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Brief Paper

An analysis and design method for linear systems subject to actuator saturation and disturbance[☆]

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Abstract

We present a method for estimating the domain of attraction of the origin for a system under a saturated linear feedback. A simple condition is derived in terms of an auxiliary feedback matrix for determining if a given ellipsoid is contractively invariant. This condition is shown to be less conservative than the existing conditions which are based on the circle criterion or the vertex analysis. Moreover, the condition can be expressed as linear matrix inequalities (LMIs) in terms of all the varying parameters and hence can easily be used for controller synthesis. This condition is then extended to determine the invariant sets for systems with persistent disturbances. LMI based methods are developed for constructing feedback laws that achieve disturbance rejection with guaranteed stability requirements. The effectiveness of the developed methods is illustrated with examples.

Keywords: Actuator saturation; Stability; Domain of attraction; Invariant set; Disturbance rejection

1. Introduction

In this paper, we are interested in the control of linear systems subject to actuator saturation and persistent disturbances:

$$\dot{x} = Ax + B\sigma(u) + Ew, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \quad w \in \mathbf{R}^q, \tag{1}$$

where x is the state, u is the control, w is the disturbance and $\sigma(\cdot)$ is the standard saturation function. Our first concern is the closed-loop stability (when $w = 0$) under a given linear state feedback $u = Fx$. There has been a lot of work on this topic (see, e.g., Davison & Kurak, 1971; Gilbert & Tan, 1991; Hindi & Boyd, 1998; Khalil, 1996; Loparo & Blankenship, 1978; Pittet, Tarbouriech, & Burgat, 1997; Vanelli & Vidyasagar, 1985; Weissenberger, 1968 and the references therein). In particular, various simple and general methods for estimating the domain of attraction have been developed by applying the absolute stability analysis tools, such as the circle and Popov criteria (see, e.g., Hindi & Boyd, 1998; Khalil, 1996; Pittet et al., 1997 ; Weissenberger, 1968), where the saturation is treated as a locally sector bounded nonlinearity and the domain of attraction is estimated by use of quadratic and Lur'e type Lyapunov functions. In Hindi and Boyd (1998) and Pittet et al. (1997), the condition for local stability and some performance problems are expressed in terms of (nonlinear) matrix inequalities in system parameters and other auxiliary optimization parameters. By fixing some of the parameters, these matrix inequalities simplify to linear matrix inequalities (LMIs) and can be treated with the LMI software.

Since the circle criterion is applicable to general memoryless sector bounded nonlinearities, we can expect the conservatism in estimating the domain of attraction when it is applied to the saturation nonlinearity. In this paper, a less conservative estimation of the domain of attraction is obtained by using a quadratic Lyapunov function. This is made possible by exploring the special property of saturation. Moreover, since this condition is given in terms of LMIs, it is very easy to handle in both analysis and design.

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In the presence of disturbance, we are interested in knowing if there exists a bounded invariant set such that all the trajectories starting from inside of it will remain in it. This problem was addressed by Blanchini (1990) and Blanchini (1994). In this paper, we would further like to synthesize feedback laws that have the ability to reject the disturbance. Here disturbance rejection is in the sense that, there is a small (as small as possible) neighborhood of the origin such that all the trajectories starting from the origin will remain in it. This performance was analyzed by Hindi and Boyd (1998) for the class of disturbances with finite energy. In this paper, we will deal with persistent disturbances and propose a controller design method.

Furthermore, we are also interested in the problem of asymptotic disturbance rejection with nonzero initial states. A related problem was addressed by Hu and Lin (2001b) and Saberi, Lin, and Teel (1996), where the disturbances are input additive and enter the system before the saturating actuator, i.e., the system has a state equation: $\dot{x} = Ax + B\sigma(u + w)$. It is shown in these papers that given any positive number D, any compact subset X_0 of the null controllable region and any arbitrarily small neighborhood X_∞ of the origin, there is a feedback control such that any trajectory starting from within X_0 will enter X_∞ in a finite time for all disturbances w:$\|w\|_\infty \leqslant D$. We, however, could not expect to have this nice result for system (1), where the disturbance enters the system after the saturating actuator. If w or E is sufficiently large, it may even be impossible to keep the state bounded. What we can expect is to have a set X_0 (as large as we can get) and a set X_∞ (as small as we can get) such that all the trajectories starting from X_0 will enter X_∞ in a finite time and remain in it thereafter.

This paper is organized as follows. Section 2 addresses the analysis of and design for closed-loop stability. Section 3 addresses issues related to disturbance rejection. A brief concluding remark is given in Section 4.

