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Automatica 41 (2005) 1949–1956
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Brief paper

# Conjugate Lyapunov functions for saturated linear systems☆

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Received 11 October 2004; received in revised form 18 February 2005; accepted 12 May 2005
Available online 24 August 2005

**Abstract**

Based on a recent duality theory for linear differential inclusions (LDIs), the condition for stability of an LDI in terms of one Lyapunov function can be easily derived from that in terms of its conjugate function. This paper uses a particular pair of conjugate functions, the convex hull of quadratics and the maximum of quadratics, for the purpose of estimating the domain of attraction for systems with saturation nonlinearities. To this end, the nonlinear system is locally transformed into a parametertized LDI system with an effective approach which enables optimization on the parameter of the LDI along with the optimization of the Lyapunov functions. The optimization problems are derived for both the convex hull and the max functions, and the domain of attraction is estimated with both the convex hull of ellipsoids and the intersection of ellipsoids. A numerical example demonstrates the effectiveness of this paper's methods.


*Keywords:* Saturation; Domain of attraction; Linear differential inclusions; Conjugate Lyapunov functions; Duality

## 1. Introduction

One practical way to study nonlinear systems, and also hybrid, switched, or uncertain time-varying linear ones, is to obtain an approximate description of a given system in terms of linear differential/difference inclusions (LDIs). Such a practice can be traced back to the early development of the absolute stability theory, where the nonlinearity and uncertainties were described with conic sectors and the systems were treated with tools adapted from those for linear systems (see, e.g., Aizerman & Gantmacher, 1964; Narendra & Taylor, 1973). The effectiveness of the LDI approach depends on two factors: how close the LDI approximation is, and what tools are used for analyzing it. One of the tools is a common Lyapunov function for all the member systems of the LDI. In the early development, the search for such a function was often restricted to quadratics. In fact, the circle criterion for absolute stability gives a necessary and sufficient condition for the existence of a common quadratic Lyapunov function for all convex combinations of two linear systems. The theory based on quadratic Lyapunov functions for LDIs was completed by the LMI optimization technique (see, e.g., Boyd, El Ghaoui, Feron, & Balakrishnan, 1994).

While the search for a common Lyapunov function has been justified (e.g., Dayawansa & Martin, 1999; Molchanov, 1989), evidence has accumulated to indicate that the search should be widened beyond quadratic forms (see, e.g., Chesi, Garulli, Tesi, & Vicino, 2003; Dayawansa & Martin, 1999; Jarvis-Wloszek & Packard, 2002; Power & Tsoi, 1973; Zelentsovsky, 1995). Recent years have witnessed an extensive search for nonquadratic and/or homogeneous Lyapunov functions, among which are piecewise quadratic functions

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Faryar Jabbari under the direction of Editor Roberto Tempo. Work supported in part by NSF under grant ECS-0324679, AFOSR under grant F49620-03-01-0203, and NSF under grant CMS-0324329.

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doi:10.1016/j.automatica.2005.05.021

(Johansson & Rantzer, 1998; Xie, Shishkin, & Fu, 1997), polyhedral functions (Blanchini, 1995; Brayton & Tong, 1979), and homogeneous polynomial Lyapunov functions (HPLFs) (Chesi et al., 2003; Jarvis-Wloszek & Packard, 2002; Zelentsovsky, 1995).

An important contribution was made in Goebel, Teel, Hu, and Lin (2005), making available in the LDI framework tools based on duality, similar to those that are well-appreciated in linear systems. The exponential stability of an LDI was shown to be equivalent to that of its dual LDI (which is given by transposes of the matrices describing the original one, see also Barabanov, 1995). Such results are derived by using convex Lyapunov/storage functions, and their conjugates in the sense of convex analysis. For example, if $V(x)$ is a Lyapunov function for a given LDI, then its conjugate $V^*(x)$ is a Lyapunov function for the dual LDI. As is demonstrated by examples in Goebel, Teel, Hu, and Lin (2005) and Goebel, Hu, and Teel (2005), numerical results based on one type of Lyapunov functions can be strikingly different from those based on the conjugate type. This is one of the strengths of the duality theory—it doubles the number of tools and presents different results for one to choose. In this paper, we use a particular pair of conjugate functions: the convex hull of a family of quadratic functions $V_c(x)$ and the pointwise maximum of a family of quadratics $V_c^*(x)$. In what follows, we will often refer to $V_c(x)$ as the convex hull function, and to $V_c^*(x)$ as the max function.

