Nonlinear Control Design for Linear Differential Inclusions via Convex Hull Quadratic Lyapunov Functions

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***Abstract*— This paper presents a nonlinear control design method for robust stabilization and robust performance of linear differential inclusions. A recently introduced non-quadratic Lyapunov function, the convex hull quadratic function, will be used for the construction of nonlinear state feedback laws. Design objectives include stabilization with maximal convergence rate, disturbance rejection with minimal reachable set and least $\mathcal{L}_2$ gain. Conditions for stabilization and performances are derived in terms of bilinear matrix inequalities (BMIs), which cover the existing linear matrix inequality (LMI) conditions as special cases. Optimization problems with BMI constraints are formulated and solved effectively by combining the path-following algorithm and the direct iterative algorithm. The design results are guaranteed to be at least as good as the existing results obtained from LMI conditions. In most examples, significant improvements on system performances have been achieved, which demonstrate the advantages of using nonlinear feedback control over linear feedback control for linear differential inclusions. It is also observed through numerical computation that nonlinear control strategies may help to reduce control effort substantially.**

Keywords: Linear differential inclusion, nonlinear feedback, Lyapunov functions, robust stability, robust performance

I. Introduction

A simple and practical approach to describe systems with nonlinearities and time-varying uncertainties is to use linear differential inclusions (LDIs). Such practice can be traced back to the earlier development (in the 1940s) of absolute stability theory, where a component with these complicated properties was described with a conic sector and the resulting closed-loop system was actually an LDI. The advantages of using LDIs to describe complicated systems are fully demonstrated in [3], where a wide variety of control problems for LDIs are interpreted with linear matrix inequalities (LMIs). The mechanism behind the LMI framework is a systematic application of Lyapunov theory through quadratic functions.

While the LMI technique has been well appreciated and has been widely applied to various control problems, the conservatism introduced by quadratic Lyapunov functions has been revealed in some literature including [3], e.g., in [2], [3], [4], [5], [14]. Considerable efforts have been devoted to the construction and development of non-quadratic Lyapunov functions (e.g., see [4], [14], [15], [17], [19]). In [17], a necessary and sufficient condition for stability of polytopic LDIs was derived as bilinear matrix equations (although it is not clear how these matrix equations can be solved). Numerically tractable stability conditions were derived as LMIs in [4], [14], [19] from piecewise quadratic functions and homogeneous polynomial functions.

Recently, a pair of conjugate Lyapunov functions have demonstrated great potential in stability and performance analysis of LDIs [7], [6], [10]. These functions are composed from more than one positive-definite matrices and are natural extensions of quadratic functions. Through these functions, stability and performances of LDIs are characterized in terms of bilinear matrix inequalities (BMIs) which cover the existing LMI conditions in [3] as special cases. Since extra degrees of freedom for optimization are injected through the bilinear terms, the analysis results are guaranteed to be at least as good as those obtained by corresponding LMI conditions. Extensive examples in [7], [6], [10] have shown that these non-quadratic Lyapunov functions can effectively reduce conservatism in various stability and performance analysis problems, e.g., in the characterization of the convergence rate, the reachable set, the $\mathcal{L}_2$ gain and some dissipativity properties.

With the effectiveness of non-quadratic Lyapunov functions demonstrated on a number of analysis problems, they can further be applied to the construction of feedback laws. Although it is straightforward to derive algorithms for designing *linear* feedback laws based on analysis results on LDIs, the full strength of these non-quadratic functions will be released when they are used as control Lyapunov functions for the construction of *nonlinear* feedback laws. For linear time-invariant systems, it is well known that nonlinear controls have no advantage over linear controls when it comes to stabilization or minimization of the $\mathcal{L}_2$ gain (e.g., see [16]). For systems with time-varying uncertainties and LDIs, it is now accepted that nonlinear control can work better than linear control. In [2], an example was constructed to demonstrate this aspect and it was suggested that non-quadratic Lyapunov functions would facilitate the construction of nonlinear feedback laws. In [1], piecewise linear Lyapunov functions was used for robust stabilization and rejection of bounded persistent disturbances.

In this paper, we use one of the pair of conjugate Lyapunov functions recently developed in [7], [6], [10], the convex hull quadratic function, for the construction of nonlinear state feedback laws. This function was introduced in [11] for the characterization of stability region for constrained control systems. Its level set is the convex hull of a family of ellipsoids. Its conjugate function is called the max (quadratic) function whose level set is the intersection of a family of ellipsoids. Of these two functions, the convex hull function is continuously differentiable whereas the max function is

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not. For this reason, the max function appears to be more cumbersome when applied to the construction of nonlinear feedback laws for continuous-time systems (differential inclusions).

