Non-quadratic Lyapunov functions for performance analysis of saturated systems

Tingshu Hu, Andrew R. Teel and Luca Zaccarian

Abstract— **In a companion paper [14], we developed a systematic Lyapunov approach to the regional stability and performance analysis of saturated systems via quadratic Lyapunov functions. The corresponding conditions are expressed in terms of LMIs but can be too conservative in some cases. To obtain less conservative conditions, we use in this paper two types of conjugate Lyapunov functions: the convex hull quadratic function and the max quadratic function. These functions yield bilinear matrix inequalities (BMIs) as conditions for stability and guaranteed performance level. The BMI conditions cover the LMI conditions for quadratic stability as special cases and hence the reduction of conservativeness is guaranteed. A numerical example demonstrates the effectiveness of this paper's methods and the great potential of the non-quadratic Lyapunov functions.**

keywords: saturation, deadzone, nonlinear $\mathcal{L}_2$ gain, reachable set, domain of attraction, Lyapunov functions.

I. Introduction

Saturation nonlinearities are very common in control systems. The development of analysis and synthesis tools for stability and performance within this context has captured increasing attention from the control community in the past ten years (see the companion paper [14] for some key references in this field). One important approach to characterizing stability and performances for systems with saturation is the Lyapunov approach. Generally the Lyapunov approach consists of two main steps. In the first step the saturation or the deadzone functions are bounded locally or globally with sectors. As a result, the system is described with a linear differential inclusion (LDI). In the second step, the LDI or the system satisfying a sector condition is analyzed with tools based on (or extended from) absolute stability theory which uses Lyapunov functions to characterize stability or performances.

Many existing results on systems with saturation adopt the Lyapunov approach and most of them use quadratic Lyapunov functions. While significant results have been developed for the characterization of global stability and L_2 gain, more recent efforts have been devoted to regional stability and performance analysis (see the extensive references in [14]). For example, in our recent work [13], we characterized nonlinear L_2 gains for saturated systems, and used an example to illustrate that the global characterization of the L_2 gain can be misleading when the system operates within a bounded region. The regional performance analysis in [13] was made possible through an effective tool of describing a saturated system with a parameterized LDI. In our companion paper [14], we enhanced the LDI description by proposing two forms of parameterized LDIs. One of them is polytopic LDI (PLDI) and the other is norm-bounded LDI (NLDI). The NLDI is derived from the PLDI and is generally more conservative but could be easier to handle in some cases. It turns out that the analysis results based on the NLDI is equivalent to those in [13]. With these LDI descriptions, there is yet another great potential to be explored in the second step about the analysis of LDIs. It is now generally accepted that quadratic Lyapunov functions could be very conservative even for stability analysis of LDIs (see, e.g., [3], [5], [15]). For this reason, considerable attention has been paid to the construction and development of non-quadratic Lyapunov functions (e.g., see [1], [2], [3], [15], [16], [17], [20]). In [18], piecewise quadratic Lyapunov functions were used for the study of saturated systems.

Recently, two types of conjugate Lyapunov functions have both demonstrated a great potential in the analysis of LDIs and saturated linear systems [7], [6], [9], [12]. One is called the convex hull quadratic function since its level set is the convex hull of a family of ellipsoids. The other is called max quadratic function since it is obtained by taking the pointwise maximum over a family of quadratic functions and its level set is the intersection of a family of ellipsoids. Some conjugate relationships about these two functions were established in [7], [6]. Since these functions are natural extensions to quadratic functions, they can also be used to perform quantitative performance analysis beyond stability, such as to estimate the L_2 gain, the reachable set, and the dissipativity, for LDIs. A handful of dual bilinear matrix inequalities (BMIs) have been derived for these purposes in [6]. As compared to the corresponding LMIs resulting from quadratic Lyapunov functions, these BMIs contain extra degrees of freedom in the bilinear terms, which are injected through the non-quadratic functions. Experience with low order systems shows that these BMIs can be effectively solved with the path-following method in [8]. Although it is possible that numerical difficulties may arise for higher order systems, the great potential of these non-quadratic Lyapunov functions has been demonstrated in [7], [6], [12] through

Work supported in part by AFOSR grant number F49620- 03-1-0203, NSF under Grants ECS-9988813 and ECS-0324679, by ENEA-Euratom, ASI and MIUR under PRIN and FIRB projects.

A.R. Teel is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, USA teel@ece.ucsb.edu

T. Hu is with the Dept. of ECE, Umass Lowell, Lowell, MA 01854, USA tingshu_hu@uml.edu

L. Zaccarian is with the Dipartimento di Informatica, Sistemi e Produzione, University of Rome, Tor Vergata, 00133 Rome, Italy zack@disp.uniroma2.it

a set of numerical examples. In this paper, we will use these two conjugate Lyapunov functions to enhance regional performance analysis of saturated systems.

This paper is organized as follows. In Section I-A we describe the problems to be studied and briefly summarize the key tools developed in [14] which are crucial to the description of a saturated system with Polytopic Differential Inclusions (PDI). Section II contains the main results on the characterization of stability and performances via the two non-quadratic Lyapunov functions. Section III uses the same example from [14] to show that tighter estimations of the nonlinear L_2 gain can be achieved by using the non-quadratic functions. The paper is concluded with proofs of key and technical results.

Notation

- $I[k_1,k_2]$: For two integers $k_1,k_2,k_1<k_2$, $I[k_1,k_2]=\{k_1,k_1+1,\cdots,k_2\}$.
- $\mathrm{sat}(\cdot)$: The standard saturation function. For $u\in\mathbb{R}^m$, $[\mathrm{sat}(u)]_i=\mathrm{sign}(u_i)\min\{1,|u_i|\}$.
- $\mathrm{dz}(u)$: The deadzone function, $\mathrm{dz}(u)=u-\mathrm{sat}(u)$.
- $\mathrm{co}\, S$: The convex hull of a set S.
- $\mathrm{He}X$: For a square matrix X, $\mathrm{He}X:=X+X^T$.
- $\mathcal{E}(P)$: For $P\in\mathbb{R}^{n\times n}, P=P^T>0$,

$$\mathcal{E}(P):=\{x\in\mathbb{R}^n:\ x^TPx\le 1\}.$$

- $\mathcal{L}(H)$: For $H\in\mathbb{R}^{m\times n}$,

$$\mathcal{L}(H):=\{x\in\mathbb{R}^n:\ |Hx|_\infty\le 1\}.$$

About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(H)$, for a given $s>0$, we have (see, e.g., [11]),

$$s\mathcal{E}(P)\subset\mathcal{L}(H)\Longleftrightarrow\begin{bmatrix}1/s^2 & H_\ell\\ H_\ell^T & P\end{bmatrix}\ge 0 \tag{1}$$

for all $\ell\in I[1,m]$, where H_ℓ is the ℓth row of H.