The functions $V_c(x)$, $V_c^*(x)$ and their conjugacy are used here to study stability of a special type of nonlinear systems—systems with saturation nonlinearities. The stability analysis of such systems, as with many other nonlinear systems, has been mostly performed through the LDI framework. As we have stated, the effectiveness of such approach depends on how the nonlinear system is represented by an LDI. The straightforward way, which has been adopted by most of the literature, is to bound each saturation function locally with a conic sector, i.e., to find a $k \in (0, 1)$ such that $\mathrm{sat}(f_i x) \in \mathrm{co}\{f_i x, k f_i x\}$. This approach was shown to be awkward and was replaced with a more effective and flexible one in Hu, Lin, and Chen(2002a,2002b), where an auxiliary feedback matrix $H$ was constructed such that $\mathrm{sat}(f_i x) \in \mathrm{co}\{f_i x, h_i x\}$ in the local region of interest. In the new approach, the parameter of the LDI description (the matrix $H$), is subject to optimization. This optimization can be tightly integrated with that of the Lyapunov function, to form a single optimization problem with the objective of maximizing the estimated domain of attraction. In Hu et al. (2002a, 2002b), level sets of quadratic functions (ellipsoids) were used as estimates. An effort was made in Hu and Lin (2003) to use level sets of other types of Lyapunov functions, in particular, of the convex hull function (in Hu et al., 2003, the convex hull function was referred to as the composite quadratic function). Level sets of the convex hull function are convex hulls of ellipsoids. The results of Hu and Lin (2003) improve on those of Hu et al. (2002a) but are still based on the concept of quadratic stability: each individual ellipsoid has to be an invariant set. With the duality theory in Goebel et al. (2005), a much weaker condition for stability will be derived.

This paper integrates the LDI description from Hu et al. (2002a), the convex hull function from Hu and Lin (2003), and the duality theory of Goebel, Teel, Hu, and Lin (2005) to perform stability analysis on systems with saturation nonlinearities. Section 2 presents preliminaries on conjugate Lyapunov functions. Section 3 derives conditions for stability of saturated systems by using the conjugate pair $V_c(x)$ and $V_c^*(x)$. Improvement on the estimation of the domain of attraction is observed in Section 4.

**Notation.** For two integers $k_1, k_2, k_1 < k_2$, we denote $I[k_1, k_2] = \{k_1, k_1 + 1, ..., k_2\}$. We use $\mathrm{sat}(\cdot)$ to denote the standard saturation function, i.e., for $u \in \mathbb{R}^m$, the $i$th component of $\mathrm{sat}(u)$ is $\mathrm{sign}(u_i)\min\{1, |u_i|\}$. For a matrix $H \in \mathbb{R}^{m\times n}$,

$$\mathscr{L}(H) := \{x \in \mathbb{R}^n : \|Hx\|_\infty \leqslant 1\}. \tag{1}$$

For a positive-definite matrix $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$,

$$\mathscr{E}(Q) := \{x \in \mathbb{R}^n \ : \ (1/2)x^{\mathrm{T}}Qx \leqslant 1\}. \tag{2}$$

## 2. Preliminaries

### 2.1. Duality in linear differential inclusions

Let $\Omega$ be a compact subset of $\mathbb{R}^{n\times n}$. Consider the following LDI:

$$\dot{x} \in \Omega x, \quad x(0) = x_0. \tag{3}$$

Solutions of (3) are $x(\cdot)$'s satisfying $\dot{x}(t) = A(t)x(t)$ and $x(0) = x_0$, where $A(t) \in \Omega$, for all $t \geqslant 0$. For LDIs, asymptotic stability (at 0) and exponential stability (at 0) are equivalent to each other and each implies global uniform exponential stability (e.g., Dayawansa & Martin, 1999). Thus, we simply use exponential stability to represent all these notions. The LDI is said to be exponentially stable if there exist $K > 0, \beta > 0$ such that the solutions satisfy

$$\|x(t)\| \leqslant K\|x_0\|\mathrm{e}^{-\beta t} \quad \forall x_0 \in \mathbb{R}^n, \ \ t \geqslant 0.$$

The number $\beta$ is called the decay rate of the LDI. It is well-known that exponential stability for (3) is equivalent to exponential stability for

$$\dot{x} \in \mathrm{co}\{\Omega\}x, \tag{4}$$

where $\mathrm{co}\{\Omega\}$ is the convex hull of $\Omega$. In Barabanov (1995) and Goebel et al. (2005), it is shown that exponential stability for (3) is equivalent to exponential stability for

$$\dot{x} \in \Omega^{\mathrm{T}}x, \tag{5}$$

where $\Omega^{\mathrm{T}} = \{A^{\mathrm{T}} : A \in \Omega\}$. In Goebel et al. (2005), this result is established through conjugate Lyapunov functions.

Given any function $V : \mathbb{R}^n \to \mathbb{R}$, its conjugate, in the sense of convex analysis, is defined as

$$V^*(z) := \sup_x \{z^{\mathrm{T}} x - V(x)\}.$$

The conjugate is always a convex function. If $V$ is convex, then the conjugate of $V^*$ is the original $V$, and the positive definiteness and positive homogeneity of degree 2 (i.e., $V(\alpha x) = \alpha^2 V(x)$ for all $\alpha > 0$ and $x \in \mathbb{R}^n$) of $V$ is equivalent to those properties of $V^*$. Furthermore, for such functions, the differentiability of $V$ is equivalent to the strict convexity of $V^*$. Below, $\partial V$ and $\partial V^*$ stand for the subdifferentials, in the sense of convex analysis, of $V$ and $V^*$. At points where $V$ is differentiable, $\partial V$ is single valued and equal to the gradient. When we say that $\partial V(x)^{\mathrm{T}} f \leqslant c$, we mean $\xi^{\mathrm{T}} f \leqslant c$ for all $\xi \in \partial V(x)$.