The design objectives to be addressed in this paper include stabilization with maximal convergence rate, disturbance rejection via minimizing the reachable set and the $\mathcal{L}_2$ gain. As will be demonstrated by numerical examples, the nonlinear control design methods can significantly improve robust stability and performances for LDIs.

Notation

- $|\cdot|_\infty$: For $x \in \mathbb{R}^n$, $|x|_\infty := \max_i |x_i|$.
- $\|\cdot\|_2$: For $u \in \mathcal{L}_2$, $\|u\|_2 := \left(\int_0^\infty u^T(t)u(t)dt\right)^{\frac{1}{2}}$.
- $I[k_1, k_2]$: For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1+1, \cdots, k_2\}$.
- $\mathrm{co}\, S$: The convex hull of a set S.
- $\mathcal{E}(P)$: For $P \in \mathbb{R}^{n\times n}, P = P^{\mathrm{T}} > 0$,

$$\mathcal{E}(P) := \{x \in \mathbb{R}^n : \ x^{\mathrm{T}}Px \le 1\}.$$

- L_V: 1-level set of a positive definite function V, $L_V := \{x \in \mathbb{R}^n : V(x) \le 1\}$.
- $\mathcal{L}(H)$: For $H \in \mathbb{R}^{r\times n}$,

$$\mathcal{L}(H) := \left\{x \in \mathbb{R}^n : \ |H_\ell x| \le 1, \ell \in I[1,r]\right\},$$

where H_ℓ is the ℓth row of H. About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(H)$, we have,

$$\mathcal{E}(P) \subseteq \mathcal{L}(H) \Longleftrightarrow H_\ell P^{-1} H_\ell^{\mathrm{T}} \le 1 \quad \forall\, \ell \in I[1,r]. \tag{1}$$

II. Problem statement and preliminaries

A. Problem statement

Consider the following polytopic linear differential inclusion (PLDI),

$$\left[\begin{array}{c} \dot{x} \\ y \end{array}\right] \in \mathrm{co}\left\{\left[\begin{array}{c} A_i x + B_i u + T_i w \\ C_i x + D_i w \end{array}\right] : \ i \in I[1,N]\right\}, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the disturbance and $y \in \mathbb{R}^q$ is the output. A_i, B_i, T_i, C_i and D_i are given real matrices of compatible dimensions. This type of LDI can be used to describe a wide variety of nonlinear systems, possibly with time-varying uncertainties. Control design problems for LDIs via linear state feedback of the form $u = Fx$ have been extensively addressed in [3], where quadratic Lyapunov functions are used as constructive tools and the control problems are transformed into LMIs. While the LMI technique has gained tremendous popularity and its applications are still expanding to different types of systems, such as constrained control systems and time-delay systems, the conservatism resulting from quadratic Lyapunov functions has been recognized and efforts have been devoted to the construction of non-quadratic Lyapunov functions.

The convex hull quadratic Lyapunov function initiated in [11], along with its conjugate function, have shown to be effective in reducing conservatism in stability analysis and evaluation of various performances [7], [6], [10], [13]. In this paper, we use the convex hull quadratic function to construct nonlinear feedback laws to achieve a few objectives of robust stabilization and performance. In particular, we would like to construct a nonlinear state feedback law $u = f(x)$ so that

- the closed-loop system is asymptotically stable in the absence of disturbance and the convergence rate is as fast as possible;
- the state will stay in a small neighborhood of the origin in the presence of a class of $\mathcal{L}_2$ norm bounded disturbances;
- the state will stay in a small neighborhood of the origin in the presence of a class of magnitude bounded disturbances;
- the $\mathcal{L}_2$ gain from w to y is as small as possible.

As will be demonstrated by examples, nonlinear feedback laws may require much less control effort or capacity than linear feedback laws.

