Nonlinear Control Design for Linear Differential Inclusions via Convex Hull of Quadratics [⋆]

Tingshu Hu [a]

[a] *Department of Electrical and Computer Engineering, University of Massachusetts, Lowell, MA 01854.*

Abstract

This paper presents a nonlinear control design method for robust stabilization and robust performance of linear differential inclusions (LDIs). A recently introduced non-quadratic Lyapunov function, the convex hull of quadratics, will be used for the construction of nonlinear state feedback laws. Design objectives include stabilization with maximal convergence rate, disturbance rejection with minimal reachable set and least $\mathcal{L}_2$ gain. Conditions for stabilization and performances are derived in terms of bilinear matrix inequalities (BMIs), which cover the existing linear matrix inequality (LMI) conditions as special cases. Numerical examples demonstrate the advantages of using nonlinear feedback control over linear feedback control for LDIs. It is also observed through numerical computation that nonlinear control strategies help to reduce control effort substantially.

Key words: Linear differential inclusion; nonlinear feedback; Lyapunov functions; robust stability; robust performance.

1 Introduction

A simple and practical approach to describe systems with nonlinearities and time-varying uncertainties is to use linear differential inclusions (LDIs). Such practice can be traced back to the earlier development of absolute stability theory. The advantages of using LDIs to describe complicated systems are fully demonstrated in Boyd, El Ghaoui, Feron,& Balakrishnan (1994), where a wide variety of control problems for LDIs are interpreted with linear matrix inequalities (LMIs). The mechanism behind the LMI framework is a systematic application of Lyapunov theory through quadratic functions.

While the LMI technique has been well appreciated and has been widely applied to various control problems, the conservatism introduced by quadratic Lyapunov functions has been revealed in some literature including Boyd et al (1994). Considerable efforts have been devoted to the construction and development of non-quadratic Lyapunov functions (see e.g. Blanchini, 1995; Chesi, Garulli, Tesi & Vicino, 2003; Jarvis-Wloszek & Packard, 2002; Molchanov, 1989; Polanski, 1997; Xie, Shishkin & Fu, 1997; Yfoulis & Shorten, 2004). In (Molchanov, 1989), a necessary and sufficient condition for stability of polytopic LDIs was derived as bilinear matrix equations (computational methods for solving these matrix equations are still under development, see Polanski, 1997; Polanski, 2000; Yfoulis & Shorten, 2004). More numerically tractable stability conditions were derived as LMIs in (Chesi et al 2003; Jarvis-Wloszek & Packard, 2002; Xie et al, 1997) from piecewise quadratic functions and homogeneous polynomial functions.

Recently, a pair of conjugate Lyapunov functions have demonstrated great potential in stability and performance analysis of LDIs, saturated systems and uncertain systems with generalized sector condition (Goebel, Hu & Teel, 2005; Goebel, Teel, Hu & Lin, 2006; Hu, Goebel, Teel & Lin, 2005; Hu & Lin, 2005; Hu, Teel & Zaccarian, 2006). Through these functions, stability and performances of LDIs are characterized in terms of bilinear matrix inequalities (BMIs) which cover the existing LMI conditions in (Boyd et al, 1994) as special cases. Since extra degrees of freedom for optimization are injected through the bilinear terms, the analysis results are guaranteed to be at least as good as those obtained by corresponding LMI conditions. Extensive examples have shown that these non-quadratic Lyapunov functions can effectively reduce conservatism in various stability and performance analysis problems.

With the effectiveness of non-quadratic Lyapunov functions demonstrated on a number of analysis problems, they can further be applied to the construction of feedback laws. For linear time-invariant systems, it is well known that nonlinear controls have no advantage over

⋆ This paper was not presented at any IFAC meeting. Corresponding author Tingshu Hu. Tel. +01-978-9344374. Fax +01-978-9343027. This work was supported by NSF under Grant ECS-0621651.

Email address: `tingshu@gmail.com` (Tingshu Hu).

linear controls when it comes to stabilization or minimization of the $\mathcal{L}_2$ gain (see e.g. Khargonekar, Petersen & Rotea, 1988). For systems with time-varying uncertainties and LDIs, it is now accepted that nonlinear control can work better than linear control. In (Blanchini & Megretski, 1999), examples were constructed to demonstrate this aspect and it was suggested that non-quadratic Lyapunov functions would facilitate the construction of nonlinear feedback laws. In (Blanchini, 1995), piecewise linear (or polyhedral) Lyapunov functions was used to guide the construction of variable-structure control laws for robust stability and rejection of bounded persistent disturbances.

In this paper, we use one of the pair of conjugate Lyapunov functions considered in (Goebel et al, 2005; Hu et al, 2005), the convex hull function (i.e., the convex hull of quadratics), for the construction of nonlinear state feedback laws. This paper is organized as follows. Section 2 describes the problems to be studied and presents some preliminaries on the convex hull function. Section 3 applies the convex hull function to the construction of nonlinear state feedback laws for robust stabilization. Section 4 constructs nonlinear feedback laws to achieve a couple of robust performance objectives. Section 5 uses a few examples to demonstrate the effectiveness of non-quadratic Lyapunov functions and nonlinear feedback design. Section 6 concludes the paper.

Notation

- $|\cdot|_\infty$: For $x \in \mathbb{R}^n$, $|x|_\infty := \max_i |x_i|$.
- $\|\cdot\|_2$: For $u \in \mathcal{L}_2$, $\|u\|_2 := \left(\int_0^\infty u^T(t)u(t)dt\right)^{\frac{1}{2}}$.
- $I[k_1, k_2]$: For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1+1, \cdots, k_2\}$.
- $\operatorname{co} S$: The convex hull of a set S.
- $\mathcal{E}(P) := \{x \in \mathbb{R}^n : \ x^{\mathrm{T}}Px \le 1\}$.
- $L_V := \{x \in \mathbb{R}^n : V(x) \le 1\}$.
- $\mathcal{L}(H) := \{x \in \mathbb{R}^n : |Hx|_\infty \le 1\}$.