**Theorem 1** (*Goebel, Teel, Hu, and Lin (2005)*). *Let $V : \mathbb{R}^n \to \mathbb{R}$ be convex, positive definite, and positively homogeneous of degree* 2. *Then $V^* : \mathbb{R}^n \to \mathbb{R}$ is convex, positive definite, and positively homogeneous of degree* 2, *and*

$$\partial V(x)^{\mathrm{T}} f \leqslant -\beta V(x) \quad \forall x \in \mathbb{R}^n, \quad f \in \Omega x$$

*if and only if*

$$\partial V^*(z)^{\mathrm{T}} g \leqslant -\beta V^*(z) \quad \forall z \in \mathbb{R}^n, \quad g \in \Omega^{\mathrm{T}} z.$$

Theorem 1 along with results of Dayawansa and Martin (1999) and Molchanov (1989) yields:

**Theorem 2.** *Let $\Omega$ be compact. Then the following statements are equivalent*:

(1) *The origin of system* (3) *is exponentially stable with a decay rate $\beta$.*
(2) *The origin of system* (4) *is exponentially stable with a decay rate $\beta$.*
(3) *The origin of system* (5) *is exponentially stable with a decay rate $\beta$.*
(4) *There exists a convex positive definite function $V(x)$ that is differentiable and positively homogeneous of degree* 2 *such that*

$$\partial V(x)^{\mathrm{T}} f \leqslant -2\beta V(x) \quad \forall x \in \mathbb{R}^n, \quad f \in \Omega x.$$

### 2.2. Convex hull of quadratic functions and its conjugate

Consider $N$ symmetric and positive definite matrices $Q_j$, $j \in I[1, N]$. Let

$$\Gamma = \{\gamma \in \mathbb{R}^N : \gamma_1 + \gamma_2 + \cdots + \gamma_N = 1, \gamma_j \geqslant 0\}.$$

The composite quadratic function was defined in Hu and Lin (2003) as

$$V_{\mathrm{c}}(x) := \frac{1}{2} \min_{\gamma \in \Gamma} x^{\mathrm{T}} \left( \sum_{j=1}^{N} \gamma_j Q_j \right)^{-1} x. \tag{6}$$

Relevant properties of this function are as follows:

(1) $V_{\mathrm{c}}$ is convex, positive definite, positively homogeneous of degree 2, and continuously differentiable.
(2) The level sets of $V_{\mathrm{c}}$ are convex hulls of unions of level sets of quadratics $\frac{1}{2} x^{\mathrm{T}} Q_j^{-1} x$. In particular, for

$$L_{V_{\mathrm{c}}} := \{x \in \mathbb{R}^n : V_{\mathrm{c}}(x) \leqslant 1\}$$

we have $L_{V_{\mathrm{c}}} = \mathrm{co}\{\mathscr{E}(Q_j^{-1}) : j \in I[1, N]\}$.

Alternatively, one can describe $V_{\mathrm{c}}$ as the convex hull function of $\frac{1}{2} x^{\mathrm{T}} Q_j^{-1} x$, $j \in I[1, N]$ (i.e. the greatest convex function majorized by each of these quadratics); see Goebel et al. (2005). For this reason, we call $V_{\mathrm{c}}$ the convex hull of quadratic functions, or simply, the convex hull function. From Goebel et al. (2005), the conjugate function of $V_{\mathrm{c}}$ is

$$V_{\mathrm{c}}^*(\xi) := \frac{1}{2} \max_{j \in I[1,N]} \xi^{\mathrm{T}} Q_j \xi. \tag{7}$$

Since this function is obtained by taking the pointwise maximum of a family of quadratic functions, it is called the maximum of quadratic functions, or simply, the max function. It has the following properties:

(1) $V_{\mathrm{c}}^*$ is strictly convex, positive definite, and positively homogeneous of degree 2.
(2) The level sets of $V_{\mathrm{c}}^*$ are strictly convex and for

$$L_{V_{\mathrm{c}}^*} := \{\xi \in \mathbb{R}^n : V_{\mathrm{c}}^*(\xi) \leqslant 1\},$$

we have $L_{V_{\mathrm{c}}^*} = \bigcap_{j=1}^{N} \mathscr{E}(Q_j)$.

The following set inclusion properties will be needed for the essential step of obtaining local LDI descriptions for a saturated linear system.

**Lemma 1.** *Let $H \in \mathbb{R}^{m \times n}$ and denote the $\ell$th row of $H$ as $h_\ell$. Let $\mathscr{L}(H)$ be defined as in* (1). *We have,*

(1) *$L_{V_{\mathrm{c}}} \subset \mathscr{L}(H)$ if and only if $2h_\ell^{\mathrm{T}} \in L_{V_{\mathrm{c}}^*}$ for all $\ell \in I[1, m]$;*
(2) *$L_{V_{\mathrm{c}}^*} \subset \mathscr{L}(H)$ if and only if $2h_\ell^{\mathrm{T}} \in L_{V_{\mathrm{c}}}$ for all $\ell \in I[1, m]$.*