B. The convex hull quadratic function

In this section, we give a brief review of the definition and some properties of the convex hull (quadratic) function that will be necessary for the development of the main results. The convex hull function is constructed from a family of positive definite matrices. Given $Q_j \in \mathbb{R}^{n\times n}, Q_j = Q_j^{\mathrm{T}} > 0, j \in I[1,J]$. Let

$$\Gamma := \left\{\gamma \in \mathbb{R}^J : \ \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \ge 0\right\},$$

the convex hull function is defined as

$$V_c(x) := \min_{\gamma\in\Gamma} x^{\mathrm{T}} \left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x. \tag{3}$$

For simplicity, we say that V_c is composed from Q_j's. This function was first used in [11] to study constrained control systems, where it was called the composite quadratic function. We later called it convex hull function in [7], [6], [10] to differentiate it from other functions which are composed from a family of quadratic functions. If we define the 1-level set of V_c as

$$L_{V_c} := \left\{x \in \mathbb{R}^n : \ \ V_c(x) \le 1\right\},$$

and denote the 1-level set of the quadratic function $x^{\mathrm{T}}Px$ as

$$\mathcal{E}(P) := \left\{x \in \mathbb{R}^n : \ \ x^{\mathrm{T}}Px \le 1\right\},$$

then

$$L_{V_c} = \left\{\sum_{j=1}^J \gamma_j x_j : \ x_j \in \mathcal{E}(Q_j^{-1}), \gamma \in \Gamma\right\},$$

which means that L_{V_c} is the convex hull of the family of ellipsoids, $\mathcal{E}(Q_j^{-1}), j \in I[1,J]$.

It is evident that V_c is homogeneous of degree 2, i.e., $V_c(\alpha x) = \alpha^2 V_c(x)$. Also established in [7], [11] is that V_c is convex and continuously differentiable.

For a compact convex set S, a point x on the boundary of S (denoted as ∂S) is called an extreme point if it cannot be represented as the convex combination of any other points in S. A compact convex set is completely determined by its extreme points. In what follows, we characterize the set of extreme points of L_{V_c}. Since L_{V_c} is the convex hull of $\mathcal{E}(Q_j^{-1})$, $j \in I[1,J]$, an extreme point must be on the boundaries of both L_{V_c} and $\mathcal{E}(Q_j^{-1})$ for some $j \in I[1,J]$. Denote

$$\begin{aligned} E_k &:= \partial L_{V_c} \cap \partial \mathcal{E}(Q_k^{-1}) \\ &= \left\{ x \in \mathbb{R}^n : V_c(x) = x^{\mathrm{T}} Q_k^{-1} x = 1 \right\}. \end{aligned}$$

Then $\bigcup_{k=1}^{J} E_k$ contains all the extreme points of L_{V_c}. The exact description of E_k is given as follows.

Lemma 1: [13] For each $k \in I[1,J]$,

$$E_k = \{x \in \partial L_{V_c} : x^{\mathrm{T}} Q_k^{-1}(Q_j - Q_k) Q_k^{-1} x \le 0, j \in I[1,J]\}. \tag{4}$$

For $x \in \mathbb{R}^n$, define

$$\gamma^*(x) := \arg\min_{\gamma \in \Gamma} x^{\mathrm{T}} \left(\sum_{j=1}^{J} \gamma_j Q_j \right)^{-1} x. \tag{5}$$

From the definition, γ^* can be computed by solving a simple LMI problem obtained via Schur complements [11]. As discussed in [12], γ^* is generally uniquely determined by x. It is evident that $\gamma^*(\alpha x) = \gamma^*(x)$.

Detailed properties about γ^* were characterized in [12]. Conditions for $\gamma^*(x)$ to be continuous were provided and numerical results revealed that γ^* is generally continuous except for some degenerated cases. The following lemma combines some results from [11], [12].

Lemma 2: Given $x \in \mathbb{R}^n$. For simplicity and without loss of generality, assume that $\gamma_k^*(x) > 0$ for $k \in I[1,J_0]$ and $\gamma_k^*(x) = 0$ for $k \in I[J_0+1,J]$. Denote

$$Q(\gamma^*) = \sum_{k=1}^{J_0} \gamma_k^* Q_k, \quad x_k = Q_k Q(\gamma^*)^{-1} x, \quad k \in I[1,J_0].$$

Then $V_c(x_k) = V_c(x) = x_k^{\mathrm{T}} Q_k^{-1} x_k$. Hence $x_k \in (V_c(x))^{\frac{1}{2}} E_k, k \in I[1,J_0]$. Moreover,

$$x = \sum_{k=1}^{J_0} \gamma_k^* x_k, \tag{6}$$

and for all $k \in I[1,J_0]$,

$$\nabla V_c(x) = \nabla V_c(x_k) = 2 Q_k^{-1} x_k = 2 Q(\gamma^*)^{-1} x, \tag{7}$$

where $\nabla V_c(x)$ denotes the gradient of V_c at x.