About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(H)$, we have

$$\mathcal{E}(P) \subseteq \mathcal{L}(H) \Longleftrightarrow H_\ell P^{-1} H_\ell^{\mathrm{T}} \le 1 \quad \forall \ell \in I[1, r], \quad (1)$$

where H_ℓ is the ℓth row of H.

2 Problem statement and preliminaries

Consider the following polytopic linear differential inclusion (PLDI),

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \operatorname{co}\left\{ \begin{bmatrix} A_i x + B_i u + T_i w \\ C_i x + D_i w \end{bmatrix} : \ i \in I[1, N] \right\}, \quad (2)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the disturbance and $y \in \mathbb{R}^q$ is the output. A_i, B_i, T_i, C_i and D_i are given real matrices of compatible dimensions. This type of LDIs can be used to describe a wide variety of nonlinear systems, possibly with time-varying uncertainties (see Boyd et al, 1994).

Control design problems for LDIs via linear state feedback of the form $u = Fx$ have been extensively addressed in (Boyd et al, 1994), where quadratic Lyapunov functions are used as constructive tools and the control problems are transformed into LMIs. While the LMI technique has gained tremendous popularity and its applications are still expanding to different types of systems, the conservatism resulting from quadratic Lyapunov functions has been recognized and efforts have been devoted to the construction of non-quadratic Lyapunov functions. On the other hand, it has also been recognized (e.g., Blanchini et al, 1999) that restriction to linear feedback laws may also impose unnecessary limitations to the achievable performances.

In this paper, we use the convex hull function to construct nonlinear feedback laws to achieve a few objectives of robust stabilization and performance. In what follows, we give a brief review of the definition and some properties of the convex hull function that will be necessary for the development of the main results.

The convex hull function is constructed from a family of positive definite matrices. Let $Q_j \in \mathbb{R}^{n\times n}, Q_j = Q_j^{\mathrm{T}} > 0, j \in I[1, J]$. Let

$$\Gamma^J := \left\{\gamma \in \mathbb{R}^J : \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \ge 0\right\}.$$

The convex hull function is defined as

$$V_c(x) := \min_{\gamma \in \Gamma^J} x^{\mathrm{T}} \left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x. \quad (3)$$

From the definition, $V_c(x)$ and the optimal γ can be computed by solving a simple LMI problem obtained via Schur complements. This function was first used in (Hu & Lin, 2003) to study constrained control systems, where it was called the composite quadratic function. It was later called convex hull function, or convex hull of quadratics, in (Goebel et al, 2006; Hu et al, 2005) since it is the convex hull (see Rockafellar, 1970) of the family of quadratics $x^T Q_k^{-1} x$, or equivalently, the convex hull of $g(x) = \min\{x^T Q_k^{-1} x : k \in I[1, J]\}$. As observed in (Goebel et al, 2006)

$$\begin{aligned} V_c(x) &= \operatorname{co} g(x) \\ &= \min\left\{ \sum_{k=1}^{n+1} \lambda_k g(x_k) : \sum_{k=1}^{n+1} \lambda_k x_k = x, \ \lambda \in \Gamma^{n+1} \right\}. \end{aligned}$$

By (Rockafellar, 1970), $g(x_k)$ and $n+1$ in the above equation can be replaced with $x_k^{\mathrm{T}} Q_k^{-1} x_k$ and J, respectively. It turns out that the level set of V_c is the convex hull of a family of ellipsoids. If we define the 1-level set of V_c as

$$L_{V_c} := \{x \in \mathbb{R}^n : \ V_c(x) \le 1\},$$

and denote the 1-level set of the function $x^{\mathrm{T}} P x$ as

$$\mathcal{E}(P) := \left\{x \in \mathbb{R}^n : \ x^{\mathrm{T}} P x \le 1\right\},$$

then

$$L_{V_c} = \left\{ \sum_{j=1}^{J} \gamma_j x_j : \ x_j \in \mathcal{E}(Q_j^{-1}), \gamma \in \Gamma^J \right\}.$$

It is established in (Goebel et al, 2006; Hu & Lin, 2003) that V_c is convex and continuously differentiable. From the definition, it can be verified that V_c is homogeneous of degree 2, i.e., $V_c(\alpha x) = \alpha^2 V_c(x)$.

For a compact convex set S, a point x on the boundary of S (denoted as ∂S) is called an extreme point if it cannot be represented as the convex combination of any other points in S. A compact convex set is completely determined by its extreme points. In what follows, we characterize the set of extreme points of L_{V_c}. Since L_{V_c} is the convex hull of $\mathcal{E}(Q_j^{-1})$, $j \in I[1, J]$, an extreme point must be on the boundaries of both L_{V_c} and $\mathcal{E}(Q_j^{-1})$ for some $j \in I[1, J]$. Denote

$$E_k := \partial L_{V_c} \cap \partial\mathcal{E}(Q_k^{-1}) = \left\{x : V_c(x) = x^{\mathrm{T}} Q_k^{-1} x = 1\right\}.$$

Then $\bigcup_{k=1}^{J} E_k$ contains all the extreme points of L_{V_c}. The exact description of E_k is given as follows.

Lemma 1 *(Hu et al, 2006) For each $k \in I[1, J]$,*

$$E_k = \{x \in \partial L_{V_c} : x^T Q_k^{-1}(Q_j - Q_k)Q_k^{-1} x \leq 0, j \in I[1, J]\}.$$

For $x \in \mathbb{R}^n$, define

$$\gamma^*(x) := \arg\min_{\gamma\in\Gamma^J} x^{\mathrm{T}} \left(\sum_{j=1}^{J} \gamma_j Q_j\right)^{-1} x. \tag{4}$$

Generally, γ^* is uniquely determined by x and is a continuous function of x except for some degenerated cases (Hu & Lin, 2004). It is evident that $\gamma^*(\alpha x) = \gamma^*(x)$ for any $\alpha \neq 0$. Detailed properties about γ^* were characterized in (Hu & Lin, 2004). The following lemma combines some results from (Hu & Lin, 2003, 2004).