A. Problem statement

The type of closed-loop system that we address in this paper corresponds to the following general representation of a linear system subject to saturation:

$$\begin{cases}\dot{x} &= Ax+B_qq+B_ww\\ y &= C_yx+D_{yq}q+D_{yw}w\\ z &= C_zx+D_{zq}q+D_{zw}w\\ q &= \mathrm{dz}(y)\ .\end{cases} \tag{2}$$

where $x\in\mathbb{R}^n,q,y\in\mathbb{R}^m,w\in\mathbb{R}^r,z\in\mathbb{R}^p$ and "dz" is the standard vector-valued deadzone function. This system can be graphically depicted as in Fig. 1, where w is the exogenous input or disturbance and z is the performance output. Many linear systems with saturation/deadzone components can be transformed into the above general form through loop transformation. Due to this fact, this type of closed-loop system has received great attention from the control community over the past decade. In most of the literature, various restrictive assumptions are made on the general configuration (2), such as the absence of algebraic loops (namely, $D_{yq}=0$), or the exponential stability of certain subsystem (typically, an open-loop plant driven by saturated signals), and so on. More detailed discussions on the background can be found in our companion paper [14]. What distinguishes our current effort from most of the existing results is that the only assumptions we make on system (2) are that A is Hurwitz and that the nonlinear algebraic loop is well posed (these are clearly basic requirements for the system to be functional). In [14], great attention has been devoted to the characterization of the well-posedness of (2) and the development of two forms of parameterized differential inclusions for (2): the polytopic differential inclusion (PDI) and the norm-bounded differential inclusion (NDI). In this paper, we will continue to use the PDI description, which is summarized in the following proposition.

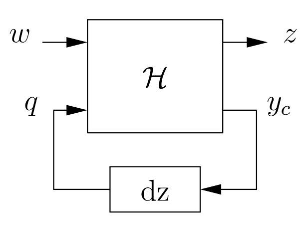

Fig. 1. Compact representation of a system with saturation/deadzone.

Proposition 1: **(Polytopic differential inclusion (PDI))** Let $\{K_i: i\in I[1,2^m]\}$ be the set of diagonal $m\times m$ matrices with 0 or 1 at the diagonal elements. For $i\in I[1,2^m]$, denote

$$\begin{aligned}&T_i=(I-K_iD_{yq})^{-1}K_i,\\ &A_i=A+B_qT_iC_y, \qquad B_i=B_w+B_qT_iD_{yw},\\ &C_i=C_z+D_{zq}T_iC_y, \qquad D_i=D_{zw}+D_{zq}T_iD_{yw}.\end{aligned}$$

Let $h:\mathbb{R}^n\to\mathbb{R}^m$ be a given map and let h_ℓ be the ℓth component of h. For system (2), if $|h_\ell(x)|\le 1$ for all $\ell\in I[1,m]$, then

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix}\in\underset{i\in I[1,2^m]}{\mathrm{co}}\left\{\begin{bmatrix}A_ix+B_iw-B_qT_ih(x)\\ C_ix+D_iw-D_{zq}T_ih(x)\end{bmatrix}\right\}. \tag{3}$$

The NDI description in [14] is more conservative than (3) but may simplify the computation under certain situations. However, we will not consider the NDI description in this paper since detailed investigation reveals that the two non-quadratic Lyapunov functions will yield the same results as those by quadratic functions when NDIs are concerned.

We will consider the same stability and performance analysis problems as in [14]. Instead of using quadratic Lyapunov functions as in [14], we will apply non-quadratic Lyapunov functions to address the following problems:

1. *Estimation of the domain of attraction*: (in the absence of w) by using invariant level sets of the non-quadratic Lyapunov functions.
2. *Estimation of the nonlinear L_2 gain from w to z*: With $\|w\|_2\le s$ for a given s, we would like to determine a number $\gamma>0$ as small as possible, so that under the condition $x(0)=0$, we have $\|z\|_2\le\gamma\|w\|_2$. Performing this analysis for each $s\in(0,\infty)$, we obtain an estimate of the nonlinear L_2 gain.
3. *Estimation of the reachable set (from L_2 bounded inputs)*: With a given bound on the L_2 norm of w, i.e, $\|w\|_2\le s$ for a given s, we would like to determine a set S as small as possible so that under the condition

$x(0) = 0$, we have $x(t) \in S$ for all t. This set S will be considered as an estimate of the reachable set.

II. Analysis with non-quadratic Lyapunov functions

In this section, we will use a pair of conjugate functions, the convex hull quadratic function and the max quadratic function to perform stability and performance analysis of system (2). We first review some results about this pair of conjugate functions.

A. The max quadratic function and the convex hull quadratic function

Given a family of positive definite matrices $P_j \in \mathbb{R}^{n\times n}, P_j = P_j^T > 0, j \in I[1,J]$, the pointwise maximum quadratic function is defined as

$$V_{\max}(x) := \max\{x^T P_j x : \ j \in I[1,J]\}. \qquad (4)$$

Given $Q_j \in \mathbb{R}^{n\times n}, Q_j = Q_j^T > 0, j \in I[1,J]$. Let

$$\Gamma = \left\{\gamma \in \mathbb{R}^J : \ \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \geq 0\right\}.$$

The convex hull quadratic function is defined as

$$V_c(x) := \min_{\gamma\in\Gamma} x^T \left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x. \qquad (5)$$

For simplicity, we say that V_c is composed from Q_j's. It was shown in [7] that $\frac{1}{2}V_{\max}$ is conjugate to $\frac{1}{2}V_c$ if $Q_j = P_j$ for each $j \in I[1,J]$. It is evident that V_c and $V_{\max}$ are homogeneous of degree 2, i.e., $V_c(\alpha x) = \alpha^2 V_c(x), V_{\max}(\alpha x) = \alpha^2 V_{\max}(x)$. Also established in [7], [9] are that V_c is convex and continuously differentiable and that $V_{\max}$ is strictly convex.

The 1-level set of $V_{\max}$ and that of V_c are respectively

$$\begin{aligned} L_{V_{\max}} &:= \left\{x \in \mathbb{R}^n : \ \ V_{\max}(x) \leq 1\right\}, \\ L_{V_c} &:= \left\{x \in \mathbb{R}^n : \ \ V_c(x) \leq 1\right\}. \end{aligned}$$

Since $V_{\max}$ and V_c are homogeneous of degree 2, we have

$$\begin{aligned} sL_{V_{\max}} &= \left\{x \in \mathbb{R}^n : \ \ V_{\max}(x) \leq s^2\right\}, \\ sL_{V_c} &= \left\{x \in \mathbb{R}^n : \ \ V_c(x) \leq s^2\right\}. \end{aligned}$$

It is easy to see that $L_{V_{\max}}$ is the intersection of the ellipsoids $\mathcal{E}(P_j)$'s. In [9], It was established that L_{V_c} is the convex hull of the ellipsoids $\mathcal{E}(Q_j^{-1})$'s, i.e.,

$$L_{V_c} = \left\{\sum_{j=1}^J \gamma_j x_j : \ x_j \in \mathcal{E}(Q_j^{-1}), \gamma \in \Gamma\right\}.$$

For a compact convex set S, a point x on the boundary of S (denoted as ∂S) is called an extreme point if it cannot be represented as the convex combination of any other points in S. For a strictly convex set, such as $L_{V_{\max}}$, every boundary point is an extreme point. In what follows, we characterize the set of extreme points of L_{V_c}. Since L_{V_c} is the convex hull of $\mathcal{E}(Q_j^{-1})$, $j \in I[1,J]$, an extreme point must be on the boundaries of both L_{V_c} and $\mathcal{E}(Q_j^{-1})$ for some $j \in I[1,J]$. Define $E_j := \partial L_{V_c} \cap \partial\mathcal{E}(Q_j^{-1})$, namely

$$E_j = \left\{x \in \mathbb{R}^n : V_c(x) = x^T Q_j^{-1} x = 1\right\}. \qquad (6)$$

Then $\bigcup_{j=1}^J E_j$ contains all the extreme points of L_{V_c}. The exact description of E_j is given by the following lemma (which is proved in the Appendix).