**Proof.** As $V_{\mathrm{c}}$ and $V_{\mathrm{c}}^*$ are positive definite, symmetric, and positively homogeneous of degree 2, they lead to a pair of polar norms (see Rockafellar, 1973, 15.3.1): $\|x\|_{\mathrm{c}} := (2V_{\mathrm{c}}(x))^{1/2}$ and $\|x\|_{\mathrm{c}}^* := (2V_{\mathrm{c}}^*(x))^{1/2}$. In particular, for any $z \in \mathbb{R}^n$, any $\delta > 0$, $|z^{\mathrm{T}} x| \leqslant 1$ for all $\|x\|_{\mathrm{c}} \leqslant \delta$ if and only if $\|z\|_{\mathrm{c}}^* \leqslant 1/\delta$. Taking $\delta = \sqrt{2}$ and $z = h_l^{\mathrm{T}}$ shows that $|h_\ell x| \leqslant 1$

for all $V_c(x) \leqslant 1$ if and only if $V_c^*(2h_\ell^T) = 4V_c^*(h_\ell^T) \leqslant 1$. This shows item 1. Item 2 is shown similarly. □

### 2.3. Dual stability conditions

Consider the following differential inclusion

$$\dot{x} \in \mathrm{co}\{A_i x : i \in I[1, M]\}, \tag{8}$$

where $A_i \in \mathbb{R}^{n\times n}$ are given constant matrices.

**Theorem 3** (*Goebel, Teel, Hu, and Lin (2005)*). *Let $Q_k \in \mathbb{R}^{n\times n}$, $k \in I[1, N]$ be given positive definite matrices and let $V_c$ and $V_c^*$ be the functions as defined in* (6) *and* (7).

(1) *For $\beta \in \mathbb{R}$, if there exist $\lambda_{ijk} \geqslant 0$, $j, k \in I[1, N]$, $i \in I[1, M]$ such that*

$$A_i^T Q_k + Q_k A_i \leqslant \sum_{j=1}^{N} \lambda_{ijk}(Q_j - Q_k) - \beta Q_k \quad \forall k \in I[1, N], \quad i \in I[1, M], \tag{9}$$

*then for all $x \in \mathbb{R}^n$, $f \in \mathrm{co}\{A_i x : i \in I[1, M]\}$,*

$$\partial V_c^*(x)^T f \leqslant -\beta V_c^*(x). \tag{10}$$

(2) *For $\beta \in \mathbb{R}$, if there exist $\lambda_{ijk} \geqslant 0$, $j, k \in I[1, N]$, $i \in I[1, M]$ such that*

$$Q_k A_i^T + A_i Q_k \leqslant \sum_{j=1}^{N} \lambda_{ijk}(Q_j - Q_k) - \beta Q_k \quad \forall k \in I[1, N], \quad i \in I[1, M], \tag{11}$$

*then for all $x \in \mathbb{R}^n$, $f \in \mathrm{co}\{A_i x : i \in I[1, M]\}$,*

$$\partial V_c(x)^T f \leqslant -\beta V_c(x). \tag{12}$$

*If $N = 2$, then the condition in each of the above items is also necessary.*

The conditions (9) for exponential stability certified by the function on the right-hand side of (7) were outlined in Boyd et al. (1994, pp. 73–74). With those conditions in hand, the conditions for stability certified by $V_c$ follow from Theorem 1 and the conjugacy between $V_c$ and $V_c^*$.

## 3. Stability analysis for systems with saturation nonlinearities

Consider the system

$$\dot{x} = Ax + B\,\mathrm{sat}(Fx), \tag{13}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}^{m\times n}$ are given and $\mathrm{sat}(\cdot)$ is the standard saturation function. In this section, we use the dual stability conditions for LDIs to improve the earlier stability analysis results in Hu et al. (2002a) and Hu and Lin (2003). First, we need a tool developed in Hu et al. (2002b) to obtain a local LDI description for system (13).

Consider the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are $2^m$ such matrices and we label them as $D_i$, $i \in I[1, 2^m]$. Denote $D_i^- = I - D_i$. Given two vectors, $u, v \in \mathbb{R}^m$, $\{D_i u + D_i^- v : i \in [1, 2^m]\}$ is the set of vectors obtained by choosing some elements from $u$ and the rest from $v$.

**Lemma 2** (*Hu et al., 2002b*). *Let $H \in \mathbb{R}^{m\times n}$ be given. Then for all $x \in \mathscr{L}(H)$,*

$$\mathrm{sat}(Fx) \in \mathrm{co}\{(D_i F + D_i^- H)x : i \in I[1, 2^m]\}.$$

Consider any $H \in \mathbb{R}^{m\times n}$. By Lemma 2, the saturated system (13) satisfies

$$\dot{x} \in \mathrm{co}\{(A + B(D_i F + D_i^- H))x : i \in I[1, 2^m]\} \tag{14}$$

for all $x \in \mathscr{L}(H)$. We call (14) a parameterized LDI because of the degree of freedom injected through $H$. To obtain an estimate of the domain of attraction, we may determine an invariant level set of $V_c$ or $V_c^*$ within $\mathscr{L}(H)$ by using Theorem 3. In fact, both $H$ and $V_c$ or $V_c^*$ can be optimized to produce "a large" estimate. The following theorem gives a sufficient condition for $L_{V_c}$ to be a contractively invariant set for (13).