Lemma 2 *Let $x \in \mathbb{R}^n$. For simplicity and without loss of generality, assume that $\gamma_k^*(x) > 0$ for $k \in I[1, J_0]$ and $\gamma_k^*(x) = 0$ for $k \in I[J_0 + 1, J]$. Denote*

$$Q(\gamma^*) = \sum_{k=1}^{J_0} \gamma_k^* Q_k, \quad x_k = Q_k Q(\gamma^*)^{-1} x, \quad k \in I[1, J_0].$$

Then $V_c(x_k) = V_c(x) = x_k^T Q_k^{-1} x_k$ and $x_k \in (V_c(x))^{\frac{1}{2}} E_k$ for $k \in I[1, J_0]$. Moreover, $x = \sum_{k=1}^{J_0} \gamma_k^* x_k$, and for all $k \in I[1, J_0]$,

$$\nabla V_c(x) = \nabla V_c(x_k) = 2Q_k^{-1} x_k = 2Q(\gamma^*)^{-1} x, \tag{5}$$

where $\nabla V_c(x)$ denotes the gradient of V_c at x.

Since $\gamma^*(\alpha x) = \gamma^*(x)$, by (5), we have $\nabla V_c(\alpha x) = \alpha \nabla V_c(x)$. Since V_c is homogeneous of degree two, to obtain some geometric interpretation of Lemma 2, we may restrict our attention to a point $x \in \partial L_{V_c}$. Then by the lemma, x can always be expressed as a convex combination of a family of x_k's, $x_k \in \partial\mathcal{E}(Q_k^{-1})$ (note $x_k \in E_k$). Furthermore, the gradient of V_c at these x_k's are the same and they all equal to the gradient of V_c at x. In other words, x and x_k's are in the same hyperplane which is tangential to L_{V_c}. In fact, the intersection of the hyperplane with L_{V_c} is a polygon whose vertices include x_k's (see Hu & Lin, 2004).

Properties in Lemma 1 and Lemma 2 are essential to system analysis and design via the convex hull function. They have been used in (Hu & Lin, 2005; Hu et al, 2006) for stability and performance analysis of saturated systems and uncertain systems with generalized sector conditions.

3 Nonlinear feedback for robust stabilization

In the absence of disturbance, the LDI (2) reduces to,

$$\dot{x} \in \mathrm{co}\{A_i x + B_i u : \ i \in I[1, N]\}. \tag{6}$$

For stability design, we only consider the state inclusion. We would like to construct a nonlinear state feedback law to achieve robust stabilization via the convex hull function $V_c(x)$. The main result is given as follows.

Theorem 1 *Consider V_c composed from $Q_k \in \mathbb{R}^{n\times n}$, $Q_k = Q_k^T > 0, k \in I[1, J]$. If there exist $\beta > 0$, $Y_k \in \mathbb{R}^{m\times n}$, and $\lambda_{ijk} \geq 0$, $i \in I[1, N], j, k \in I[1, J]$ such that*

$$Q_k A_i^T + A_i Q_k + Y_k^T B_i^T + B_i Y_k \leq \sum_{j=1}^{J} \lambda_{ijk}(Q_j - Q_k) - \beta Q_k \quad \forall\, i,\, k, \tag{7}$$

then a stabilizing nonlinear feedback law can be constructed as follows. For each $x \in \mathbb{R}^n$, let $\gamma^*(x) \in \Gamma^J$ be defined as in (4). Let

$$Y(\gamma^*) = \sum_{k=1}^{J} \gamma_k^* Y_k, \quad Q(\gamma^*) = \sum_{k=1}^{J} \gamma_k^* Q_k, \tag{8}$$

$$F(\gamma^*) = Y(\gamma^*) Q(\gamma^*)^{-1}. \tag{9}$$

Define $f(x) = F(\gamma^*(x))x$. Then for all $x \in \mathbb{R}^n$, we have

$$\max\{\nabla V_c(x)^T (A_i x + B_i f(x)) : i \in I[1, N]\} \leq -\beta V_c(x), \tag{10}$$

which implies that the closed-loop system under $u = f(x)$ is stable. If the vector function $\gamma^*(x)$ is continuous in x, then $u = f(x)$ is a continuous feedback law. ⋄

Proof. See Appendix A. □

Since $\gamma^*(\alpha x) = \gamma^*(x)$, we have $f(\alpha x) = \alpha f(x)$ and the resulting closed-loop system is homogeneous of degree one. Here we give some explanation on the construction of $f(x)$. Let $F_k = Y_k Q_k^{-1}$. If $x \in E_k$, then $f(x) = F_k x$. For a general $x \in \partial L_{V_c}$, we can express it as the convex combination of a family of $x_k \in E_k, k = 1, 2, \cdots, J_0$, i.e., $x = \Sigma_{k=1}^{J_0} \gamma_k^* x_k$ by Lemma 2. Then $f(x)$ is the convex combination of $f(x_k)'s$ with the same coefficients, i.e., $f(x) = \Sigma_{k=1}^{J_0} \gamma_k^* f(x_k)$.