Since $\gamma^*(\alpha x) = \gamma^*(x)$, by (7), we have $\nabla V(\alpha x) = \alpha \nabla V(x)$. Since V_c is homogeneous of degree two, to obtain some geometric interpretation of Lemma 2, we may restrict our attention to a point $x \in \partial L_{V_c}$. Then by the lemma, x can always be expressed as a convex combination of a family of x_k's, $x_k \in \partial \mathcal{E}(Q_k^{-1})$ (note $x_k \in E_k$). Furthermore, the gradient of V_c at these x_k's are the same and they all equal to the gradient of V_c at x. In other words, x and x_k's are in the same hyperplane which is tangential to L_{V_c}. In fact, the intersection of the hyperplane with L_{V_c} is a polygon whose vertices include x_k's (see [12]).

III. **Nonlinear feedback law for robust stabilization**

In the absence of disturbance, the LDI (2) reduces to,

$$\dot{x} \in \mathrm{co}\{A_i x + B_i u : \ i \in I[1,N]\}. \tag{8}$$

For stability design, we only consider the state inclusion. We would like to construct a nonlinear state feedback law to achieve robust stabilization via the convex hull quadratic function $V_c(x)$. The main result is given as follows.

Theorem 1: Consider V_c composed from $Q_k \in \mathbb{R}^{n \times n}$, $Q_k = Q_k^{\mathrm{T}} > 0, k \in I[1,J]$. If there exist $\beta > 0$, $Y_k \in \mathbb{R}^{m \times n}, k \in I[1,J]$ and $\lambda_{ijk} \ge 0$, $i \in I[1,N], j,k \in I[1,J]$ such that

$$\begin{aligned} & Q_k A_i^{\mathrm{T}} + A_i Q_k + Y_k^{\mathrm{T}} B_i^{\mathrm{T}} + B_i Y_k \\ & \qquad \le \sum_{j=1}^{J} \lambda_{ijk}(Q_j - Q_k) - \beta Q_k \quad \forall\, i,\, k. \end{aligned} \tag{9}$$

Then a stabilizing nonlinear feedback law can be constructed as follows. For each $x \in \mathbb{R}^n$, let $\gamma^*(x) \in \Gamma$ be defined as in (5). Let

$$Y(\gamma^*) = \sum_{k=1}^{J} \gamma_k^* Y_k, \quad Q(\gamma^*) = \sum_{k=1}^{J} \gamma_k^* Q_k, \tag{10}$$

$$F(\gamma^*) = Y(\gamma^*) Q(\gamma^*)^{-1}. \tag{11}$$

Define $f(x) = F(\gamma^*(x))x$. Then for all $x \in \mathbb{R}^n$, we have

$$\max\{\nabla V_c(x)^{\mathrm{T}}(A_i x + B_i f(x)) : i \in I[1,N]\} \le -\beta V_c(x), \tag{12}$$

which implies that the closed-loop system under $u = f(x)$ is stable. If the vector function $\gamma^*(x)$ is continuous in x, then $u = f(x)$ is a continuous feedback law. $\diamond$

Since $\gamma^*(\alpha x) = \gamma^*(x)$, we have $f(\alpha x) = \alpha f(x)$ and the resulting closed-loop system is homogeneous of degree one.

When the inequality (12) is satisfied, $V_c(x(t))$ is strictly decreasing and we have $V_c(x(t)) \le V_c(x(0)) e^{-\beta t}$. Hence β is a measure of convergence rate. Moreover, a trajectory starting from the boundary of a level set will go to boundaries of smaller and smaller level sets. Suppose $x_0 \in \alpha_0 \partial L_{V_c}$. Then $x(t) \in \alpha(t) \partial L_{V_c}$ with $\alpha(t)$ strictly decreasing. In this case, we say that the level sets are contractively invariant.

To increase the convergence rate, an optimization problem can be formulated to maximize β as follows:

$$\sup_{\lambda_{ijk} \ge 0, Q_k = Q_k^{\mathrm{T}} > 0, Y_k} \beta \quad \text{s.t. } (9). \tag{13}$$

The constraint (9) consists of a family of bilinear matrix inequalities (BMIs) which contain some bilinear terms as the product of a full matrix and a scalar, i.e., $\lambda_{ijk}(Q_j - Q_k)$. We implemented a two-step iterative algorithm which combines the path-following method in [9] and the direct iterative method. The first step of each iteration uses the path-following method to update all the parameters at the same time. The second step fixes λ_{ijk}'s and solves the resulting LMI problem which includes Q_j's and Y_j's as variables. This two-step iterative method proves very effective on the BMI problems in [6], [7], [10], [13] and also works well on the examples in Section V.