**Theorem 4.** *Let $Q_j$, $j \in I[1, N]$, be positive definite matrices. Let $V_c(x)$ be the convex hull function as defined in* (6), *and take $\beta > 0$. If there exist an $H \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \geqslant 0$, $i \in I[1, 2^m]$, $j, k \in I[1, N]$ such that*

$$Q_k(A + B(D_i F + D_i^- H))^T + (A + B(D_i F + D_i^- H))Q_k \leqslant \sum_{j=1}^{N} \lambda_{ijk}(Q_j - Q_k) - \beta Q_k, \quad i \in I[1, 2^m], \quad k \in I[1, N], \tag{15}$$

$$2h_\ell Q_k h_\ell^T \leqslant 1, \quad \ell \in I[1, m], \quad k \in I[1, N], \tag{16}$$

*where $h_\ell$ is the $\ell$th row of $H$, then for* (13), *we have*

$$\partial V_c(x)^T(Ax + B\,\mathrm{sat}(Fx)) \leqslant -\beta V_c(x) \quad \forall x \in L_{V_c}. \tag{17}$$

**Proof.** Note that (16) implies

$$\frac{1}{2}\max_{k\in[1,N]}(2h_\ell)Q_k(2h_\ell)^T \leqslant 1, \quad \forall \ell \in I[1, m], \quad k \in I[1, N],$$

i.e., $2h_\ell^T \in L_{V_c^*}$. By Lemma 1, we have $L_{V_c} \subset \mathscr{L}(H)$. By Lemma 2, we have

$$\mathrm{sat}(Fx) \in \mathrm{co}\{(D_i F + D_i^- H)x : i \in I[1, 2^m]\} \quad \forall x \in L_{V_c}.$$

If we let $A_i = A + B(D_i F + D_i^- H)$, then for all $x \in L_{V_c}$,

$$\dot{x} = Ax + B\,\mathrm{sat}(Fx) \in \mathrm{co}\{A_i x : i \in I[1, 2^m]\}.$$

By (2) of Theorem 3, under the condition (15), we have

$$\partial V_c(x)^T(Ax + B\,\mathrm{sat}(Fx)) \leqslant -\beta V_c(x) \quad \forall x \in L_{V_c}. \qquad \square$$

Since $\beta > 0$, the property (17) implies that $L_{V_c}$ is a contractively invariant set and any trajectory starting from it will remain inside and converge to the origin with a decay rate $\beta/2$. Hence $L_{V_c}$ is an estimate of the domain of attraction. Given $x_0 \in \mathbb{R}^n$, we want to optimize the function $V_c(x)$ such that $L_{V_c}$ satisfies the condition of Theorem 4 for certain $\beta > 0$ and $L_{V_c}$ contains $\alpha x_0$ with largest $\alpha$ possible. This objective can be described as

$$\begin{aligned} &\sup_{Q_j,\beta,\lambda_{ijk},H} \quad \alpha \\ &\quad \text{s.t.}(a)\ \alpha x_0 \in L_{V_c} \\ &\qquad (b)\ (15), (16), \\ &\qquad (c)\ \beta > 0,\ Q_j > 0, \quad \lambda_{ijk} \geqslant 0\ \ \forall i, j, k. \end{aligned} \tag{18}$$

The optimization problem (18) can be modified to maximize $L_{V_c}$ with respect to other kinds of shape reference set $X_0 \subset \mathbb{R}^n$ (see Hu et al., 2002a). One only needs to replace (a) with $\alpha X_0 \subset L_{V_c}$. If $X_0$ is a polytope, this condition can be equivalently stated as $\alpha x_i \in L_{V_c}$ for finite many $x_i$'s. Note that $\alpha x_0 \in L_{V_c}$, or, $V_c(\alpha x_0) \leqslant 1$, is equivalent to the existence of $\gamma \in \Gamma$ such that (see Hu et al., 2003)

$$\begin{bmatrix} 2 & \alpha x_0^{\mathrm{T}} \\ \alpha x_0 & \sum_{j=1}^N \gamma_j Q_j \end{bmatrix} \geqslant 0. \tag{19}$$

Also note that (16) is equivalent to

$$\begin{bmatrix} \frac{1}{2} & h_\ell Q_k \\ Q_k h_\ell^{\mathrm{T}} & Q_k \end{bmatrix} \geqslant 0 \quad \forall \ell, k. \tag{20}$$

Hence (18) can be rewritten as

$$\begin{aligned} &\sup_{Q_j,\beta,\lambda_{ijk},H} \quad \alpha \\ &\quad \text{s.t., } (19), (15), (20), \\ &\qquad \beta > 0, \gamma \in \Gamma, Q_j > 0, \quad \lambda_{ijk} \geqslant 0 \quad \forall i, j, k. \end{aligned} \tag{21}$$