When the inequality (10) is satisfied, $V_c(x(t))$ is strictly decreasing and we have $V_c(x(t)) \leq V_c(x(0))e^{-\beta t}$ for every solution $x(\cdot)$. Hence β is a measure of convergence rate. To increase the convergence rate, an optimization problem can be formulated to maximize β as follows:

$$\sup_{\lambda_{ijk} \geq 0, Q_k = Q_k^{\mathrm{T}} > 0, Y_k} \beta \quad \text{s.t. } (7). \tag{11}$$

The constraint (7) consists of a family of bilinear matrix inequalities (BMIs) which contain some bilinear terms as the product of a full matrix and a scalar, i.e., $\lambda_{ijk}(Q_j - Q_k)$. Similar bilinear terms are contained in the optimization problems in (Goebel et al, 2005, 2006; Hu et al, 2005, 2006). In the aforementioned works, we adopted the path-following method from (Hassibi, How & Boyd, 1999) and our extensive numerical experience shows that the path-following method is very effective. We actually implemented a two-step iterative algorithm which combines the path-following method and the direct iterative method. The first step of each iteration uses the path-following method to update all the parameters at the same time. The second step fixes λ_{ijk}'s and solves the resulting LMI problem which includes Q_j's and Y_j's as variables. In (Hu et al, 2006), a 12-th order anti-windup system was used to demonstrate nonlinear $\mathcal{L}_2$ gain analysis via convex hull functions. The two-step iterative algorithm converges very well and the results show significant improvement on those obtained via quadratic functions.

We note that when $J = 1$, V_c reduces to a quadratic function and $F(\gamma^*(x))$ reduces to a constant gain. And the optimization problem (11) reduces to a generalized eigenvalue problem (GEVP) which can be solved under the LMI framework. In our computation, we first solve the optimization problem for $J = 1$ and then use the optimal Q^* and Y^* to start the two-step iterative algorithm for some $J > 1$, with $Q_j = Q^*$ and $Y_j = Y^*$ for all j and $\lambda_{ijk} \geq 0$ randomly chosen. With this approach, the optimization result will be guaranteed to be at least as good as that obtained by solving the corresponding GEVP problem. Similar approaches can be derived for other optimization problems for evaluating the reachable sets and the $\mathcal{L}_2$ gain in Section 4.

4 Nonlinear feedback for robust performance

Consider the linear differential inclusion (2) in the presence of disturbances. Like in (Boyd et al, 1994), we consider two types of disturbances, the unit peak disturbances

$$w^{\mathrm{T}}(t)w(t) \leq 1 \qquad \forall t \geq 0 \tag{12}$$

and the unit energy disturbances

$$\|w\|_2 = \left(\int_0^\infty w^{\mathrm{T}}(t)w(t)dt\right)^{\frac{1}{2}} \leq 1. \tag{13}$$

Let $u = f(x)$ be a nonlinear state feedback. The closed-loop system is

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A_i x + B_i f(x) + T_i w \\ C_i x + D_i w \end{bmatrix} : \ i \in I[1, N] \right\}. \tag{14}$$

The control design objective is disturbance rejection, i.e., to keep the state close to the origin or to keep the size of the output small (in terms of certain norm) in the presence of a class of disturbances. The disturbance rejection performance can be characterized by reachable set or the maximal output norm. When the disturbance is of unit peak type, the maximal output norm is associated with the $\mathcal{L}_\infty$ gain; when the disturbance is of unit energy, the maximal output norm is associated with the $\mathcal{L}_2 - \mathcal{L}_\infty$ gain or the $\mathcal{L}_2$ gain. We first consider the reachable set.

4.1 Suppression of the reachable set

The reachable set can be estimated with a level set of a certain Lyapunov function. In (Boyd et al, 1994), quadratic Lyapunov functions are considered for LDIs and the reachable set is estimated with ellipsoids. In this section, we use the convex hull of a family of ellipsoids to characterize the reachable set and we attempt to reduce the reachable set by nonlinear feedback laws.

4.1.1 Reachable set with finite energy disturbances

Theorem 2 *Consider V_c composed from $Q_k \in \mathbb{R}^{n\times n}$, $Q_k = Q_k^T > 0, k \in I[1, J]$. Suppose that there exist $Y_k \in \mathbb{R}^{m\times n}$, and $\lambda_{ijk} \geq 0$, $i \in I[1, N], j, k \in I[1, J]$ such that*

$$\begin{bmatrix} M_{ik} & T_i \\ T_i^T & -I \end{bmatrix} \leq 0 \quad \forall i, k, \tag{15}$$

where

$$M_{ik} = Q_k A_i^T + A_i Q_k + Y_k^T B_i^T + B_i Y_k - \sum_{j=1}^{J} \lambda_{ijk}(Q_j - Q_k). \tag{16}$$

Let the nonlinear feedback $u = f(x) = F(\gamma^*(x))x$ be constructed from Y_k's and Q_k's as in (8) and (9). Then for all w bounded by $\|w\|_2 \leq 1$ and with $x_0 = 0$, the state of (14) satisfies $x(t) \in L_{V_c}$ for all $t \geq 0$. ⋄

With the feedback law constructed in Theorem 2, the level set L_{V_c} can be considered as an estimate for the reachable set. To keep the state in a small neighborhood of the origin, it is desirable that L_{V_c} satisfying the condition is as small as possible. We may use a reference polytope to measure the size of L_{V_c}. The polytope is described in terms of a prescribed matrix $H \in \mathbb{R}^{r\times n}$ as follows, $\mathcal{L}(H) := \{x \in \mathbb{R}^n : |Hx|_\infty \le 1\}$. The "outer" size of L_{V_c} is defined as

$$\alpha_{\rm out} := \min\{\alpha : L_{V_c} \subset \alpha\mathcal{L}(H)\}. \quad (17)$$