Lemma 1: For each $j \in I[1,J]$,

$$E_j = \{x \in \partial L_{V_c} : x^T Q_j^{-1}(Q_k - Q_j)Q_j^{-1}x \leq 0, \forall\, k \in I[1,J]\}.$$

It is clear from Lemma 1 that $\bigcup_{\delta\in[0,1]}\delta E_j = \{x \in L_{V_c} : x^T Q_j^{-1}(Q_k - Q_j)Q_j^{-1}x \leq 0, \ \forall\, k \in I[1,J]\}$. The following lemma combines results from [9], [10].

Lemma 2: For a given $x_0 \in \mathbb{R}^n$, let $\gamma^* \in \Gamma$ be an optimal γ such that $x_0^T\left(\sum_{j=1}^J \gamma_j^* Q_j\right)^{-1} x_0 = \min_{\gamma\in\Gamma} x_0^T\left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x_0 = V_c(x_0)$. Assume that $\gamma_j^* > 0$ for $j \in I[1,J_0]$ and $\gamma_j^* = 0$ for $j \in I[J_0+1,J]$. Denote

$$Q_0 = \sum_{j=1}^{J_0} \gamma_j^* Q_j, \quad x_j = Q_j Q_0^{-1} x_0, \quad j \in I[1,J_0].$$

Then $V_c(x_j) = V_c(x_0)$ and $x_j \in V_c(x_0)^{\frac{1}{2}} E_j, j \in I[1,J_0]$. Moreover, $x_0 = \sum_{j=1}^{J_0} \gamma_j^* x_j$, and

$$\nabla V_c(x_0) = \nabla V_c(x_j) = 2Q_j^{-1}x_j = 2Q_0^{-1}x_0, \quad j \in I[1,J_0],$$

where $\nabla V_c(x)$ denotes the gradient of V_c at x.

The following lemma is adapted from a result of [12] to the slightly different definition of V_c and $V_{\max}$ (the two functions in [12] have the coefficient $\frac{1}{2}$).

Lemma 3: [12] Let $H \in \mathbb{R}^{m\times n}$ and denote the ℓ-th row of H as H_ℓ. We have, 1) $L_{V_c} \subset \mathcal{L}(H)$ if and only if $H_\ell^T \in L_{V_{\max}}$ for all $\ell \in I[1,m]$; 2) $L_{V_{\max}} \subset \mathcal{L}(H)$ if and only if $H_\ell^T \in L_{V_c}$ for all $\ell \in I[1,m]$.

B. Analysis with convex hull quadratic functions

In this section, we apply the convex hull quadratic function to the analysis of system (2) through the polytopic differential inclusion (3), which is repeated below for easy reference:

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \underset{i\in I[1,2^m]}{\mathrm{co}} \left\{\begin{bmatrix} A_i x + B_i w - B_q T_i h(x) \\ C_i x + D_i w - D_{zq} T_i h(x) \end{bmatrix}\right\}. \qquad (7)$$

This PDI is a valid description for (2) as long as $|h(x)|_\infty \leq 1$. We will restrict our attention to the level set sL_{V_c}, where $|h(x)|_\infty \leq 1$ for all $x \in sL_{V_c}$. As with the case of using quadratic functions, the crucial point is to guarantee that $x(t) \in sL_{V_c}$ under the class of norm-bounded w and the set of initial states under consideration.

It may appear that choosing $h(x)$ as a linear function Hx within sL_{V_c} should lead to simpler results than choosing it as a nonlinear function. However, it turns out that a nonlinear $h(x)$ not only reduces conservatism but also leads to cleaner and numerically more tractable results.

Theorem 1: (Reachable set by bounded inputs) Given $Q_j = Q_j^T > 0, j \in I[1,J]$ and let V_c be composed from Q_j's. Also given $s > 0$. For system (2), with $x(0) = 0$, we have $x(t) \in sL_{V_c}$ for all $t > 0$ and for all w such that $\|w\|_2 \le s$ if there exist $Y_j \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \ge 0$, $i \in I[1,2^m]$, $j,k \in I[1,J]$ such that

$$\begin{bmatrix} \frac{1}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \ge 0, \quad \ell \in I[1,m], j \in I[1,J], \tag{8}$$

where $Y_{j,\ell}$ is the ℓth row of Y_j, and such that for all $i \in I[1,2^m], j \in I[1,J]$,

$$\mathrm{He}\begin{bmatrix} A_iQ_j - B_qT_iY_j + \sum_{k=1}^{J}\lambda_{ijk}(Q_j - Q_k) & B_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0. \tag{9}$$

Remark 1: (Optimization issues) With conditions (9) and (8), we may formulate an optimization problem to minimize the estimation of the reachable set as with the quadratic function case. We observe that the first block in (9) contains some bilinear terms as the product of a full matrix and a scalar. From our computational experience, such kind of bilinear matrix inequalities (BMIs) can be effectively addressed with the path-following method in [8]. We also see that if we take $Q_j = Q$ and $Y_j = Y$ for all j, then the bilinear terms vanish and the conditions reduce to LMIs (these LMIs coincide with the ones given in [14, Theorem 1]). In our computation, we first solve the resulting optimization problem with LMI constraints and then use the optimal Q^* and Y^* to start the new algorithm with BMI constraints, with $Q_j = Q^*$ and $Y_j = Y^*$ for all j and $\lambda_{ijk} \ge 0$ randomly chosen. This approach also proves effective for the problems of estimating the L_2 gain and the domain of attraction, which will be addressed in Theorems 2 and 3.