The constraints involve bilinear matrix inequalities (BMIs) since $V_c$ is not quadratic. Similar BMIs are derived in Goebel, Teel, Hu, and Lin (2005) and Goebel, Hu, and Teel (2005) for stability and performance analysis of linear differential/difference inclusions. A direct method to solve BMI problems is to alternatively fix one set of parameters and optimize the rest. In Goebel, Hu, and Teel (2005), we adopted the path-following method from Hassibi, How, and Boyd (1999), and our experience with several numerical examples shows that the path-following method is much more effective than the straightforward iterative method. To use the path-following method, we need to change the problem formulation in (21). Instead of directly maximizing $\alpha$, we fix $\alpha$ and maximize $\beta$ satisfying (15) and

$$\begin{bmatrix} 2 & \alpha x_0^{\mathrm{T}} \\ \alpha x_0 & \sum_{j=1}^N \gamma_j Q_j \end{bmatrix} \geqslant \beta I, \quad \begin{bmatrix} \frac{1}{2} & h_\ell Q_k \\ Q_k h_\ell^{\mathrm{T}} & Q_k \end{bmatrix} \geqslant \beta I.$$

If the maximal $\beta$ is greater than 0, then $\alpha x_0 \in L_{V_c}$ and $L_{V_c}$ is an estimate of the domain of attraction. We actually implemented a two-step iterative algorithm which combines the path-following method and the direct iterative method. The first step uses the path-following method to update all the parameters at the same time. The second step fixes $\lambda_{ijk}$'s, $H$ and $\gamma$ and solves the resulting LMI problem which only include $Q_j$'s as variables. This two-step method proves very effective on the BMI problems in Goebel, Hu, and Teel (2005) and also works well on the example in Section 4.

We note that if we impose $Q_1 = Q_2 \cdots = Q_N$, then $L_{V_c}$ is an invariant ellipsoid and (21) reduces to the corresponding optimization problem in Hu et al. (2002a) which can be transformed into LMIs with a change of variables. To guarantee that a better result is produced than that of Hu et al. (2002a), we can start the two-step algorithm with $H$ and $Q_j$'s inherited from the optimal solution of Hu et al. (2002a), i.e., by choosing the initial $H$ as the optimal solution under the restriction $Q_1 = Q_2 \cdots = Q_N$. The initial $\lambda_{ijk}$'s and $\gamma$ can be arbitrarily assigned under the constraint of (21).

We now use $V_c^*$ to estimate the domain of attraction. The following theorem gives a sufficient condition for $L_{V_c^*}$ to be a region of exponential stability for system (13).

**Theorem 5.** *Let $Q_j$, $j \in I[1, N]$, be positive definite matrices. Let $V_c^*$ be the max function as defined in* (7), *and take $\beta > 0$. If there exist an $H \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \geqslant 0$, $i \in I[1, 2^m]$, $j, k \in I[1, N]$ such that*

$$\begin{aligned} &(A + B(D_i F + D_i^- H))^{\mathrm{T}} Q_k + Q_k (A + B(D_i F + D_i^- H)) \\ &\quad \leqslant \sum_{j=1}^N \lambda_{ijk}(Q_j - Q_k) - \beta Q_k, \\ &\quad i \in I[1, 2^m], \quad k \in I[1, N], \end{aligned} \tag{22}$$

$$2h_\ell^{\mathrm{T}} \in L_{V_c}, \quad \ell \in I[1, m], \tag{23}$$

*then for system* (13), *we have*

$$\partial V_c^*(x)^{\mathrm{T}}(Ax + B\,\mathrm{sat}(Fx)) \leqslant -\beta V_c^*(x) \quad \forall x \in L_{V_c^*}.$$

**Proof.** By Lemma 1, (23) implies $L_{V_c^*} \subset \mathscr{L}(H)$. By Lemma 2, we have

$$\mathrm{sat}(Fx) \in \mathrm{co}\{(D_i F + D_i^- H)x : i \in I[1, 2^m]\} \quad \forall x \in L_{V_c^*}.$$

The rest of the proof is similar to that of Theorem 4 by following item 1 of Theorem 3. □

To optimize the function $V_c^*(x)$ such that $L_{V_c^*}$ satisfies the condition of Theorem 5 and $L_{V_c^*}$ contains $\alpha x_0$ with $\alpha$ maximized, we formulate the optimization problem

$$\begin{aligned} &\sup_{Q_j,\beta,\lambda_{ijk},H} \quad \alpha \\ &\quad \text{s.t.}(a)\ \alpha x_0 \in L_{V_c^*} \\ &\qquad (b)\ (22), (23), \\ &\qquad (c)\ \beta > 0, \quad Q_j > 0, \quad \lambda_{ijk} \geqslant 0\ \ \forall i, j, k. \end{aligned} \tag{24}$$

Note that condition (23) is equivalent to the existence of $\gamma_\ell \in \Gamma$ such that

$$\begin{bmatrix} \frac{1}{2} & h_\ell \\ h_\ell^{\mathrm{T}} & \sum_{j=1}^{N} \gamma_{\ell j} Q_j \end{bmatrix} \geqslant 0.$$

and $\alpha x_0 \in L_{V_c^*}$ is equivalent to $\frac{1}{2}\alpha^2 x_0^{\mathrm{T}} Q_j x_0 \leqslant 1$ for all $j \in I[1, N]$. Similarly to the problem (18), the two-step path-following algorithm also proves effective for (24).

**Remark 1.** The analysis results developed in this section can be readily adapted to design a feedback law $u = \mathrm{sat}(Fx)$ such that a guaranteed region of stability is maximized. This is done by considering $F$ as an additional optimization parameter. The results can also be adapted for the purpose of analyzing controlled invariant sets.