2. Stability analysis

2.1. Problem statement

Consider the open-loop system

$$\dot{x} = Ax + B\sigma(u), \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \tag{2}$$

where $\sigma(\cdot)$ is the standard saturation function of appropriate dimensions. In the above system, $\sigma : \mathbf{R}^m \to \mathbf{R}^m$, and $\sigma(u) = [\sigma(u_1)\ \sigma(u_2)\ \cdots\ \sigma(u_m)]^{\mathrm{T}}$, where $\sigma(u_i) = \mathrm{sgn}(u_i)\min\{1, |u_i|\}$. Here we have slightly abused the notation by using σ to denote both the scalar valued and the vector valued saturation functions. Suppose that a state feedback $u = Fx$ has been designed such that $A + BF$ is Hurwitz. We would like to know how the closed-loop system behaves in the presence of saturation nonlinearity, in particular, to what extent the stability is preserved. Our main objective in this section is to obtain an estimate of the domain of attraction of the origin for the closed-loop system

$$\dot{x} = Ax + B\sigma(Fx). \tag{3}$$

Denote the ith column of B as b_i and the ith row of F as f_i. Then $BF = b_1 f_1 + \cdots + b_m f_m$. For a matrix $F \in \mathbf{R}^{m\times n}$, define

$$\mathscr{L}(F) := \{x \in \mathbf{R}^n\colon\ |f_i x| \leqslant 1, i \in [1, m]\}.$$

If F is the feedback matrix, then $\mathscr{L}(F)$ is the region in the state space where the control is linear in x.

For $x(0) = x_0 \in \mathbf{R}^n$, denote the state trajectory of system (3) as $\psi(t, x_0)$. Then the *domain of attraction* of the origin is

$$\mathscr{S} := \left\{x_0 \in \mathbf{R}^n\colon\ \lim_{t\to\infty} \psi(t, x_0) = 0\right\}.$$

Let $P \in \mathbf{R}^{n\times n}$ be a positive-definite matrix. Denote

$$\mathscr{E}(P, \rho) = \{x \in \mathbf{R}^n\colon\ x^{\mathrm{T}} P x \leqslant \rho\}.$$

Let $V(x) = x^{\mathrm{T}} P x$. The ellipsoid $\mathscr{E}(P, \rho)$ is said to be *contractively invariant* if $\dot{V}(x) = 2x^{\mathrm{T}} P(Ax + B\sigma(Fx)) < 0$ for all $x \in \mathscr{E}(P, \rho)\backslash\{0\}$. Clearly, if $\mathscr{E}(P, \rho)$ is contractively invariant, then it is inside the domain of attraction. We will develop conditions under which $\mathscr{E}(P, \rho)$ is contractively invariant and hence obtain an estimate of the domain of attraction.

2.2. A set invariance condition based on circle criterion

A multivariable circle criterion is presented in Khalil (1996, Theorem 10.1) and is applied to estimate the domain of attraction for system (3), with a given feedback gain F, in Hindi and Boyd (1998) and Pittet et al. (1997).

Proposition 1 (Khalil, 1996; Pittet et al., 1997). *Assume that (F, A, B) is controllable and observable. Given an ellipsoid $\mathscr{E}(P, \rho)$, if there exist positive diagonal matrices $K_1, K_2 \in \mathbf{R}^{n\times n}$ with $K_1 < I$, $K_1 + K_2 \geqslant I$ such that*

$$(A + BK_1F)^{\mathrm{T}} P + P(A + BK_1F) + \tfrac{1}{2}(F^{\mathrm{T}} K_2 + PB)(K_2 F + B^{\mathrm{T}} P) < 0 \tag{4}$$

and $\mathscr{E}(P, \rho) \subset \mathscr{L}(K_1 F)$, then $\mathscr{E}(P, \rho)$ is a contractively invariant set and hence inside the domain of attraction.

A similar condition based on circle criterion is given in Hindi and Boyd (1998). These conditions are then used for stability and performance analysis with LMI software in Hindi and Boyd (1998) and Pittet et al. (1997). Since inequality (4) is not jointly convex in K_1, K_2 and P, these parameters need to be optimized separately and there is no guarantee that the global optimal solutions can be obtained for related problems.