Although there is no guarantee that the global optimal solution can be located, the convergence of the algorithms is satisfactory. Furthermore, since the initial value of the optimizing parameters can be inherited from the optimal solution obtained with quadratic Lyapunov functions, the algorithms always improve on the results from using the quadratic functions proposed in [14]. To avoid redundancy, we will not discuss the computational issues after we present Theorems 2 or 3. ◦

Remark 2: (About the nonlinear function $h(x)$) From the proof of Theorem 1, we see that a nonlinear function $h(x_0) = H_0(x_0)x_0$ is constructed from Q_j's and Y_j's so that $|H_0(x_0)x_0| \le 1$ for all $x_0 \in sL_{V_c}$ (see (23) where H_0 is constructed and subsequent discussion up to (26)). This makes the proof more complicated than with a linear function Hx but the result turns out to be cleaner and more easily tractable numerically. If we attempt to use a linear function $h(x) = Hx$ such that $|Hx|_\infty \le 1$ for all $x \in sL_{V_c}$, we would have Y_j in (9) replaced with HQ_j and $Y_{j,\ell}$ in (8) replaced with $H_\ell Q_{j,\ell}$. When we formulate an optimization problem to estimate the reachable set by taking H and Q_j's as optimizing parameters, this would result in more complex BMI terms including HQ_j which may cause difficulties in the algorithms, such as slow convergence or getting easily stuck at a local solution. ◦

We next address the problems of estimating the L_2 gain and the domain of attraction.

Theorem 2: (Nonlinear $\mathcal{L}_2$ gain) Given $Q_j = Q_j^T > 0, j \in I[1,J]$ and let V_c be composed from Q_j's. Consider system (2). Given $s,\gamma > 0$. If there exist $Y_j \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \ge 0$, $i \in I[1,2^m]$, $j,k \in I[1,J]$ such that

$$\begin{bmatrix} \frac{1}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \ge 0, \quad \ell \in I[1,m], j \in I[1,J]. \tag{10}$$

and such that

$$\mathrm{He}\begin{bmatrix} A_iQ_j - B_qT_iY_j + \sum_{k=1}^{J}\lambda_{ijk}(Q_j - Q_k) & B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_iQ_j - D_{zq}T_iY_j & D_i & -\frac{\gamma^2}{2}I \end{bmatrix} \le 0, \tag{11}$$

for all $i \in I[1,2^m], j \in I[1,J]$. Then for all w such that $\|w\|_2 \le s$ and $x(0) = 0$, we have $\|z\|_2 \le \gamma\|w\|_2$.

The proof of the following theorem can be adapted from the proof of Theorem 1 by assuming that $B_i = 0$.

Theorem 3: (Estimation of the domain of attraction) Given $Q_j = Q_j^T > 0$, $j \in I[1,J]$. Let V_c be composed from Q_j's. Consider system (2). With $w \equiv 0$, we have $\dot{V}_c(x) < 0$ for all $x \in L_{V_c} \setminus \{0\}$ if there exist $\lambda_{ijk} \ge 0$, $Y_j \in \mathbb{R}^{m\times n}, i \in I[1,2^m], j,k \in I[1,J]$ such that

$$\begin{bmatrix} 1 & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \ge 0, \quad \ell \in I[1,m], j \in I[1,J]. \tag{12}$$

and that for all $i \in I[1,2^m], j \in I[1,J]$,

$$\mathrm{He}(A_iQ_j - B_qT_iY_j + \sum_{k=1}^{J}\lambda_{ijk}(Q_j - Q_k)) < 0. \tag{13}$$

C. Analysis with max quadratic functions

The max quadratic function is not differentiable everywhere. We use $\partial V_{\max}(x)$ to denote its generalized Jacobian at x (see, e.g., [4]). If $x^TP_jx > x^TP_kx$ for all $k \ne j$, then $V_{\max}$ is differentiable at x and $\partial V_{\max}(x)$ is single valued and equals $2P_jx$. If $x^TP_1x = x^TP_2x = \cdots = x^TP_{J_0}x$ and $x^TP_1x > x^TP_jx$ for $j > J_0$, then $\partial V_{\max}(x) = \mathrm{co}\{2P_jx : j \in I[1,J_0]\}$. For simplicity and with some abuse of notation, along the trajectory of (2), we denote

$$\dot{V}_{\max} := \max\{\xi^T\dot{x} : \xi \in \partial V_{\max}(x)\}.$$

Theorem 4: (Reachable set by bounded inputs) Given $P_j = P_j^T > 0, j \in I[1,J]$ and let $V_{\max}$ be the max quadratic function formed by P_j's. Also given $s > 0$. For the system (2), with $x(0) = 0$, we have $x(t) \in sL_{V_{\max}}$ for all $t > 0$ and for all w such that $\|w\|_2 \le s$ if there exist $H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \ge 0$, $\alpha_{\ell j} \ge 0$, $j,k \in I[1,J]$, $i \in I[1,2^m]$, $\ell \in I[1,m]$, such that $\sum_{j=1}^{J}\alpha_{\ell j} = 1$,

$$\begin{bmatrix} \frac{1}{s^2} & H_\ell \\ H_\ell^T & \sum_{j=1}^{J}\alpha_{\ell j}P_j \end{bmatrix} \ge 0, \quad \ell \in I[1,m]. \tag{14}$$

and for all $i \in I[1,2^m], j \in I[1,J]$,

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH + \sum_{k=1}^{J}\lambda_{ijk}(P_j - P_k) & P_jB_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0. \tag{15}$$

Theorem 5: (Nonlinear $\mathcal{L}_2$ gain) Consider system (2). Given $P_j = P_j^T > 0, j \in I[1,J]$ and $s,\gamma > 0$, if there exist

$H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j,k \in I[1,J]$, $i \in I[1,2^m]$, $\ell \in I[1,m]$, such that $\sum_{j=1}^{J} \alpha_{\ell j} = 1$,

$$\begin{bmatrix} \frac{1}{s^2} & H_\ell \\ H_\ell^T & \sum_{j=1}^{J} \alpha_{\ell j} P_j \end{bmatrix} \geq 0, \quad \ell \in I[1,m], \tag{16}$$

and for all $i \in I[1,2^m], j \in I[1,J]$,

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH + \sum_{k=1}^{J}\lambda_{ijk}(P_j-P_k) & P_jB_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq}T_iH & D_i & -\frac{\gamma^2}{2}I \end{bmatrix} \leq 0, \tag{17}$$

then for all w such that $\|w\|_2 \leq s$ and $x(0) = 0$, we have $\|z\|_2 \leq \gamma\|w\|_2$.

The following result can be derived by adapting the proof of Theorem 4.

Theorem 6: (Estimation of the domain of attraction) Consider system (2) and any $P_j = P_j^T > 0, j \in I[1,J]$. With $w \equiv 0$, then $\dot{V}_{\max}(x) < 0$ for all $x \in L_{V_{\max}} \setminus \{0\}$ if there exist $H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j,k \in I[1,J]$, $i \in I[1,2^m]$, $\ell \in I[1,m]$, such that $\sum_{j=1}^{J} \alpha_{\ell j} = 1$,

$$\begin{bmatrix} 1 & H_\ell \\ H_\ell^T & \sum_{j=1}^{J} \alpha_{\ell j} P_j \end{bmatrix} \geq 0, \quad \ell \in I[1,m]. \tag{18}$$

and for all $i \in I[1,2^m], j \in I[1,J]$,

$$\mathrm{He}\left[P_jA_i - P_jB_qT_iH + \sum_{k=1}^{J}\lambda_{ijk}(P_j-P_k) \right] < 0. \tag{19}$$

As compared to the counterpart results from using convex hull quadratic functions, the conditions (15), (17) and (19) in Theorems 4 to 6 appear to be less tractable because of the bilinear term $P_jB_qT_iH$ in the first blocks of the matrices. Also, the same H for all P_j's seem to offer fewer degrees of freedom as compared to the different Y_j for different Q_j in Theorems 1 to 3. However, numerical examples show that Theorems 4 to 6 may produce better results in some cases.