## 4. Numerical example

The following system is taken from Hu et al. (2002a):

$$\dot{x} = Ax + B\,\mathrm{sat}(Fx)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad F = [-2 \;\; -1].$$

We use the convex hull and the max functions to estimate the domain of attraction. The functions $V_{c1}$ and $V_{c2}^*$ are constructed from two quadratic functions, i.e.,

$$V_{c1}(x) = \frac{1}{2} \min_{\gamma \in \Gamma} x^{\mathrm{T}} (\gamma_1 Q_1 + \gamma_2 Q_2)^{-1} x,$$
$$V_{c2}^*(x) = \tfrac{1}{2} \max\{x^{\mathrm{T}} \tilde{Q}_1 x, x^{\mathrm{T}} \tilde{Q}_2 x\}.$$

We use $V_{c1}$ and $V_{c2}^*$ to denote these two functions to avoid confusion. We note that Theorems 4 and 5 have dual structure but the resulting Lyapunov functions need not be conjugate to each other, they only belong to classes of functions "conjugate" to each other.

The reference point is taken as $x_0 = [1 \;\; 0]^{\mathrm{T}}$. For each $\alpha$, we use the two-step path-following algorithm to test if $\alpha x_0$ is feasible for (21). The algorithm quickly produces a result for each $\alpha$ and it is determined that $\alpha$ can be made arbitrarily close to 5. Since $5x_0$ is an equilibrium point and does not belong to the domain of attraction, $\alpha = 5$ must be the optimal solution. The matrices $Q_1$ and $Q_2$ corresponding to $\alpha = 4.999$ are given as follows:

$$Q_1 = \begin{bmatrix} 15.325 & -21.273 \\ -21.273 & 38.868 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 15.989 & -2.989 \\ -2.989 & 2.992 \end{bmatrix}.$$

Fig. 1 plots the boundary of $L_{V_{c1}}$ in solid curve. Also plotted in the figure are some line segments indicating the direction of $\dot{x}$ along the boundary of $L_{V_{c1}}$. It is clearly seen that $L_{V_{c1}}$ is invariant, although some directions almost overlap the boundary, which is a result of optimization. The two ellipsoid boundaries for $\mathscr{E}(Q_1^{-1})$ and $\mathscr{E}(Q_2^{-1})$ are plotted in the figure as dotted curves.

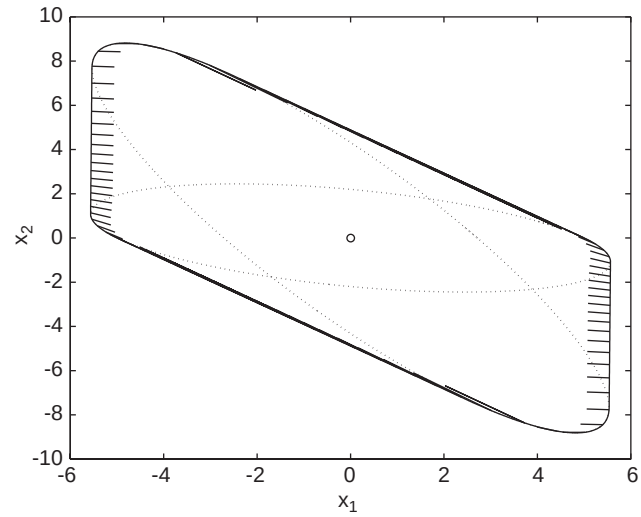


Fig. 1. Vectors along the boundary of $L_{V_{c1}}$.

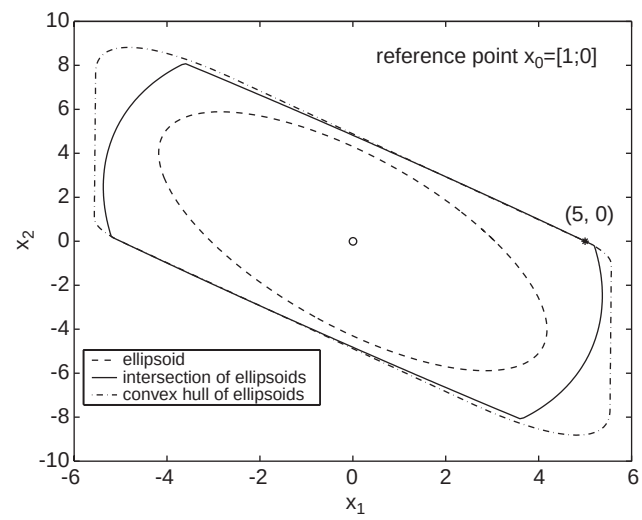


Fig. 2. Three invariant level sets.