2.3. An improved condition for set invariance

We will develop a less conservative set invariance condition by exploring the special property of the saturation nonlinearity. It is based on direct Lyapunov function analysis in terms of an auxiliary feedback matrix $H \in \mathbf{R}^{m\times n}$. This condition turns out to be equivalent to some LMIs. Denote the ith row of H as h_i. For two matrices $F, H \in \mathbf{R}^{m\times n}$ and a vector $v \in \mathbf{R}^m$, denote

$$M(v,F,H) = \begin{bmatrix} v_1 f_1 + (1-v_1)h_1 \\ \vdots \\ v_m f_m + (1-v_m)h_m \end{bmatrix}. \tag{5}$$

Let $\mathscr{V} = \{v \in \mathbf{R}^m\colon v_i = 1 \text{ or } 0\}$. There are 2^m elements in $\mathscr{V}$. We will use a $v \in \mathscr{V}$ to choose from the rows of F and H to form a new matrix $M(v,F,H)$: if $v_i = 1$, then the ith row of $M(v,F,H)$ is f_i and if $v_i = 0$, then the ith row of $M(v,F,H)$ is h_i. For example, suppose $m = 2$, then

$$\{M(v,F,H)\colon v \in \mathscr{V}\} = \left\{H, \begin{bmatrix} h_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ h_2 \end{bmatrix}, F\right\}.$$

Theorem 1. *Given an ellipsoid $\mathscr{E}(P,\rho)$, if there exists an $H \in \mathbf{R}^{m\times n}$ such that*

$$(A + BM(v,F,H))^{\mathrm{T}}P + P(A + BM(v,F,H)) < 0 \tag{6}$$

for all $v \in \mathscr{V}$ and $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$, i.e., $|h_i x| \leqslant 1$ for all $x \in \mathscr{E}(P,\rho)$, $i \in [1,m]$, then $\mathscr{E}(P,\rho)$ is a contractively invariant set.

Proof. Let $V(x) = x^{\mathrm{T}}Px$, we need to show that

$$\dot V(x) = 2x^{\mathrm{T}}P(Ax + B\sigma(Fx)) < 0 \quad \forall x \in \mathscr{E}(P,\rho)\backslash\{0\}.$$

Here we have

$$\begin{aligned} \dot V(x) &= 2x^{\mathrm{T}}A^{\mathrm{T}}Px + 2x^{\mathrm{T}}PB\sigma(Fx) \\ &= 2x^{\mathrm{T}}A^{\mathrm{T}}Px + \sum_{i=1}^{m} 2x^{\mathrm{T}}Pb_i\sigma(f_i x). \end{aligned}$$

For each term $2x^{\mathrm{T}}Pb_i\sigma(f_i x)$,

(1) If $x^{\mathrm{T}}Pb_i \geqslant 0$ and $f_i x \leqslant -1$, then $2x^{\mathrm{T}}Pb_i\sigma(f_i x) = -2x^{\mathrm{T}}Pb_i \leqslant 2x^{\mathrm{T}}Pb_i h_i x$. Here we note that $-1 \leqslant h_i x$ $\forall x \in \mathscr{E}(P,\rho)$.
(2) If $x^{\mathrm{T}}Pb_i \geqslant 0$ and $f_i x \geqslant -1$, then $\sigma(f_i x) \leqslant f_i x$ and $2x^{\mathrm{T}}Pb_i\sigma(f_i x) \leqslant 2x^{\mathrm{T}}Pb_i f_i x$.
(3) If $x^{\mathrm{T}}Pb_i \leqslant 0$ and $f_i x \geqslant 1$, then $2x^{\mathrm{T}}Pb_i\sigma(f_i x) = 2x^{\mathrm{T}}Pb_i \leqslant 2x^{\mathrm{T}}Pb_i h_i x$. Here we note that $1 \geqslant h_i x$ $\forall x \in \mathscr{E}(P,\rho)$.
(4) If $x^{\mathrm{T}}Pb_i \leqslant 0$ and $f_i x \leqslant 1$, then $\sigma(f_i x) \geqslant f_i x$ and $2x^{\mathrm{T}}Pb_i\sigma(f_i x) \leqslant 2x^{\mathrm{T}}Pb_i f_i x$.

Combining all the four cases, we have

$$2x^{\mathrm{T}}Pb_i\sigma(f_i x) \leqslant \max\{2x^{\mathrm{T}}Pb_i h_i x,\ 2x^{\mathrm{T}}Pb_i f_i x\}$$

for every $x \in \mathscr{E}(P,\rho)$ and each $i \in [1,m]$. Therefore, for every $x \in \mathscr{E}(P,\rho)$,

$$\dot V(x) \leqslant 2x^{\mathrm{T}}A^{\mathrm{T}}Px + \sum_{i=1}^{m} \max\{2x^{\mathrm{T}}Pb_i h_i x,\ 2x^{\mathrm{T}}Pb_i f_i x\}.$$