III. An Example

Consider system (2) with the following parameters:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \left[\begin{array}{ccc|cc|cc} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right].$$

This system is the same as the one considered in [14], where quadratic Lyapunov functions are applied to the PDI description and the NDI description, respectively, to obtain two estimates of the nonlinear L_2 gain. Using this paper's methods, we obtain two improved estimates of the nonlinear L_2 gain, one by applying the convex hull function to the PDI description and the other by applying the max quadratic function to the PDI description. In what follows, we compare the two estimates from [14] and the two improved estimates by this paper's methods.

Fig. 2 presents these four estimates of the nonlinear L_2 gain. The dotted curve is from applying quadratics via NDI, the dash-dotted one is from applying quadratics via PDI, the dashed one is from applying max quadratics (with $J = 2$) via PDI (Theorem 5) and the solid one is from applying convex hull quadratics ($J = 2$) via PDI (Theorem 2). Each of the

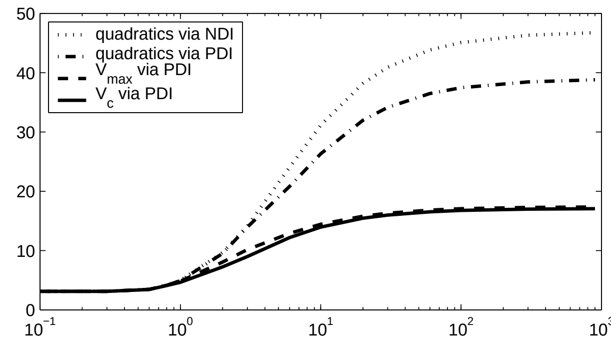

Fig. 2. Different estimates of the nonlinear L_2 gain: Case 1.

four curves tends to a constant value as $\|w\|_2$ goes to infinity. This constant value will be an estimate of the global L_2 gain. As expected, applying non-quadratic Lyapunov functions always leads to better results than applying quadratic ones. However, the relationship between the two non-quadratic functions is not definite. The situation exhibited in Fig. 2 can be reversed if we change the parameters of the system. In what follows, we present several scenarios through some adjustment of the parameters.

Case 2: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -1.3 \\ -2.3 & -4 \end{bmatrix}$, then the global L_2 gain by using quadratics via NDI is unbounded (or, global stability is not confirmed), while that by using quadratics via PDI is 170.1473. By using max quadratics and convex hull quadratics, the global L_2 gains are respectively 20.7833 and 19.3307.

Case 3: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$, then the global L_2 gain by using quadratics via either NDI or PDI is unbounded. By using max quadratics and convex hull quadratics, the global L_2 gains are respectively 42.3354 and 31.6731.

The two situations above also show how the stability and performance results by the same method can be affected by the parameter D_{yq} which describes the algebraic loop. As discussed in [19], this parameter can be adjusted through anti-windup compensation.

Case 4: Next we replace the matrix A with its transpose and take $D_{yq} = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$. The four different bounds on the L_2 gain are plotted in Fig. 3, where the dashed curve is from using max quadratics via PDI and the solid curve is from using convex hull quadratics via PDI. We see that for some range of $\|w\|_2$ around 1, the dashed curve is higher than the solid one but for $\|w\|_2 > 10$, the dashed curve is lower than the solid one.

Preprint submitted to 44th IEEE Conference on Decision and Control
and European Control Conference ECC 2005. Received March 7, 2005.

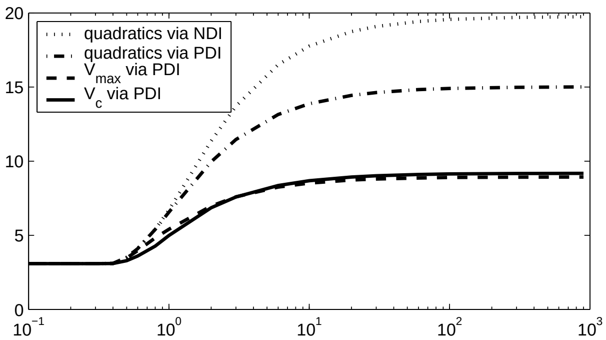

Fig. 3. Different estimates of the nonlinear L_2 gain: Case 4.

Due to space limitation, we will not present computational results about the estimation of the domain of attraction or the estimation of the reachable set. From the different situations exhibited through the L_2 gain, it is not hard to infer that the difference among the estimations by using quadratics/non-quadratics via NDI/PDI can be made arbitrarily large through adjusting the four elements of D_{yq}. For instance, Case 2 suggests that the estimate of the domain of attraction by using quadratics via NDI is bounded while that by using quadratics via PDI is the whole state space. Case 3 suggests that the domain of attraction estimated by non-quadratic functions is the whole state space while that by quadratics (via PDI or NDI) is bounded. On the other hand, the estimate of the reachable set by non-quadratics can be bounded while that by quadratics is not.

We should remark that for this particular example, the algorithm for applying convex hull quadratics converges very well for all $\|w\|_2$ and under different parameter changes. The algorithm for applying max quadratics generally converges well but for some $\|w\|_2$, it may have some difficulties and we need to stop the algorithm and restart it from different initial values of λ_{ijk} which are randomly generated. In any case, improvement is expected from the non-quadratic functions.

IV. Proof of the main results

Proof of Theorem 1. We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$, we have $\dot{V}_c(x,w) \le w^T w$, where $\dot{V}_c(x,w)$ is the derivative of V_c along the trajectory of (2), which depends on x and w.

Let $P_j = Q_j^{-1}$, $H_j = Y_j Q_j^{-1}$. Left- and right-multiplying (9) by $\mathrm{diag}\{P_j, I\}$, we have

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH_j + \sum_{k=1}^{J}\lambda_{ijk}P_j(Q_j - Q_k)P_j & P_jB_i \\ 0 & -\frac{I}{2}\end{bmatrix} \le 0.$$

This implies that for all $i \in I[1,2^m], j \in I[1,J]$,

$$\begin{array}{l} 2x^TP_j(A_ix + B_iw - B_qT_iH_jx) - w^Tw \\ \quad \le 2\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx, \end{array} \tag{20}$$

$\forall\, x \in \mathbb{R}^n, w \in \mathbb{R}^r$. Given $j \in I[1,J]$ and any $\delta > 0$. Consider $x \in \delta E_j$. By Lemma 1 we have

$$\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \le 0. \tag{21}$$

It follows from (20) that for all $x \in \delta E_j, w \in \mathbb{R}^r,\ \ \delta > 0$,

$$2x^TP_j(A_ix + B_iw - B_qT_iH_jx) - w^Tw \le 0. \tag{22}$$

(In view of (7) and condition (8), this actually shows that $\dot{V}_c(x,w) \le w^Tw$ for all $x \in s(L_{V_c} \cap E_j)$. We proceed to show that this inequality holds for all $x \in sL_{V_c}$ by exploring the properties of V_c.)