IV. **Nonlinear feedback law for robust performance**

Consider the linear differential inclusion (2) in the presence of disturbances. Like in [3], we consider two types of disturbances, the unit peak disturbances

$$w^{\mathrm{T}}(t)w(t) \le 1 \qquad \forall t \ge 0 \tag{14}$$

and the unit energy disturbances

$$\|w\|_2 = \left(\int_0^\infty w^{\mathrm{T}}(t)w(t)dt\right)^{\frac{1}{2}} \le 1. \tag{15}$$

Let $u = f(x)$ be a nonlinear state feedback. The closed-loop system is

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix} A_i x + B_i f(x) + T_i w \\ C_i x + D_i w \end{bmatrix} : \ i \in I[1,N]\right\}. \tag{16}$$

The control design objective is disturbance rejection, i.e., to keep the state close to the origin or to keep the size of the output (in terms of certain norm) small in the presence of a class of disturbances. The disturbance rejection performance can be characterized by reachable set or the maximal output norm. When the disturbance is of unit peak type, the maximal output norm is associated with the L_∞ gain; when the disturbance is of unit energy, the maximal output norm is associated with the $L_2 - L_\infty$ gain or the L_2 gain. We first consider the reachable set.

A. Suppression of the reachable set

The reachable set can be estimated with a level set of a certain Lyapunov function. In [3], quadratic Lyapunov functions are considered for linear differential inclusions and the reachable set is estimated with ellipsoids through solving some LMI problems. In this section, we use the convex hull of a family of ellipsoids to characterize the reachable set and we attempt to reduce the reachable set by nonlinear feedback laws.

1) Reachable set with finite energy disturbances:

Theorem 2: Consider V_c composed from $Q_k \in \mathbb{R}^{n\times n}$, $Q_k = Q_k^{\mathrm{T}} > 0, k \in I[1,J]$. Suppose that there exist $Y_k \in \mathbb{R}^{m\times n}, k \in I[1,J]$ and $\lambda_{ijk} \ge 0,\ i \in I[1,N], j,k \in I[1,J]$ such that

$$\begin{bmatrix} M_{ik} & T_i \\ T_i^{\mathrm{T}} & -I \end{bmatrix} \le 0 \quad \forall i,k, \tag{17}$$

where

$$M_{ik} = Q_k A_i^{\mathrm{T}} + A_i Q_k + Y_k^{\mathrm{T}} B_i^{\mathrm{T}} + B_i Y_k - \sum_{j=1}^{J} \lambda_{ijk}(Q_j - Q_k). \tag{18}$$

Let the nonlinear feedback $u = f(x) = F(\gamma^*(x))x$ be constructed from Y_k's and Q_k's as in (10) and (11). Then for all w bounded by $\|w\|_2 \le 1$ and with $x_0 = 0$, the state of (16) satisfies $x(t) \in L_{V_c}$ for all $t \ge 0$. ⋄

Under the condition of Theorem 2, the level set L_{V_c} includes the actual reachable set and can be considered as an estimate for the reachable set. To keep the state in a small neighborhood of the origin, it is desirable that L_{V_c} satisfying the condition is as small as possible. We may use a reference polytope to measure the size of L_{V_c}. The polytope is described in terms of a matrix $H \in \mathbb{R}^{r\times n}$ as follows,

$$\mathcal{L}(H) := \{x \in \mathbb{R}^n : |H_\ell x| \le 1, \ell \in I[1,r]\}.$$

The "outer" size of L_{V_c} is defined as

$$\alpha_{\mathrm{out}} := \min\{\alpha : L_{V_c} \subset \alpha\mathcal{L}(H)\}. \tag{19}$$

The matrix H can be chosen such that $H_\ell x$ is a certain quantity that we would like to keep small. If we have $L_{V_c} \subset \alpha\mathcal{L}(H)$, then $|H_\ell x(t)| \le \alpha$ for all t in the presence of the class of disturbances. Since $\mathcal{L}(H)$ is a convex set and L_{V_c} is the convex hull of the ellipsoids $\mathcal{E}(Q_k^{-1})$, it is easy to see that $L_{V_c} \subset \alpha\mathcal{L}(H) = \mathcal{L}(H/\alpha)$ if and only if $\mathcal{E}(Q_k^{-1}) \subset \mathcal{L}(H/\alpha)$ for all k. By (1), this is equivalent to

$$H_\ell Q_k H_\ell^{\mathrm{T}} \le \alpha^2 \quad \forall \ell \in I[1,r], k \in I[1,J]. \tag{20}$$