Now we associate every $x \in \mathscr{E}(P,\rho)$ with a vector $v(x) \in \mathscr{V}$ as follows: if $2x^{\mathrm{T}}Pb_i h_i x < 2x^{\mathrm{T}}Pb_i f_i x$, then we set $v_i = 1$, otherwise we set $v_i = 0$. It follows that

$$\begin{aligned} \dot V(x) &\leqslant 2x^{\mathrm{T}}A^{\mathrm{T}}Px + 2\sum_{i=1}^{m}(v_i x^{\mathrm{T}}Pb_i f_i x + (1-v_i)x^{\mathrm{T}}Pb_i h_i x) \\ &= 2x^{\mathrm{T}}A^{\mathrm{T}}Px + 2x^{\mathrm{T}}P\left(\sum_{i=1}^{m} b_i(v_i f_i + (1-v_i)h_i)\right)x \\ &= 2x^{\mathrm{T}}(A + BM(v,F,H))^{\mathrm{T}}Px. \end{aligned}$$

In view of (6), we have that $\dot V(x) < 0$ for all $x \in \mathscr{E}(P,\rho)\backslash\{0\}$. $\square$

A geometric interpretation of Theorem 1 can be found in Hu and Lin (2001a). If we restrict H to be K_1F, where K_1 is the same as that in Proposition 1, then we have

Corollary 1. *Given an ellipsoid $\mathscr{E}(P,\rho)$, if there exists a positive diagonal matrix $K_1 \in \mathbf{R}^{n\times n}$ such that*

$$(A + BM(v,F,K_1F))^{\mathrm{T}}P + P(A + BM(v,F,K_1F)) < 0$$

for all $v \in \mathscr{V}$ and $\mathscr{E}(P,\rho) \subset \mathscr{L}(K_1F)$, then $\mathscr{E}(P,\rho)$ is a contractively invariant set.

This corollary is equivalent to Theorem 10.4 in Khalil (1996) when applied to saturation nonlinearity. Obviously, the condition in Corollary 1 is more conservative than that in Theorem 1 because the latter provides more freedom in choosing the H matrix. However, it is evident from Khalil (1996) that the condition in Proposition 1 is even more conservative than that in Corollary 1. Computations show that in general, for a fixed P, Theorem 1 allows for a larger ρ than Corollary 1. Therefore, Theorem 1 offers a wider choice of invariant ellipsoids for optimization and will lead to less conservative estimation of the domain of attraction.

2.4. Estimation of the domain of attraction

With all the ellipsoids satisfying the set invariance condition, we would like to choose from among them the "largest" one to get a least conservative estimation of the domain of attraction. In the literature (see, e.g., Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Davison &

Kurak, 1971; Pittet et al., 1997), the largeness of an ellipsoid is usually measured by its volume. Here, we will take its shape into consideration. Let $X_{\rm R} \subset \mathbf{R}^n$ be a prescribed bounded convex set. For a set $S \subset \mathbf{R}^n$, define

$$\alpha_{\rm R}(S) := \sup\{\alpha > 0:\ \alpha X_{\rm R} \subset S\}.$$

If $\alpha_{\rm R}(S) \geqslant 1$, then $X_{\rm R} \subset S$. Two typical types of $X_{\rm R}$ are the ellipsoids

$$X_{\rm R} = \mathscr{E}(R,1) = \{x \in \mathbf{R}^n:\ x^{\rm T} R x \leqslant 1\}, \quad R > 0$$

and the polyhedrons

$$X_{\rm R} = \mathrm{co}\{x_1, x_2, \ldots, x_l\},$$

where "co" denotes the convex hull.

Theorem 1 gives a condition for an ellipsoid to be inside the domain of attraction. Now we would like to choose from all the $\mathscr{E}(P,\rho)$'s that satisfy the condition such that the quantity $\alpha_{\rm R}(\mathscr{E}(P,\rho))$ is maximized. This problem can be formulated as

$$\begin{aligned} \sup_{P>0,\rho,H} \quad & \alpha \\ \text{s.t.} \quad & \text{(a) } \alpha X_{\rm R} \subset \mathscr{E}(P,\rho), \\ & \text{(b) } (A + BM(v,F,H))^{\rm T} P \\ & \qquad + P(A + BM(v,F,H)) < 0 \ \ \forall v \in \mathscr{V}, \\ & \text{(c) } \mathscr{E}(P,\rho) \subset \mathscr{L}(H). \end{aligned} \tag{7}$$

If we replace α with $\log\det(P/\rho)^{-1}$ and remove constraint (a), then we obtain the problem of maximizing the volume of $\mathscr{E}(P,\rho)$. Similar modification can be made to other optimization problems to be formulated in this paper. Moreover, the following procedure to transform (7) into a convex optimization problem with LMI constraints can be adapted to the corresponding volume maximization (or minimization) problems.

