \omega_n^2 - a = \frac{\sigma^2}{\zeta^2} - a$$

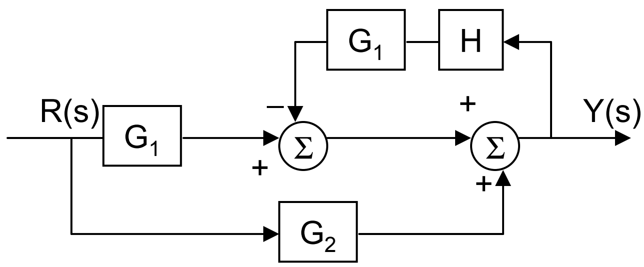
If we pick the minimum value, i.e., $a = 8.2$

$$K = \frac{4.6^2}{0.591^2} - 8.2 = 52.4$$

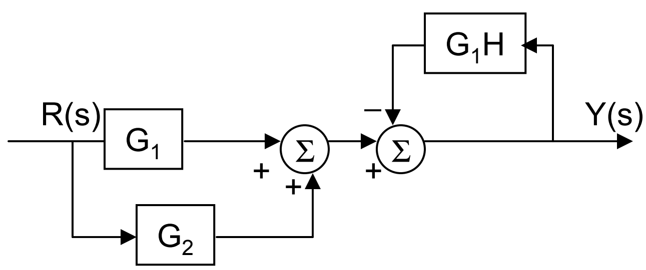
8. (2) Find the transfer function from R(s) to Y(s):



Solution: Move G_1 pass the summer,



Exchange the order of the two summer,



So the transfer function is

$$T(s)=\frac{Y(s)}{R(s)}=\frac{1}{1+G_1H}(G_1+G_2)=\frac{G_1+G_2}{1+G_1H}$$