Now consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0,s]$. By Lemma 2, there exist $x_j \in \delta E_j$, $\gamma_j > 0, j \in I[1,J_0]$ with $J_0 \le J$ such that $\sum_{j=1}^{J_0}\gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0}\gamma_jx_j$. Let

$$Q_0 = \sum_{j=1}^{J_0}\gamma_jQ_j, \quad Y_0 = \sum_{j=1}^{J_0}\gamma_jY_j, \quad H_0 = Y_0Q_0^{-1}. \tag{23}$$

Then we also have $x_0^TQ_0^{-1}x_0 = V_c(x_0) = \delta^2$ and

$$\nabla V_c(x_0) = 2Q_0^{-1}x_0 = 2Q_j^{-1}x_j, \quad j \in I[1,J_0]. \tag{24}$$

Applying convex combination to the inequalities in (8), we have for $\ell \in I[1,m]$,

$$\begin{bmatrix} 1/s^2 & Y_{0,\ell} \\ Y_{0,\ell}^T & Q_0 \end{bmatrix} \ge 0 \Leftrightarrow \begin{bmatrix} 1/s^2 & H_{0,\ell} \\ H_{0,\ell}^T & Q_0^{-1} \end{bmatrix} \ge 0. \tag{25}$$

By (1), this implies that $s\mathcal{E}(Q_0^{-1}) \subset \mathcal{L}(H_0)$. Since $x_0^TQ_0^{-1}x_0 = \delta^2 \le s^2$, we have $|H_0x_0| \le 1$. Applying (7) at x_0 with $h(x_0) = H_0x_0$, we have

$$\dot{x} \in \mathrm{co}\{A_ix_0 + B_iw - B_qT_iH_0x_0 :\ i \in I[1,2^m]\} \tag{26}$$

and

$$\begin{array}{l} \dot{V}_c(x_0,w) - w^Tw \in \\ \underset{i\in I[1,2^m]}{\mathrm{co}}\{\nabla^TV_c(x_0)(A_ix_0 + B_iw - B_qT_iH_0x_0) - w^Tw\}. \end{array} \tag{27}$$

Recall that, for all $x_j \in \delta E_j$,

$$\begin{array}{rcl} x_0 &=& \sum_{j=1}^{J_0}\gamma_jx_j, \\ \nabla V_c(x_0) &=& 2Q_0^{-1}x_0 = 2Q_j^{-1}x_j = 2P_jx_j. \end{array} \tag{28}$$

Applying (22) to x_j and replacing $2x_j^TP_j$ by $\nabla^TV_c(x_0)$, we obtain for all $w \in \mathbb{R}^r$,

$$\nabla^TV_c(x_0)(A_ix_j + B_iw - B_qT_iH_jx_j) - w^Tw \le 0. \tag{29}$$

By the definition of Q_0, H_0 and Y_0 in (23),

$$H_0x_0 = Y_0Q_0^{-1}x_0 = \left(\sum_{j=1}^{J_0}\gamma_jY_j\right)Q_0^{-1}x_0 \tag{30}$$

and from (24) we have

$$H_jx_j = Y_jQ_j^{-1}x_j = Y_jQ_0^{-1}x_0, \quad j \in I[1,J_0]. \tag{31}$$

Combining (28), (30) and (31) we have

$$\begin{array}{l} A_ix_0 + B_iw - B_qT_iH_0x_0 = \\ \sum_{j=1}^{J_0}\gamma_j(A_ix_j + B_iw - B_qT_iH_jx_j) \quad \forall\, w \in \mathbb{R}^r. \end{array} \tag{32}$$

Note that this is satisfied for all $i \in I[1,2^m]$. It follows from (29) that for each $i \in I[1,2^m]$ and $w \in \mathbb{R}^r$,

$$\begin{array}{l} \nabla^TV_c(x_0)(A_ix_0 + B_iw - B_qT_iH_0x_0) - w^Tw \\ = \sum_{j=1}^{J_0}\gamma_j[\nabla^TV_c(x_0)(A_ix_j + B_iw - B_qT_iH_jx_j) - w^Tw] \\ \le 0 \end{array}$$

By (27), we have

$$\dot{V}_c(x_0, w) - w^T w \le 0 \quad \forall\, w \in \mathbb{R}^r. \tag{33}$$

Note that x_0 is an arbitrary point in sL_{V_c}.

Hence we have that $\dot{V}_c(x, w) \le w^T w$ for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$. Now suppose $x(0) = 0$ and $\|w\|_2^2 \le s^2$. Then for any $t_0 > 0$, as long as $x(t) \in sL_{V_c}$ for all $t \in (0, t_0)$, we have $V_c(x(t_0)) \le \int_0^{t_0} w^T(\tau)w(\tau)d\tau \le s^2$, i.e., $x(t_0) \in sL_{V_c}$. On the other hand, if there exists $t_0 > 0$ such that $V_c(x(t)) \le s^2$ for all $t \in (0, t_0)$ and $V_c(x(t_0)) = s^2$ then we must have $\int_{t_0}^{\infty} w^T(\tau)w(\tau)d\tau = 0$ and $\dot{V}_c(x(t), w(t)) \le 0$ for almost all $t > t_0$. Hence $V_c(x(t)) \le s^2$ for all $t > t_0$. Therefore, we conclude that $x(t) \in sL_{V_c}$ for all $t > 0$. □

Proof of Theorem 2. We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$,

$$\dot{V}_c(x, w) + \frac{1}{\gamma^2} z^T z \le w^T w.$$

Since (11) implies (9), by Theorem 1, we have $x(t) \in sL_{V_c}$ for all t and for all $\|w\|_2 \le s$, $x(0) = 0$. Also, all the relationships established in the proof of Theorem 1 are true under the conditions of the current theorem.

Let $P_j = Q_j^{-1}$ and $H_j = Y_j Q_j^{-1}$. Left- and right-multiplying (11) by $\mathrm{diag}\{P_j, I, I\}$, and applying Schur complement, similar to the proof of Theorem 1, we obtain that for all $x \in \delta E_j, w \in \mathbb{R}^r, \delta > 0$

$$2x^T P_j f_{ij}(x, w) + \frac{1}{\gamma^2} g_{ij}^T(x, w) g_{ij}(x, w) - w^T w \le 0, \tag{34}$$

where

$$\begin{aligned} f_{ij}(x, w) &= A_i x + B_i w - B_q T_i H_j x, \\ g_{ij}(x, w) &= C_i x + B_i w - D_{zq} T_i H_j x. \end{aligned}$$

Relation (34) holds for all $i \in I[1, 2^m]$ and $j \in I[1, J]$.