Now we transform the constraints of (7) into LMIs. If $X_{\rm R}$ is a polyhedron, then by Schur complement, (a) is equivalent to

$$\alpha^2 x_i^{\rm T}\left(\frac{P}{\rho}\right) x_i \leqslant 1 \quad \Leftrightarrow \quad \begin{bmatrix} \dfrac{1}{\alpha^2} & x_i^{\rm T} \\ x_i & \left(\dfrac{P}{\rho}\right)^{-1} \end{bmatrix} \geqslant 0 \tag{8}$$

for all $i \in [1,l]$. If $X_{\rm R}$ is an ellipsoid $\mathscr{E}(R,1)$, then (a) is equivalent to

$$\frac{R}{\alpha^2} \geqslant \left(\frac{P}{\rho}\right) \quad \Leftrightarrow \quad \begin{bmatrix} \dfrac{1}{\alpha^2} R & I \\ I & \left(\dfrac{P}{\rho}\right)^{-1} \end{bmatrix} \geqslant 0. \tag{9}$$

Constraint (b) is equivalent to

$$\left(\frac{P}{\rho}\right)^{-1} (A + BM(v,F,H))^{\rm T} + (A + BM(v,F,H))\left(\frac{P}{\rho}\right)^{-1} < 0 \quad \forall v \in \mathscr{V}. \tag{10}$$

From Hindi and Boyd (1998), constraint (c) is equivalent to

$$\rho h_i P^{-1} h_i^{\rm T} \leqslant 1 \Leftrightarrow \begin{bmatrix} 1 & h_i\left(\dfrac{P}{\rho}\right)^{-1} \\ \left(\dfrac{P}{\rho}\right)^{-1} h_i^{\rm T} & \left(\dfrac{P}{\rho}\right)^{-1} \end{bmatrix} \geqslant 0 \tag{11}$$

for all $i \in [1,m]$. Let $\gamma = 1/\alpha^2, Q = (P/\rho)^{-1}$ and $G = H(P/\rho)^{-1}$. Also let the ith row of G be g_i, i.e., $g_i = h_i(P/\rho)^{-1}$. Note that $M(v,F,H)Q = M(v,FQ,HQ) = M(v,FQ,G)$. If $X_{\rm R}$ is a polyhedron, then from (8),(10) and (11) optimization problem (7) can be rewritten as

$$\begin{aligned} \inf_{Q>0,G} \quad & \gamma \\ \text{s.t.} \quad & \text{(a1) } \begin{bmatrix} \gamma & x_i^{\rm T} \\ x_i & Q \end{bmatrix} \geqslant 0, \ \ i \in [1,l], \\ & \text{(b) } QA^{\rm T} + AQ + M(v,FQ,G)^{\rm T} B^{\rm T} \\ & \qquad + BM(v,FQ,G) < 0 \ \ \forall v \in \mathscr{V}, \\ & \text{(c) } \begin{bmatrix} 1 & g_i \\ g_i^{\rm T} & Q \end{bmatrix} \geqslant 0, \ \ i \in [1,m], \end{aligned} \tag{12}$$

where all the constraints are given in LMIs. If $X_{\rm R}$ is an ellipsoid, we just need to replace (a1) with

$$\text{(a2)} \quad \begin{bmatrix} \gamma R & I \\ I & Q \end{bmatrix} \geqslant 0.$$

Note that there are 2^m matrix inequalities in constraint (b) corresponding to all $v \in \mathscr{V}$.

Example 1. We use an example of Pittet et al. (1997) to illustrate our results. The system is described by (3) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad F = [-2 \ \ -1].$$

With $x_1 = [-1 \ \ 0.8]^{\rm T}$ and $X_{\rm R} = \mathrm{co}\{x_1, -x_1\}$, we solve (12) and get $\alpha^* = 1/(\gamma^*)^{1/2} = 4.3711$. The maximal ellipsoid is $\mathscr{E}(P^*,1)$,

$$P^* = \begin{bmatrix} 0.1170 & 0.0627 \\ 0.0627 & 0.0558 \end{bmatrix}$$

(see the solid ellipsoid in Fig. 1). The inner dashed ellipsoid is an invariant set obtained by the circle criterion method in Pittet et al. (1997) and the region bounded by the dash–dotted curve is obtained by the Popov method, also in Pittet et al. (1997). We see that both the regions obtained by the circle criterion and by the Popov method can be actually enclosed in a single invariant ellipsoid.

To get a better estimation, we vary x_1 over a unit circle, and solve (12) for each x_1. Let the optimal α be $\alpha^*(x_1)$. The outermost dotted boundary in Fig. 1 is formed by the points $\alpha^*(x_1)x_1$ as x_1 varies along the unit circle.