We then solved the optimization problem (24). The optimal $\alpha$ is also found to be 5. The matrices $\tilde{Q}_1$ and $\tilde{Q}_2$ corresponding to $\alpha = 4.999$ are given as follows:

$$\tilde{Q}_1 = \begin{bmatrix} 0.0749 & 0.0122 \\ 0.0122 & 0.0264 \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} 0.0800 & 0.0800 \\ 0.0800 & 0.0861 \end{bmatrix}.$$

In Fig. 2, we compare these two estimates with an earlier one which is the maximal invariant ellipsoid with respect to $x_0$, obtained by the method of Hu et al. (2002a). The maximal $\alpha$ such that $\alpha x_0$ can be enclosed by an invariant ellipsoid is 3.0573 (the maximal $\alpha$ such that $\alpha x_0$ is in the invariant level set of $V_{c1}$ or $V_{c2}^*$ is 5). The outermost boundary in Fig. 2

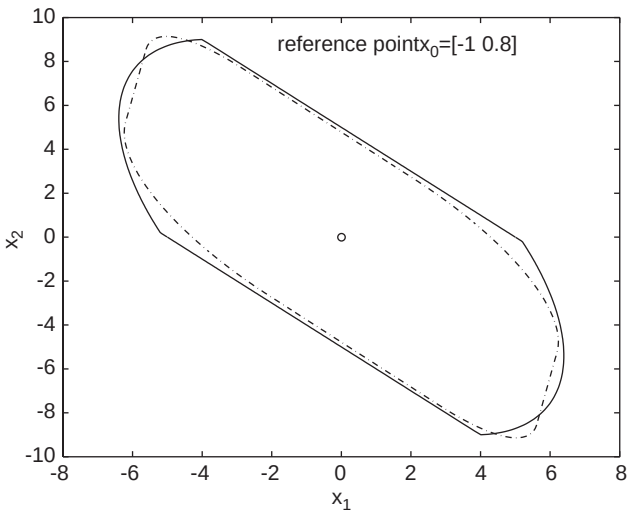


Fig. 3. Optimized level sets for the reference $x_0 = [-1\ \ 0.8]^{\mathrm{T}}$.

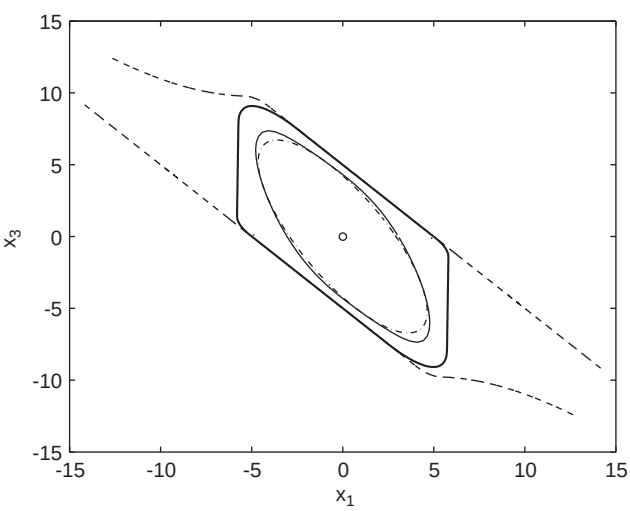


Fig. 4. Domain of attraction and its estimates.

is that of $L_{V_{c1}}$ and the solid curve is the boundary of the level set $L_{V^*_{c2}}$. The boundary of the invariant ellipsoid is the innermost dashed curve.

For the two estimates resulting from the reference point $x_0 = [1\ \ 0]^{\mathrm{T}}$, we have $L_{V_{c1}} \supset L_{V^*_{c2}}$. This is not always the case. For instance, for $x_0 = [-1\ \ 0.8]^{\mathrm{T}}$, the resulting $L_{V^*_{c2}}$ is not a subset of $L_{V_{c1}}$ and the union of these two sets is larger than each of them (see Fig. 3 where $L_{V^*_{c2}}$ is bounded by the solid curve and $L_{V_{c1}}$ by the dash–dotted curve.) This shows the advantage of applying the conjugate Lyapunov functions.

The exact boundary of the domain of attraction can be obtained from simulation. The comparison between $L_{V_{c1}}$ for the reference $x_0 = [1\ \ 0]^{\mathrm{T}}$ (the thick solid curve) and the exact domain of attraction (the dashed curve) is made in Fig. 4. We see that parts of the boundaries almost overlap. Also plotted in the figure is one invariant ellipsoid (the innermost dash-dotted curve) and the boundary of the convex hull of all the invariant ellipsoids (the thin solid curve), the best that can be obtained by the methods in Hu and Lin (2003) and Hu et al. (2002a).

## 5. Conclusions

This paper revisits the problems of stability analysis for systems with saturation nonlinearities. The recently developed duality theory for LDIs is utilized to enhance the stability results in Hu and Lin (2003) and Hu et al. (2002a). Apart from using the convex hull function introduced in Hu and Lin (2003) as a Lyapunov function, we also developed dual stability results by using its conjugate function, the max (of quadratics) function. Optimization problems are derived for the purpose of maximizing the estimate of the domain of attraction. These optimization problems involve bilinear matrix inequalities and experience shows that they can be effectively solved with the path-following method in Hassibi et al. (1999). Although the global optimal solutions are not guaranteed to be determined, the algorithm always improves on earlier results based on quadratic functions.

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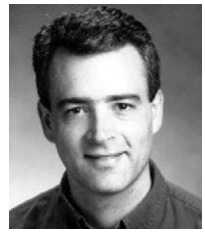

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