Now consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0, s]$. Similarly to the proof of Theorem 1, there exist $x_j \in \delta E_j$, $\gamma_j > 0, j \in I[1, J_0]$ such that $\sum_{j=1}^{J_0} \gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0} \gamma_j x_j$. Let H_0, Q_0, Y_0 be defined as in (23). Then we also have $|H_0 x_0| \le 1$. Applying Proposition 1 at x_0, we have

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \underset{i \in I[1,2^m]}{\mathrm{co}} \left\{ \begin{bmatrix} A_i x_0 + B_i w - B_q T_i H_0 x_0 \\ C_i x_0 + D_i w - D_{zq} T_i H_0 x_0 \end{bmatrix} \right\}.$$

Let

$$\begin{aligned} f_{i0}(x_0, w) &= A_i x_0 + B_i w - B_q T_i H_0 x_0, \\ g_{i0}(x_0, w) &= C_i x_0 + D_i w - D_{zq} T_i H_0 x_0. \end{aligned}$$

Then

$$\begin{aligned} &\dot{V}_c(x_0, w) + \frac{1}{\gamma^2} z^T z - w^T w \le \\ &\max_{i \in I[1,2^m]} \{\nabla^T V_c(x_0) f_{i0}(x_0, w) + \frac{1}{\gamma^2} \|g_{i0}(x_0, w)\|^2 - w^T w\}. \end{aligned} \tag{35}$$

Since $2x_j^T P_j = 2x_0^T Q_0^{-1} = \nabla^T V_c(x_0)$ (see (24)), applying (34) to x_j, we obtain for all $w \in \mathbb{R}^r,\ i \in I[1, 2^m]$,

$$\nabla^T V_c(x_0) f_{ij}(x_j, w) + \frac{1}{\gamma^2} \|g_{ij}(x_j, w)\|^2 - w^T w \le 0. \tag{36}$$

Similar to (32), we have

$$\begin{aligned} f_{i0}(x_0, w) &= \textstyle\sum_{j=1}^{J_0} \gamma_j f_{ij}(x_j, w), \\ g_{i0}(x_0, w) &= \textstyle\sum_{j=1}^{J_0} \gamma_j g_{ij}(x_j, w). \end{aligned} \tag{37}$$

It follows that

$$\nabla^T V_c(x_0) f_{i0}(x_0, w) + \frac{1}{\gamma^2} \|g_{i0}(x_0, w)\|^2 - w^T w \le 0,$$

and from (35)

$$\dot{V}_c(x_0, w) + \frac{1}{\gamma^2} z^T z - w^T w \le 0, \tag{38}$$

which is satisfied for all $x_0 \in sL_{V_c}$ and $w \in \mathbb{R}^r$. Since $x(0) = 0$, $x(t) \in sL_{V_c}$ for all t and for all $\|w\|_2 \le s$, integrating both sides of (38), we have $\|z\|_2^2 \le \gamma^2 \|w\|_2^2$. This completes the proof. □

Proof of Theorem 4. By the definition of V_c, condition (14) implies that $V_c(sH_\ell) \le 1$ for all $\ell \in I[1, m]$. By Lemma 3, this implies that $L_{V_{\max}} \subset \mathcal{L}(sH) = (1/s)\mathcal{L}(H)$, i.e., $sL_{V_{\max}} \subset \mathcal{L}(H)$. Hence $|Hx|_\infty \le 1$ for all $x \in sL_{V_{\max}}$. By Proposition 1, we have for all $x \in sL_{V_{\max}}$,

$$\dot{x} \in \mathrm{co}\left\{A_i x + B_i w - B_q T_i H x :\ i \in I[1, 2^m]\right\}. \tag{39}$$

On the other hand, it can be verified that (15) implies that

$$\begin{aligned} &2x^T P_j (A_i x + B_i w - B_q T_i H x) - w^T w \le \\ &2 \sum_{k=1}^{J} \lambda_{ijk} x^T (P_k - P_j) x,\ \forall j \in I[1, J], i \in I[1, 2^m]. \end{aligned} \tag{40}$$

The state space of x can be partitioned in the following subsets (for all $j \in I[1, J]$):

$$S_j = \{x \in \mathbb{R}^n : x^T (P_k - P_j) x \le 0, k \in I[1, J]\}. \tag{41}$$

If $x \in S_j \backslash \cup_{k \ne j} S_k$, then $V_{\max}(x) = x^T P_j x$ and $\partial V_{\max}(x) = 2P_j x$. If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^{J} S_j$, then $V_{\max}(x) = x^T P_j x, j \in I[1, J_0]$ and $\partial V_{\max}(x) = \mathrm{co}\{2P_j x : j \in I[1, J_0]\}$.

We first consider $x \in S_j \setminus \cup_{k \ne j} S_k$. Then

$$\sum_{k=1}^{J} \lambda_{ijk} x^T (P_k - P_j) x \le 0, \tag{42}$$

and

$$\begin{aligned} &\dot{V}_{\max}(x, w) - w^T w \le \\ &\max_{i \in I[1,2^m]} (2x^T P_j (A_i x + B_i w - B_q T_i H x) - w^T w). \end{aligned} \tag{43}$$

If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^{J} S_j$, then (42) is satisfied for all $j \in I[1, J_0]$ and we have

$$\begin{aligned} &\dot{V}_{\max}(x, w) - w^T w \le \\ &\max_{j \in I[1,J_0]} \max_{i \in I[1,2^m]} (2x^T P_j (A_i x + B_i w - B_q T_i H x) - w^T w). \end{aligned} \tag{44}$$

It follows from (40) and (42) that $\dot{V}_{\max}(x, w) - w^T w \le 0$. The remaining part of the proof is similar to the proof of Theorem 1. □

Proof of Theorem 5. Similarly to the proof of Theorem 4, we have $x(t) \in sL_{V_{\max}}$ for all $t > 0$ under the condition $\|w\|_2 \le s$ and $x(0) = 0$. Also, we have $|Hx|_\infty \le 1$ for all

$x(t) \in sL_{V_{\max}}$. By Proposition 1,

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \text{co}\left\{ \begin{bmatrix} f_i(x,w) \\ g_i(x,w) \end{bmatrix} : \ i \in I[1, 2^m] \right\}, \qquad (45)$$

where

$$\begin{aligned} f_i(x,w) &= A_i x + B_i w - B_q T_i H x, \\ g_i(x,w) &= C_i x + D_i w - D_{yq} T_i H x. \end{aligned}$$

By Schur complement it can be verified that (17) implies

$$\begin{aligned} &2x^T P_j f_i(x,w) + \tfrac{1}{\gamma^2}\|g_i(x,w)\|^2 - w^T w \\ &\leq 2\textstyle\sum_{k=1}^J \lambda_{ijk} x^T (P_k - P_j) x \end{aligned} \qquad (46)$$

for all $j \in I[1,J], i \in I[1,2^m]$. With similar arguments as in the proof of Theorem 4, it can be shown that for all $x \in sL_{V_{\max}}$ and $w \in \mathbb{R}^r$,

$$\dot{V}_{\max}(x,w) + \frac{1}{\gamma^2} z^T z - w^T w \leq 0. \qquad (47)$$

The remaining part of the proof is similar to the proof of Theorem 2. □

V. CONCLUSIONS

In this paper we addressed the stability and performance analysis of linear systems with saturation elements. Two conjugate Lyapunov functions are used to improved the performance analysis results from our companion paper [14]. The arising BMI conditions reduce the conservatism of the LMI conditions of [14]. Numerical experience with low order systems shows that these BMI conditions can be effectively solved with the path following method. Although there is no guarantee that the global optimal solutions will be obtained, the great potential of these non-quadratic Lyapunov functions has been revealed by a numerical example. The effectiveness demonstrated through this example motivates further investigation on these non-quadratic Lyapunov functions and the development of more efficient algorithms to handle them for more complicated situations.