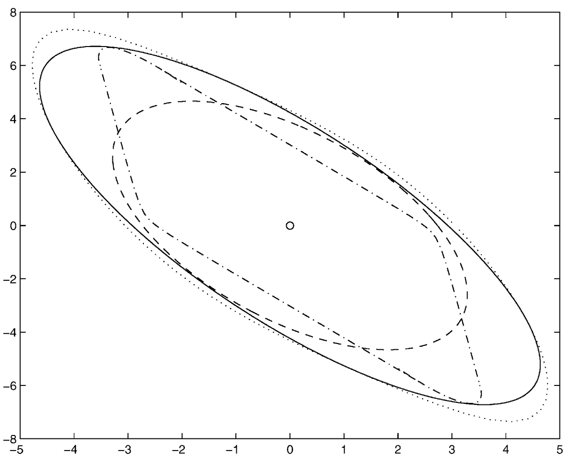

Fig. 1. The invariant sets obtained with different methods.

2.5. Controller design

Now our objective is to choose a feedback matrix $F \in \mathbf{R}^{m\times n}$ such that the estimated domain of attraction as obtained by the method of the last subsection is maximized with respect to X_R. This can be simply done by taking the F in (12) as an extra optimization parameter. To make the optimization easy, we use a new parameter Y to replace FQ in (12) and the resulting LMI problem is

$$\begin{aligned} \inf_{Q>0,Y,G} \quad & \gamma \\ \text{s.t.} \quad & (12\text{a}1), (12\text{c}) \text{ and} \\ & (\text{b})\ QA^{\mathrm{T}} + AQ + M(v,Y,G)^{\mathrm{T}}B^{\mathrm{T}} \\ & \quad +BM(v,Y,G) < 0, \ \forall v \in \mathscr{V}. \end{aligned} \tag{13}$$

The optimal F will be recovered from YQ^{-1}. Consider a simpler optimization problem

$$\begin{aligned} \inf_{Q>0,G} \quad & \gamma \\ \text{s.t.} \quad & (12\text{a}1), (12\text{c}) \quad \text{and} \\ & (\text{b}) \quad QA^{\mathrm{T}} + AQ + G^{\mathrm{T}}B^{\mathrm{T}} + BG < 0. \end{aligned} \tag{14}$$

If $Y = G$, then all the 2^m inequalities in (13b) are the same as (14b). Hence the new problem (14) can be considered as a result from forcing $Y = G$ in (13). On this account, (14) should have an infimum no less than (13). On the other hand, since the 2^m inequalities in (13b) include (14b), problem (14) can also be considered as a result from discarding $2^m - 1$ inequality constraints of (13b). Because of this, (14) should have an infimum no larger than (13). These arguments show that the optimal values of (14) and (13) must be the same. From the above analysis, we see that, if our only purpose is to enlarge the domain of attraction, we might as well solve the simpler optimization problem (14). The freedom in choosing Y can be used to improve other performances beyond large domain of attraction.

3. Disturbance rejection

3.1. Problem statement

Consider the open-loop system

$$\dot{x} = Ax + B\sigma(u) + Ew, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \quad w \in \mathbf{R}^q, \tag{15}$$

where, without loss of generality, we assume that the bounded disturbance w belongs to the set

$$\mathscr{W} := \{w\colon\ w(t)^{\mathrm{T}}w(t) \leqslant 1 \ \forall t \geqslant 0\}.$$

Let the state feedback be $u = Fx$. The closed-loop system is

$$\dot{x} = Ax + B\sigma(Fx) + Ew. \tag{16}$$

For an initial state $x(0) = x_0$, denote the state trajectory of the closed-loop system under w as $\psi(t, x_0, w)$. A set in $\mathbf{R}^n$ is said to be *invariant* if all the trajectories starting from it will remain in it regardless of $w \in \mathscr{W}$. An ellipsoid $\mathscr{E}(P,\rho)$ is said to be *strictly invariant* if $\dot{V} = 2x^{\mathrm{T}}P(Ax + B\sigma(Fx) + Ew) < 0$ for all w such that $w^{\mathrm{T}}w \leqslant 1$ and all $x \in \partial\mathscr{E}(P,\rho)$, the boundary of $\mathscr{E}(P,\rho)$. The notion of invariant set plays an important role in studying the stability and other performances of a system (see, e.g., Blanchini, 1994; Boyd et al., 1994 and the references therein).

Our primary concern is the boundedness of the trajectories for some set of initial states (may be as large as possible). This requires a large invariant set. On the other hand, for the purpose of disturbance rejection, we would also like to have a small invariant set containing the origin in its interior so that a trajectory starting from the origin will stay close to the origin.

To formally state the objectives of this section, we need to extend the notion of the domain of attraction as follows.

Definition 1. Let $\mathscr{B}$ be a bounded invariant set of (16). The domain of attraction of $\mathscr{B}$ is

$$\mathscr{S}(\mathscr{B}) := \left\{x_0 \in \mathbf{R}^n\colon \lim_{t\to\infty} d(\psi(t,x_0,w),\mathscr{B}) = 0 \ \forall w \in \mathscr{W}\right\},$$

where $d(\psi(t, x_0, w), \mathscr{B}) = \inf_{x\in\mathscr{B}} \|\psi(t, x_0, w) - x\|$ is the distance from $\psi(t, x_0, w)$ to $\mathscr{B}$.