The matrix H can be chosen such that $H_\ell x$ is a certain quantity that we would like to keep small. If we have $L_{V_c} \subset \alpha\mathcal{L}(H)$, then $|H_\ell x(t)| \le \alpha$ for all t in the presence of the class of disturbances. Since $\mathcal{L}(H)$ is a convex set and L_{V_c} is the convex hull of the ellipsoids $\mathcal{E}(Q_k^{-1})$, it is easy to see that $L_{V_c} \subset \alpha\mathcal{L}(H) = \mathcal{L}(H/\alpha)$ if and only if $\mathcal{E}(Q_k^{-1}) \subset \mathcal{L}(H/\alpha)$ for all k. By (1), this is equivalent to

$$H_\ell Q_k H_\ell^{\rm T} \le \alpha^2 \quad \forall \ell \in I[1,r], k \in I[1,J]. \quad (18)$$

In view of the above arguments, the problem of reducing the reachable set can be formulated as

$$\inf_{\lambda_{ijk}\ge 0, Q_k=Q_k^{\rm T}>0, Y_k} \alpha \quad \text{s.t. } (15), (18). \quad (19)$$

Proof of Theorem 2. It suffices to show that, under condition (15), we have $\dot V_c \le w^{\rm T}w$ for all x and w satisfying (14). Then by integrating both sides, we have $V_c(x(t)) \le \int_0^\infty w^{\rm T}w dt \le 1$ and hence $x(t) \in L_{V_c}$ for all t. We need to prove that

$$\nabla V_c(x)^{\rm T}(A_i x + B_i f(x) + T_i w) \le w^{\rm T}w \quad \forall x, w, i. \quad (20)$$

Similarly to the proof of Theorem 1, we can first verify (20) for every $x \in E_k$ by using (15). Then extend the results to all other x by expressing it as a convex combination of $x_k \in E_k, k = 1, 2, \cdots, J_0$ with $f(x)$ as the same convex combination of $f(x_k)'s$. □

4.1.2 Reachable set with unit peak disturbances

Theorem 3 *Consider V_c composed from $Q_k \in \mathbb{R}^{n\times n}$, $Q_k = Q_k^T > 0, k \in I[1,J]$. Suppose that there exist $Y_k \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \ge 0$, $i \in I[1,N], j,k \in I[1,J]$ and $\beta > 0$ such that*

$$\begin{bmatrix} M_{ik} + \beta Q_k & T_i \\ T_i^T & -\beta I \end{bmatrix} \le 0 \quad \forall i, k, \quad (21)$$

where M_{ik} is given by (16). Let the nonlinear feedback law $u = f(x) = F(\gamma^*(x))x$ be constructed from Y_k's and Q_k's as in (8) and (9). Then L_{V_c} is an invariant set, which means that all trajectories starting from L_{V_c} will stay inside for any possible disturbance satisfying $w(t)^T w(t) \le 1, \forall t \ge 0$. Moreover, for all $x_0 \in \mathbb{R}^n$ and all possible disturbances, $x(t)$ will converge to L_{V_c}. ◇

Proof. With similar arguments as in the proof of Theorem 1, it can be shown that under the condition (21), we have $\dot V_c \le -\beta V_c(x) + \beta w^{\rm T}w$, for all $x \in \mathbb{R}^n, w \in \mathbb{R}^p$ satisfying (14). Since $w^{\rm T}w \le 1$, for $V_c(x) = 1$, we have $\dot V_c \le 0$ and V_c is nonincreasing. Hence L_{V_c} is an invariant set. If $V_c(x) > 1$, then $\dot V_c$ is strictly decreasing. Hence any trajectory starting from outside of L_{V_c} will converge to L_{V_c}. □

Similarly to the unit energy disturbance case, we can formulate the following optimization problem for minimizing the reachable set or the maximal output norm,

$$\inf_{\lambda_{ijk},\beta\ge 0, Q_k=Q_k^{\rm T}>0, Y_k} \alpha \quad \text{s.t. } (21), (18). \quad (22)$$

4.2 Suppression of the $\mathcal{L}_2$ gain

For the type of energy bounded disturbances, we have the following result:

Theorem 4 *Let $Q_k \in \mathbb{R}^{n\times n}, Q_k = Q_k^T > 0$, $k \in I[1,J]$. Let $\delta > 0$. Suppose that there exist $Y_k \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \ge 0$, $i \in I[1,N], j,k \in I[1,J]$ such that*

$$\begin{bmatrix} M_{ik} & T_i & Q_k C_i^T \\ T_i^T & -I & D_i^T \\ C_i Q_k & D_i & -\delta^2 I \end{bmatrix} \le 0, \quad \forall\, i, k, \quad (23)$$

where M_{ik} is given by (16). Let the nonlinear feedback law $u = f(x) = F(\gamma^*(x))x$ be constructed from Y_k's and Q_k's as in (8) and (9). Then for system (14) with $x_0 = 0$, we have $\|y\|_2 \le \delta\|w\|_2$. ◇

The proof of Theorem 4 is omitted since the main ideas are similar to those for the previous theorems. We just need to show $\dot V_c + \frac{1}{\delta^2}y^{\rm T}y - w^{\rm T}w \le 0$, first for $x \in E_k$, then use the properties of V_c and the controller to extend the result to other x. By Theorem 4, the quantity δ gives an upper bound for the $\mathcal{L}_2$ gain. The following optimization problem can be formulated for suppression of the $\mathcal{L}_2$ gain:

$$\inf_{\lambda_{ijk}\ge 0, Q_k=Q_k^{\rm T}, Y_k} \delta \quad \text{s.t. } (23). \quad (24)$$

5 Examples

Example 1 Consider a second-order LDI taken from (Blanchini & Megretski, 1999),

$$\dot x \in {\rm co}\{A_1 x + B_1 u, A_2 x + B_2 u\},$$

where

$$A_1 = A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} K \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} -K \\ 1 \end{bmatrix},$$

for some $K > 0$. For the LDI to be quadratically stabilizable by linear feedback, K has to be less than 1, i.e., there exists $\beta > 0$ and $Q > 0$ such that the inequalities

$$Q(A_i + B_iF)^{\mathrm{T}} + (A_i + B_iF)Q \le -\beta Q, \quad i = 1, 2,$$

are satisfied if and only if $K < 1$. However, at $K = 1$, the LDI can be stabilized with a positive convergence rate via nonlinear feedback. By solving (11) with $J = 2, 3, 4$, the convergence rate β can be increased to $0.5633, 0.6364$ and 0.7973, respectively. On the other hand, with $J = 2, 3, 4$, the maximal K for (11) to have a solution $\beta > 0$ is found to be greater than $2.05, 2.9$ and 3.4, respectively. As shown in (Blanchini & Megretski, 1999), for any $K > 0$, the LDI can be stabilized by nonlinear feedback which was explicitly constructed via analytical method based on the geometric structure of the vector field. However, it seems hard to extend the analytical method to other systems.