In view of the above arguments, the problem of reducing the reachable set can be formulated as

$$\inf_{\lambda_{ijk}\ge 0, Q_k = Q_k^{\mathrm{T}} > 0, Y_k} \alpha \quad \text{s.t. } (17), (20). \tag{21}$$

2) Reachable set with unit peak disturbances:

Theorem 3: Consider V_c composed from $Q_k \in \mathbb{R}^{n\times n}$, $Q_k = Q_k^{\mathrm{T}} > 0, k \in I[1,J]$. Suppose that there exist $Y_k \in \mathbb{R}^{m\times n}, k \in I[1,J]$, $\lambda_{ijk} \ge 0$, $i \in I[1,N], j,k \in I[1,J]$ and $\beta > 0$ such that

$$\begin{bmatrix} M_{ik} + \beta Q_k & T_i \\ T_i^{\mathrm{T}} & -\beta I \end{bmatrix} \le 0 \quad \forall i,k, \tag{22}$$

where M_{ik} is given by (18). Let the nonlinear feedback $u = f(x) = F(\gamma^*(x))x$ be constructed from Y_k's and Q_k's as in (10) and (11). Then L_{V_c} is an invariant set, which means that all trajectories starting from L_{V_c} will stay inside for any possible disturbance satisfying $w(t)^{\mathrm{T}}w(t) \le 1, \forall t \ge 0$. Moreover, for all $x_0 \in \mathbb{R}^n$ and all possible disturbances satisfying the constraint, $x(t)$ will converge to L_{V_c}. ⋄

Similarly to the unit energy disturbance case, we can formulate the following optimization problem for the purpose of minimizing the reachable set or to minimize the maximal output norm,

$$\inf_{\lambda_{ijk},\beta\ge 0, Q_k = Q_k^{\mathrm{T}} > 0, Y_k} \alpha \quad \text{s.t. } (22), (20). \tag{23}$$

B. Suppression of the $\mathcal{L}_2$ gain

The main result is stated in the following theorem:

Theorem 4: Given $Q_k \in \mathbb{R}^{n\times n}, Q_k = Q_k^{\mathrm{T}} > 0,\ k \in I[1,J]$. Let $\delta > 0$. Suppose that there exist $Y_k \in \mathbb{R}^{m\times n}, k \in I[1,J]$ and $\lambda_{ijk} \ge 0,\ i \in I[1,N], j,k \in I[1,J]$ such that

$$\begin{bmatrix} M_{ik} & T_i & Q_k C_i^{\mathrm{T}} \\ T_i^{\mathrm{T}} & -I & D_i^{\mathrm{T}} \\ C_i Q_k & D_i & -\delta^2 I \end{bmatrix} \le 0, \quad \forall\, i,k, \tag{24}$$

where

$$M_{ik} = Q_k A_i^{\mathrm{T}} + A_i Q_k + Y_k^{\mathrm{T}} B_i^{\mathrm{T}} + B_i Y_k - \sum_{j=1}^{J} \lambda_{ijk}(Q_j - Q_k).$$

Let the nonlinear feedback $u = f(x) = F(\gamma^*(x))x$ be constructed from Y_k's and Q_k's as in (10) and (11). Then for system (16) with $x_0 = 0$, we have $\|y\|_2 \le \delta\|w\|_2$. ⋄

By Theorem 4, the quantity δ gives an upper bound for the $\mathcal{L}_2$ gain. The following optimization problem can be formulated for suppression of the $\mathcal{L}_2$ gain:

$$\inf_{\lambda_{ijk}\geq 0, Q_k=Q_k^{\mathrm{T}}, Y_k} \delta \quad \text{s.t. } (24). \tag{25}$$

V. Numerical Examples

Example 1: Consider a second-order LDI with two vertices,

$$\dot{x} \in \mathrm{co}\{A_1x + B_1u, A_2x + B_2u\},$$

where

$$A_1 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}.$$

There exists no linear feedback such that the closed-loop system is quadratically stable. We try to maximize the convergence rate β via nonlinear feedback by solving (13) with $J = 2$. In this case, L_{V_c} is the convex hull of two ellipsoids. The maximal β that we have obtained is $0.4260 > 0$, which shows that the system can be stabilized via nonlinear feedback law. To achieve this convergence rate, the spectral norm of the feedback gain F_1 is $\|F_1\|_2 > 1.4 \times 10^4$ and the matrix Q_1 is not well conditioned. From our simulation experience, high feedback gains F_j's and badly conditioned Q_j's usually require very small sampling period, which makes digital implementation difficult. To avoid this situation, we can impose some additional constraints on the range of Q_j's. For example, we can pick $q_{\min}, q_{\max} > 0$ and impose the constraint $q_{\min}I \leq Q_j \leq q_{\max}I$ for each $j \in I[1, N]$. It turns out that this constraint also helps to limit the magnitude of F_j's. By picking $q_{\min}$ and $q_{\max}$ properly, we are able to restrict F_j's within the range $\|F_j\|_2 \leq 600$:

$$F_1 = \begin{bmatrix} -511.50 & -313.62 \end{bmatrix}, F_2 = \begin{bmatrix} -26.77 & -8.25 \end{bmatrix}.$$

Other parameters resulting from this additional restriction are

$$Q_1 = \begin{bmatrix} 0.3114 & -0.4635 \\ -0.4635 & 0.7286 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.1015 & -0.1289 \\ -0.1289 & 0.3468 \end{bmatrix},$$

and the maximal β we obtained is $\beta = 0.1334$. With these parameters, the nonlinear feedback law is

$$u = f(x) = (\gamma_1^*(x)Y_1 + \gamma_2^*Y_2)(\gamma_1^*(x)Q_1 + \gamma_2^*(x)Q_2)^{-1}.$$

where $Y_i = F_iQ_i$, $i = 1, 2$. Simulation is carried out for the closed-loop system

$$\dot{x} \in \mathrm{co}\{A_1x + B_1f(x), A_2x + B_2f(x)\}$$

under initial condition $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The switching between $\dot{x} = A_1x + B_1f(x)$ and $\dot{x} = A_2x + B_2f(x)$ is chosen so that $\dot{V}_c = (\nabla V_c(x))^{\mathrm{T}}\dot{x}$ is maximized at each time instant (This is intended to approximate the worst situation). The trajectory is plotted in Fig. 1, where the closed curve is the boundary of the level set that includes x_0.

Example 2: An LDI subject to disturbance is described as

$$\dot{x} \in \mathrm{co}\{A_1x + B_1u + Ew, A_2x + B_2u + Ew\}, \quad y = Cx, \tag{26}$$

where

$$A_1 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},$$

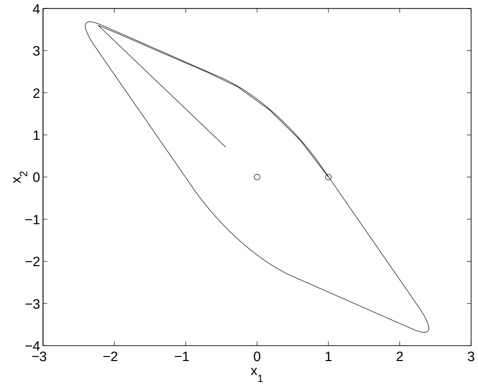

Fig. 1. A trajectory and a level set

$$A_2 = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.5 \\ -0.8 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

This LDI is quadratically stabilizable through linear state feedback. We would like to design a feedback law to suppress the $\mathcal{L}_2$ gain.

When quadratic Lyapunov function is applied to designing a linear state feedback law, the optimization problem is a special case of (25) with $J = 1$. In this case (25) reduces to an LMI problem. The optimal δ for this case is $\delta_1 = 10.7670$. When the optimal δ is approached, the norm of the feedback gain will approach infinity. If we restrict the norm of the feedback gain to be less than 5000 (via the additional constraint on Q_j's, as in Example 1), the optimal δ is $\bar{\delta}_1 = 11.8886$. The feedback gain is

$$F = \begin{bmatrix} -4.2509 & -2.6324 \end{bmatrix} \times 1000, \tag{27}$$

Next we apply the convex hull function $V_c(x)$ with $J = 2$ to the design of a nonlinear feedback law. By solving (25) with $J = 2$, the minimal δ we have obtained is $\delta_2 = 1.1947$, which is much less than δ_1. If we restrict the norm of F_k to be less than 1000, then the best δ we have computed is $\bar{\delta}_2 = 1.8477$. Other variables corresponding to this value of $\delta = \bar{\delta}_2$ are

$$F_1 = \begin{bmatrix} -815.05 & -579.43 \end{bmatrix}, F_2 = \begin{bmatrix} -58.73 & -28.49 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 15.6863 & -20.5376 \\ -20.5376 & 27.9733 \end{bmatrix}, Q_2 = \begin{bmatrix} 5.8599 & -8.3901 \\ -8.3901 & 15.8879 \end{bmatrix}.$$