The problems we are to address in this section are given as follows:

Problem 1 (Set invariance analysis).

Let F be known. Given an ellipsoid $\mathscr{E}(P,\rho)$, determine if $\mathscr{E}(P,\rho)$ is (strictly) invariant.

Problem 2 (Invariant set enlargement).

Given a bounded set $X_0 \subset \mathbf{R}^n$, design F such that the closed-loop system has an invariant set $\mathscr{E}(P,\rho) \supset \alpha_2 X_0$ with α_2 maximized.

Problem 3 (Disturbance rejection).

Given a set $X_\infty \subset \mathbf{R}^n$, design F such that the closed-loop system has an invariant set $\mathscr{E}(P,\rho) \subset \alpha_3 X_\infty$ with α_3 minimized.

Problem 4 (Disturbance rejection with guaranteed domain of attraction).

Given two reference sets, X_∞ and X_0, design F such that the closed-loop system has an invariant set $\mathscr{E}(P,1) \supset X_0$, and for all $x_0 \in \mathscr{E}(P,1)$, $\psi(t,x_0,w)$ will enter a smaller invariant set $\mathscr{E}(P,\rho_1) \subset \alpha_4 X_\infty$ with α_4 minimized.

3.2. Condition for set invariance

We consider closed-loop system (16) with a given F. The following theorem gives a sufficient condition for the invariance of a set $\mathscr{E}(P,\rho)$.

Theorem 2. *For a given set $\mathscr{E}(P,\rho)$, if there exist an $H \in \mathbf{R}^{m\times n}$ and a positive number η such that*

$$(A+BM(v,F,H))^{\mathrm{T}}P + P(A+BM(v,F,H)) + \frac{1}{\eta}PEE^{\mathrm{T}}P + \frac{\eta}{\rho}P \leqslant 0 (<0) \quad \forall v \in \mathscr{V} \tag{17}$$

and $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$, then $\mathscr{E}(P,\rho)$ is a (strictly) invariant set for system (16).

Proof. We prove the strict invariance. That is, for $V(x) = x^{\mathrm{T}}Px$, we will show that

$$\dot{V} = 2x^{\mathrm{T}}P(Ax + B\sigma(Fx) + Ew) < 0$$

for all $x \in \partial\mathscr{E}(P,\rho)$ and all w such that $w^{\mathrm{T}}w \leqslant 1$. Following the procedure of the proof of Theorem 1, we can show that for every $x \in \mathscr{E}(P,\rho)$, there exists a $v \in \mathscr{V}$ such that

$$2x^{\mathrm{T}}P(Ax + B\sigma(Fx)) \leqslant 2x^{\mathrm{T}}(A+BM(v,F,H))^{\mathrm{T}}Px.$$

Since

$$\begin{aligned} 2x^{\mathrm{T}}PEw &\leqslant \frac{1}{\eta}x^{\mathrm{T}}PEE^{\mathrm{T}}Px + \eta w^{\mathrm{T}}w \\ &\leqslant \frac{1}{\eta}x^{\mathrm{T}}PEE^{\mathrm{T}}Px + \eta \end{aligned}$$

we have

$$\dot{V} \leqslant x^{\mathrm{T}}\left(2(A+BM(v,F,H))^{\mathrm{T}}P + \frac{1}{\eta}PEE^{\mathrm{T}}P\right)x + \eta.$$

It follows from (17) that for all $x \in \mathscr{E}(P,\rho)$,

$$\dot{V} < -\frac{\eta}{\rho}x^{\mathrm{T}}Px + \eta.$$

Observing that on the boundary of $\mathscr{E}(P,\rho), x^{\mathrm{T}}Px = \rho$, hence $\dot{V} < 0$. It follows that $\mathscr{E}(P,\rho)$ is a strictly invariant set. □

Theorem 2 deals with Problem 1 and can be easily used for controller design in Problems 2 and 3. For Problem 2, we can solve the following optimization problem:

$$\begin{aligned} \sup_{P>0,\rho,\eta>0,F,H} \quad & \alpha_2 \\ \text{s.t.} \quad & \alpha_2 X_0 \subset \mathscr{E}(P,\rho), \\ & \mathscr{E}(P,\rho) \subset \mathscr{L}(H) \text{ and (17).} \end{aligned} \tag{18}$$