Example 2 Consider an LDI subject to disturbances,

$$\dot{x} \in \mathrm{co}\{A_1x + B_1u + Ew, A_2x + B_2u + Ew\}, y = Cx,$$

where

$$A_1 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ -0.8 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

When quadratic Lyapunov function is applied to designing a linear state feedback law, the problem is a special case of the one studied in Section 4.2 and the optimization problem is a special case of (24) with $J = 1$. In this case (24) reduces to an LMI problem. The optimal δ for this case is $\delta_1 = 10.7670$. When δ approaches to δ_1, the norm of the feedback gain will approach infinity. If we restrict the norm of the feedback gain to be less than 5000 (via an additional constraint on Q's, i.e., $\varepsilon_1 I < Q < \varepsilon_2 I$), the optimal δ is $\bar{\delta}_1 = 11.8886$. The feedback gain is $F = \begin{bmatrix} -4.25 & -2.63 \end{bmatrix} \times 10^3$.

Next we apply the convex hull function $V_c(x)$ with $J = 2$ to the design of a nonlinear feedback law. By solving (24) with $J = 2$, the minimal δ we have obtained is $\delta_2 = 1.1947$. Again, the norm of one of the feedback gain has to be very large (in the order of 10^{10}) to produce the value δ_2. If we restrict the norm of $F_k = Y_k Q_k^{-1}$ to be less than 1000, then the best δ we have computed is $\bar{\delta}_2 = 1.8477$. Other variables corresponding to this value of $\delta = \bar{\delta}_2$ are

$$F_1 = \begin{bmatrix} -815.05 & -579.43 \end{bmatrix}, F_2 = \begin{bmatrix} -58.74 & -28.49 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 15.68 & -20.53 \\ -20.5376 & 27.9733 \end{bmatrix}, Q_2 = \begin{bmatrix} 5.86 & -8.39 \\ -8.39 & 15.89 \end{bmatrix}.$$

Here we compare the output responses for the two designs under the disturbance $w(t) = 1$ for $t \in [0, 1]$ and $w(t) = 0$ for $t > 1$. The switching between $\dot{x} = A_1x + B_1f(x) + Ew$ and $\dot{x} = A_2x + B_2f(x) + Ew$ is chosen such that $\dot{V}_c$ is maximized at each time instant. The two time responses are compared in Fig. 1, where the dashed curve is produced by the linear state feedback $u = Fx$ and the solid curve is produced by the nonlinear feedback constructed from Q_1, Q_2 and $Y_1 = F_1Q_1, Y_2 = F_2Q_2$. For the dashed curve, we have $(\int_0^{50} y^2(t)dt)^{\frac{1}{2}} = 2.6858$,

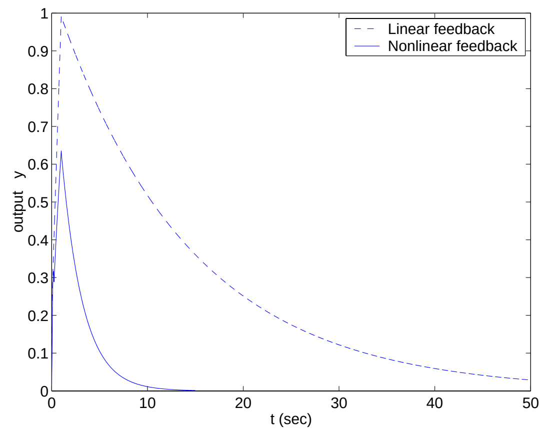

Fig. 1. Two output responses

and for the solid curve, we have $(\int_0^{15} y^2(t)dt)^{\frac{1}{2}} = 0.7984$.

Example 3 Consider the same LDI as in Example 2. Assume that the disturbance is of unit peak type, i.e., $w^{\mathrm{T}}(t)w(t) \le 1$ for all t. We would like to design a control law such that the peak of the output is suppressed. This is achieved by solving (22). When $J = 1$, V_c is a quadratic function and the resulting control law is linear. The optimal solution can be obtained by running β from 0 to ∞. If no restriction on the magnitude of the feedback matrix is imposed, the optimal α is 11.9529. This would require F to go to infinity. If a bound on the norm of F is imposed, say, $\|F\| \le 5000$, we obtain $\alpha = 12.8287 := \alpha_1$ and $F = \begin{bmatrix} -4.22 & -2.61 \end{bmatrix} \times 10^3$. For $J = 2$, we impose a bound $\|F_k\| \le 1000$ and the best α is $2.4573 := \alpha_2$. The other parameters are

$$F_1 = \begin{bmatrix} -813.89 & -577.48 \end{bmatrix}, F_2 = \begin{bmatrix} -62.46 & -30.39 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 18.12 & -23.47 \\ -23.47 & 31.87 \end{bmatrix}, Q_2 = \begin{bmatrix} 6.89 & -9.99 \\ -9.99 & 19.13 \end{bmatrix}.$$

The two level sets resulting from $J = 1$ and $J = 2$ are plotted in Fig. 2, where the outer dashed boundary is that of the ellipsoid $\mathcal{E}(Q^{-1})$ and the inner boundary (in thick curve) is that of L_{V_c} composed from Q_1 and Q_2. A trajectory under the linear control $u = Fx$ is plotted. It starts from near the origin and ends very close to $\partial\mathcal{E}(Q^{-1})$. The switching strategy and the value of w

are chosen such that $\dot{V}_c$ is maximized. Another output response is generated under the nonlinear control. The two responses are plotted in Fig. 3.