Here we compare the output responses (with $x(0) = 0$) for the two designs under the disturbance $w(t) = 1$, for $t \in [0, 1]$ and 0 elsewhere. We have $\|w\|_2 = 1$. The switching between $\dot{x} = A_1x + B_1f(x) + Ew$ and $\dot{x} = A_2x + B_2f(x) + Ew$ is chosen such that $\dot{V}_c$ is maximized at each time instant. The two time responses are compared in Fig. 2, where the dashed curve is produced by the linear state feedback with gain in (27) and the solid curve is produced by the nonlinear feedback constructed from Q_1, Q_2 and $Y_1 = F_1Q_1, Y_2 = F_2Q_2$. For the dashed curve, we have $(\int_0^{50} y^2(t)dt)^{\frac{1}{2}} = 2.6858$, and for the solid curve, we have $(\int_0^{15} y^2(t)dt)^{\frac{1}{2}} = 0.7984$.

Example 3: Consider the same LDI as in Example 2. Assume that the disturbance is of unit peak type, i.e.,

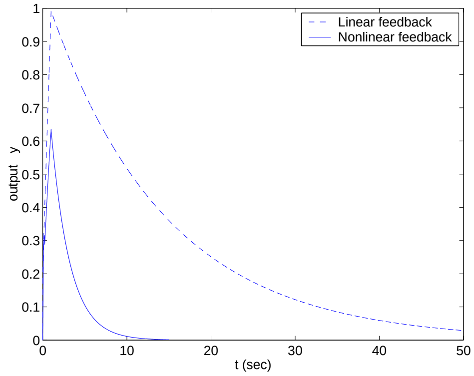

Fig. 2. Two output responses under unit energy disturbances

$w^{\mathrm{T}}(t)w(t) \le 1$ for all t. We would like to design a control law such that the peak of the output is suppressed. This is achieved by solving (23). When $J = 1$, V_c is a quadratic function and the resulting control law is linear. If no restriction on the magnitude of the feedback matrix is imposed, the optimal α is 11.9529. If a bound on the norm of F_k is imposed, say, $\|F_k\| \le 5000$, we obtain $\alpha = 12.8287 := \alpha_1$ and

$$F = \begin{bmatrix} -4.2244 & -2.6093 \end{bmatrix} \times 10^3.$$

For $J = 2$, we impose a bound $\|F_k\| \le 1000$ and the best α is $2.4573 := \alpha_2$. Other parameters for the resulting nonlinear feedback law are

$$F_1 = \begin{bmatrix} -813.88 & -577.47 \end{bmatrix}, F_2 = \begin{bmatrix} -62.46 & -30.39 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} 18.1203 & -23.4665 \\ -23.4665 & 31.8746 \end{bmatrix}, Q_2 = \begin{bmatrix} 6.8899 & -9.9904 \\ -9.9904 & 19.1287 \end{bmatrix}.$$

In simulation, the switching strategy and w are chosen such that $\dot{V}_c$ is maximized at each time instant to approximate the worst situation. The two output responses corresponding to the worst w and the worst switching are plotted in Fig. 3. It can be seen that the magnitude of the output is substantially suppressed by the nonlinear feedback control.

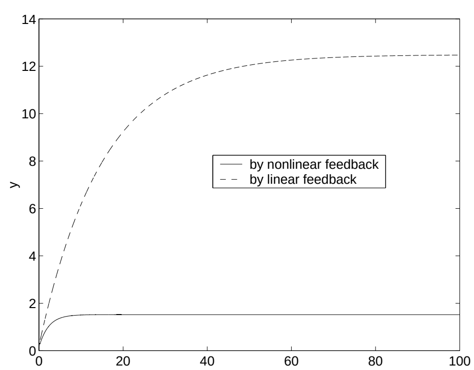

Fig. 3. Two output responses under unit peak disturbances

VI. **Conclusions**

We developed LMI-based methods for the construction of nonlinear feedback laws for linear differential inclusions. The convex hull quadratic Lyapunov functions are used to guide the design for achieving a few objectives of robust stabilization and performance. The advantages of nonlinear feedback over linear feedback has been demonstrated through some numerical examples. It is expected that the design methods can be extended to deal with other performances, such as the input-to-state, input-to-output and state-to-output performances studied in [3]. The max quadratic Lyapunov function may also be suitable for the design of nonlinear control if its non-differentiability can be handled properly. Other non-quadratic Lyapunov functions, such as the homogeneous polynomial function, may also be explored for the design of nonlinear control laws.

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