APPENDIX

Proof of Lemma 1. Without loss of generality, consider $j = 1$. Note that

$$V_c(x) = \frac{1}{2} \min_{\gamma_k \geq 0 : \sum_{k=2}^N \gamma_k \leq 1} \left\{ x^{\mathrm{T}} \left(Q_1 + \sum_{j=2}^N \gamma_k (Q_k - Q_1) \right)^{-1} x \right\}.$$

It is implied here that $\gamma_1 = 1 - \sum_{k=2}^N \gamma_k$. For a fixed x, define

$$\phi(\gamma_2, \gamma_3, \cdots, \gamma_N) := \frac{1}{2} x^{\mathrm{T}} \left(Q_1 + \sum_{k=2}^N \gamma_k (Q_k - Q_1) \right)^{-1} x.$$

Then by Schur complement, for any $c > 0$, the set

$$\begin{aligned} &\{(\gamma_2, \gamma_3, \cdots, \gamma_N) : \\ &\phi(\gamma_2, \gamma_3, \cdots, \gamma_N) \leq c, \textstyle\sum_{k=2}^N \gamma_k \leq 1, \gamma_k \geq 0\} \end{aligned} \qquad (48)$$

is convex. Hence the optimal $(\gamma_2, \gamma_3, \cdots, \gamma_N)$'s that minimize ϕ form a convex set.

If $x \in \bigcup_{\delta \in [0,1]} \delta E_1$, then $V_c(x) = \frac{1}{2} x^{\mathrm{T}} Q_1^{-1} x$, implying that the minimal value of ϕ is reached at $(\gamma_2, \gamma_3, \cdots, \gamma_N) = (0, 0, \cdots, 0)$. This means that at this point, $\partial\phi/\partial\gamma_k \geq 0$ for all $k \in I[2,N]$, i.e.,

$$x^{\mathrm{T}} Q_1^{-1} (Q_k - Q_1) Q_1^{-1} x \leq 0 \quad \forall\, k \in [2, N]. \qquad (49)$$

On the other hand, it is also clear that (49) implies that the minimal value of ϕ is reached at $(0, 0, \cdots, 0)$ by the convexity of the set in (48). Hence, (49) is equivalent to $V_c(x) = \frac{1}{2} x^{\mathrm{T}} Q_1^{-1} x$. In summary, we have $\bigcup_{\delta \in [0,1]} \delta E_1 = \{x \in L_{V_c} : \ x^{\mathrm{T}} Q_1^{-1} (Q_k - Q_1) Q_1^{-1} x \leq 0, k \in I[1,N]\}$. □

REFERENCES

[1] F. Blanchini. Nonquadratic lyapunov functions for robust control. *Automatica*, 31:451–461, 1995.

[2] R. K. Brayton and C. H. Tong. Stability of dynamical systems: a constructive approach. *IEEETrans. on Circuits and Systems*, 26:224–234, 1979.

[3] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. Homogeneous lyapunov functions for systems with structured uncertainties. *Automatica*, 39:1027–1035, 2003.

[4] F.H. Clarke. *Optimization and Nonsmooth Analysis*. Society for Industrial an Applied Mathematics, 1990.

[5] W.P. Dayawansa and C.F. Martin. A converse Lyapunov theorem for a class of dynamical systems with undergo switching. *IEEE Trans. Aut. Cont.*, 44(4):751–759, April 1999.

[6] R. Goebel, T. Hu, and A. R. Teel. Dual matrix inequalities in stability and performance analysis of linear differential/difference inclusions. In L. Menini, L. Zaccarian, and C. Abdallah, editors, *Current Trends in Nonlinear Systems and Control*. Birkhauser, to appear, 2005.

[7] R. Goebel, A. R. Teel, T. Hu, and Z. Lin. A collection of robust control problems leading to LMI's. In *IEEE Conference on Decision and Control*, 2004.

[8] A. Hassibi, J. How, and S. Boyd. A path-following method for solving bmi problems in control. In *American Control Conference*, pages 1385–1389, 1999.

[9] T. Hu and Z. Lin. Composite quadratic lyapunov functions for constrained control systems. *IEEE Transactions on Automatic Control*, 48(3):440–450, 2003.

[10] T. Hu and Z. Lin. Properties of convex hull quadratic lyapunov functions. *IEEE Transactions on Automatic Control*, 49(7):1162–1167, 2004.

[11] T. Hu, Z. Lin, and B.M. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38(2):351–359, 2002.

[12] T. Hu, Z. Lin, R. Goebel, and A. R. Teel. Stability regions for saturated linear systems via conjugate lyapunov functions. In *IEEE Conf. on Dec. and Contr*, 2004.

[13] T. Hu, A.R. Teel, and L. Zaccarian. Nonlinear $\mathcal{L}_2$ gain and regional analysis for linear systems with anti-windup compensation. In *American Control Conference*, Portland (OR), USA, June 2005.

[14] T. Hu, A.R. Teel, and L. Zaccarian. Performance analysis of saturated systems via two forms of differential inclusions. In *Joint CDC-ECC*, Seville, Spain, Submitted, December 2005.

[15] Z. Jarvis-Wloszek and A. K. Packard. An LMI method to demonstrate simultaneous stability using non-quadratic polynomial lyapunov functions. In *IEEE Conf. on Decision and Control*, pages 287–292, Las Vegas, 2002.

[16] M. Johansson and A. Rantzer. Computation of piecewise quadratic lyapunov functions for hybrid systems. *IEEE Trans. Aut. Cont.*, 43(4):555–559, April 1998.

[17] A. P. Molchanov. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems & Control Letters*, 13:59–64, 1989.

[18] E.F. Mulder and M.V. Kothare. Synthesis of stabilizing anti-windup controllers using piecewise quadratic Lyapunov functions. In *Proceedings of the American Control Conference*, pages 3239–3243, Chicago (IL), June 2000.

[19] E.F. Mulder, M.V. Kothare, and M. Morari. Multivariable anti-windup controller synthesis using linear matrix inequalities. *Automatica*, 37(9):1407–1416, September 2001.

[20] L. Xie, S. Shishkin, and M. Fu. Piecewise lyapunov functions for robust stability of linear time-varying systems. *Systems & Control Letters*, 31:165–171, 1997.