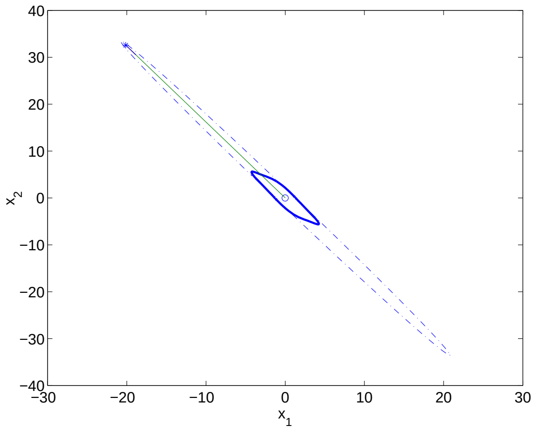

Fig. 2. Two reachable sets and a trajectory by linear control

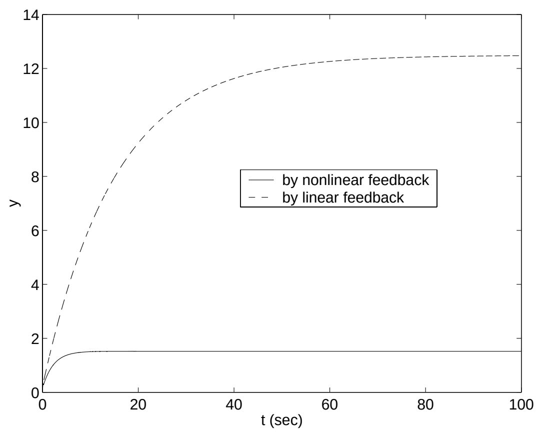

Fig. 3. Two output responses

6 Conclusions

We developed LMI-based methods for the construction of nonlinear feedback laws for linear differential inclusions. The convex hull functions are used to guide the design for achieving a few objectives of robust stabilization and performance. The advantages of nonlinear feedback over linear feedback has been demonstrated through some numerical examples. It is expected that the design methods can be extended to deal with other performances, such as the input-to-state, input-to-output and state-to-output performances studied in (Boyd et al, 1994).

A Proof of Theorem 1

Since the closed-loop system is homogeneous of degree one, V_c is homogeneous of degree two and $\nabla V_c(\alpha x) = \alpha \nabla V_c(x)$, we only need to restrict our attention to the boundary of the 1-level set, ∂L_{V_c}. The rest of the proof is proceeded with two steps. We first prove that (10) is satisfied for all extreme points of L_{V_c}, in particular, for all $x \in E_k$, $k \in I[1, J]$. Next we use Lemma 2 to express an arbitrary $x \in \partial L_{V_c}$ as a convex combination of a set of extreme points, say, $x_k \in E_k$, $k = 1, 2, \cdots, J_0$. Again by Lemma 2 the gradient of V_c at x is the same as that at each x_k. Finally (10) follows from the fact that $f(x)$ is a convex combination of $f(x_k)$'s and that $V_c(x) = V_c(x_k)$ for each k.

Now consider $x \in E_k$ for some $k \in I[1, J]$. Then $V_c(x) = x^{\mathrm{T}} Q_k^{-1} x = 1$ and $\gamma^*(x)$ is a vector whose kth element is 1 and the rest are zeros. Hence $F(\gamma^*(x)) = Y_k Q_k^{-1}$ and $\nabla V_c(x) = 2Q_k^{-1}x$. By Lemma 1,

$$\sum_{j=1}^{J} \lambda_{ijk} x^{\mathrm{T}} Q_k^{-1}(Q_j - Q_k)Q_k^{-1} x \leq 0, \ i \in I[1, N].$$

Let $F_k = Y_k Q_k^{-1}$. Then $f(x) = F_k x$. Multiply (7) from left and from right with Q_k^{-1}, we have

$$\begin{aligned}(A_i + B_i F_k)^{\mathrm{T}} Q_k^{-1} + Q_k^{-1}(A_i + B_i F_k) \\ \leq \sum_{j=1}^{J} \lambda_{ijk} Q_k^{-1}(Q_j - Q_k)Q_k^{-1} - \beta Q_k^{-1}, \ i \in I[1, N].\end{aligned}$$

It follows that

$$\begin{aligned}x^{\mathrm{T}}((A_i + B_i F_k)^{\mathrm{T}} Q_k^{-1} + Q_k^{-1}(A_i + B_i F_k))x \\ \leq -\beta x^{\mathrm{T}} Q_k^{-1} x = -\beta V_c(x), \quad i \in I[1, N].\end{aligned}$$

Hence for every $x \in E_k$ and $i \in I[1, N]$,

$$\begin{aligned}\nabla V_c(x)^{\mathrm{T}}(A_i x + B_i f(x)) &= 2x^{\mathrm{T}} Q_k^{-1}(A_i + B_i F_k)x \\ &\leq -\beta V_c(x). \qquad (A.1)\end{aligned}$$

This implies that (10) is satisfied for all $x \in E_k$.

Next we consider an arbitrary $x \in \partial L_{V_c}$. By Lemma 2, x is a convex combination of a set of x_k's, each of which belongs to a certain E_k. For simplicity, assume that $\gamma_k^*(x) > 0$ for $k = 1, 2, \cdots, J_0$ and $\gamma_k^*(x) = 0$ for $k > J_0$. Then $x = \sum_{k=1}^{J_0} \gamma_k^* x_k$. Recalling from Lemma 2 that $\nabla V_c(x) = 2Q(\gamma^*)^{-1}x$ and

$$Q(\gamma^*)^{-1}x = Q_k^{-1}x_k, \quad k \in I[1, J_0], \qquad (A.2)$$

we have

$$\begin{aligned}F(\gamma^*)x = Y(\gamma^*)Q(\gamma^*)^{-1}x &= \sum_{k=1}^{J_0} \gamma_k^* F_k Q_k Q_k^{-1} x_k \\ &= \sum_{k=1}^{J_0} \gamma_k^* F_k x_k. \qquad (A.3)\end{aligned}$$

Hence

$$A_i x + B_i F(\gamma^*) x = \sum_{k=1}^{J_0} \gamma_k^* (A_i x_k + B_i F_k x_k). \qquad (A.4)$$

It follows that

$$\begin{aligned}
&\nabla V_c(x)^{\mathrm{T}} (A_i x + B_i F(\gamma^*) x) \\
&\quad = 2x^{\mathrm{T}} Q(\gamma^*)^{-1} \sum_{k=1}^{J_0} \gamma_k^* (A_i x_k + B_i F_k x_k) \\
&\quad = 2 \sum_{k=1}^{J_0} \gamma_k^* x_k^{\mathrm{T}} Q(\gamma^*)^{-1} (A_i + B_i F_k) x_k \\
&\quad = 2 \sum_{k=1}^{J_0} \gamma_k^* x_k^{\mathrm{T}} Q_k^{-1} (A_i + B_i F_k) x_k. \qquad (A.5)
\end{aligned}$$

Since $x_k \in E_k$, by (A.1) and noting that $V_c(x_k) = V_c(x)$ for each k, we have

$$\nabla V_c(x)^{\mathrm{T}} (A_i x + B_i F(\gamma^*) x) \leq -\sum_{k=1}^{J_0} \gamma_k^* \beta V_c(x_k) = -\beta V_c(x),$$

which shows (10). Since $Y(\gamma^*)$ and $Q(\gamma^*)$ are continuous in γ^*, and $Q(\gamma) > 0$ for all $\gamma \in \Gamma^J$, the continuity of $f(x) = F(\gamma^*(x))x$ follows from that of $\gamma^*(x)$. □

References

Blanchini, F. (1995). Non-quadratic Lyapunov functions for robust control. *Automatica*, 31, 451-461.

Blanchini, F. & Megretski, A. (1999). Robust state feedback control of LTV systems: nonlinear is better than linear. *IEEE Transactions on Automatic Control*, 44(4), 802-807.

Boyd, S., El Ghaoui, L., Feron, E. & Balakrishnan, V. (1994). *Linear Matrix Inequalities in Systems and Control Theory*. SIAM Studies in Appl. Mathematics, Philadelphia.

Chesi, G., Garulli, A., Tesi, A. and Vicino, A. (2003). Homogeneous Lyapunov functions for systems with structured uncertainties. *Automatica*, 39, 1027-1035.

Goebel, R., Hu, T. & Teel, A. R. (2005). Dual matrix inequalities in stability and performance analysis of linear differential/difference inclusions. In *Current Trends in Nonlinear Systems and Control*. Birkhauser.

Goebel, R., Teel, A. R., Hu, T. & Lin, Z. (2006). Conjugate convex Lyapunov functions for dual linear differential equations. *IEEE Transactions on Automatic Control*, 51(4), 661-666.

Hassibi, A., How, J. and Boyd, S. (1999). A path-following method for solving BMI problems in control. *Proc. of American Contr. Conf.*, 1385-1389.

Hu, T., Goebel, R., Teel, A. R. & Lin, Z. (2005). Conjugate Lyapunov functions for saturated linear systems. *Automatica*, 41(11), 1949-1956.

Hu, T. & Lin, Z. (2003). Composite quadratic Lyapunov functions for constrained control systems. *IEEE Trans. Automat. Contr.*, 48, 440-450.

Hu, T. & Lin , Z. (2004). Properties of composite quadratic Lyapunov functions. *IEEE Trans. Automat. Contr.*,49, 1162-1167.

Hu, T. & Lin , Z. (2005). Absolute stability analysis of discrete-time systems with composite quadratic Lyapunov functions. *IEEE Trans. Automat. Contr.* 50, 781-798.

Hu, T., Teel, A. R. & Zaccarian, L. (2006). Stability and performance for saturated systems via quadratic and non-quadratic Lyapunov functions. *IEEE Trans. Automatic Control*, to appear.

Jarvis-Wloszek, Z. & Packard, A. K. (2002). An LMI method to demonstrate simultaneous stability using non-quadratic polynomial Lyapunov functions. *IEEE Conf. on Dec. and Contr.*, 287-292, Las Vegas, NV.

Khargonekar, P. P., Petersen, I. R. & Rotea, M. A. (1988). H_∞-optimal control with state-feedback. *IEEE Trans. Automat. Contr.*, 33(8), 786-788.

Molchanov, A. P. (1989). Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Sys. & Contr. Lett.*, 13, 59-64.

Polanski, A. (1997). Lyapunov function construction by linear programming. *IEEE Trans. Automat. Contr.*, 42, 1013-1016.

Polanski, A. (2000). On absolute stability analysis by polyhedral Lyapunov functions. *Automatica*, 36, 573-578.

Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press.

Xie, L., Shishkin, S. & Fu, M. (1997). Piecewise Lyapunov functions for robust stability of linear time-varying systems. *Sys. & Contr. Lett.*, 31, 165-171.

Yfoulis, C. A., & Shorten, R. (2004). A numerical technique for the stability analysis of linear switched systems. *Int. J. Contr.*, 77, 1019-1039.