Let $Q = (P/\rho)^{-1}$, $Y = FQ$ and $G = HQ$, then (17) is equivalent to

$$QA^{\mathrm{T}} + AQ + M(v,Y,G)^{\mathrm{T}}B^{\mathrm{T}} + BM(v,Y,G) + \frac{\rho}{\eta}EE^{\mathrm{T}} + \frac{\eta}{\rho}Q < 0 \quad \forall v \in \mathscr{V}.$$

If we fix ρ/η, then the original optimization constraints can be transformed into LMIs as with (7). The global maximum of α_2 will be obtained by running ρ/η from 0 to ∞. For Problem 3, we have

$$\begin{aligned} \inf_{P>0,\rho,\eta>0,F,H} \quad & \alpha_3 \\ \text{s.t.} \quad & \mathscr{E}(P,\rho) \subset \alpha_3 X_\infty, \\ & \mathscr{E}(P,\rho) \subset \mathscr{L}(H) \text{ and (17),} \end{aligned} \tag{19}$$

which can be solved similarly as Problem 2.

3.3. Disturbance rejection with guaranteed domain of attraction

Given $X_0 \subset \mathbf{R}^n$, if the optimal solution of Problem 2 is $\alpha_2^* > 1$, then there are infinitely many choices of the feedback matrices F's such that X_0 is contained in some invariant ellipsoid. We will use this extra freedom for disturbance rejection, that is, to construct another invariant set $\mathscr{E}(P,\rho_1)$ which is as small as possible with respect to some X_∞. Moreover, X_0 is inside the domain of attraction of $\mathscr{E}(P,\rho_1)$. In this way, all the trajectories starting from X_0 will enter $\mathscr{E}(P,\rho_1) \subset \alpha_4 X_\infty$ for some $\alpha_4 > 0$. Here the number α_4 is a measure of the degree of disturbance rejection.

Before addressing Problem 4, we need to answer the following question: Suppose that for given F and P, both $\mathscr{E}(P,\rho_1)$ and $\mathscr{E}(P,\rho_2), \rho_1 < \rho_2$ are strictly invariant, then under what conditions will the other ellipsoids $\mathscr{E}(P,\rho), \rho \in (\rho_1,\rho_2)$ also be strictly invariant? If they are, then all the trajectories starting from within $\mathscr{E}(P,\rho_2)$ will enter $\mathscr{E}(P,\rho_1)$ and remain inside it.

Theorem 3. *Given two ellipsoids, $\mathscr{E}(P,\rho_1)$ and $\mathscr{E}(P,\rho_2)$, $\rho_2 > \rho_1 > 0$, if there exist $H_1, H_2 \in \mathbf{R}^{m\times n}$ and a positive number η such that*

$$(A+BM(v,F,H_1))^{\mathrm{T}}P + P(A+BM(v,F,H_1)) + \frac{1}{\eta}PEE^{\mathrm{T}}P + \frac{\eta}{\rho_1}P < 0 \quad \forall v \in \mathscr{V}, \tag{20}$$

$$(A+BM(v,F,H_2))^{\mathrm{T}}P+P(A+BM(v,F,H_2))$$
$$+\frac{1}{\eta}PEE^{\mathrm{T}}P+\frac{\eta}{\rho_2}P<0 \quad \forall v\in\mathscr{V} \tag{21}$$

and $\mathscr{E}(P,\rho_1)\subset\mathscr{L}(H_1)$, $\mathscr{E}(P,\rho_2)\subset\mathscr{L}(H_2)$, then for every $\rho\in[\rho_1,\rho_2]$, there exists an $H\in\mathbf{R}^{m\times n}$ such that

$$(A+BM(v,F,H))^{\mathrm{T}}P+P(A+BM(v,F,H))$$
$$+\frac{1}{\eta}PEE^{\mathrm{T}}P+\frac{\eta}{\rho}P<0 \quad \forall v\in\mathscr{V} \tag{22}$$

and $\mathscr{E}(P,\rho)\in\mathscr{L}(H)$. This implies that $\mathscr{E}(P,\rho)$ is also strictly invariant.

Proof. Let $h_{1,i}$ and $h_{2,i}$ be the ith row of H_1 and H_2 respectively. The conditions $\mathscr{E}(P,\rho_1)\subset\mathscr{L}(H_1)$ and $\mathscr{E}(P,\rho_2)\subset\mathscr{L}(H_2)$ are equivalent to

$$\begin{bmatrix}\frac{1}{\rho_1} & h_{1,i}\\ h_{1,i}^{\mathrm{T}} & P\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}\frac{1}{\rho_2} & h_{2,i}\\ h_{2,i}^{\mathrm{T}} & P\end{bmatrix}\geqslant 0,\quad i\in[1,m].$$

Since $\rho\in[\rho_1,\rho_2]$, there exists a $\lambda\in[0,1]$ such that $1/\rho=\lambda(1/\rho_1)+(1-\lambda)/\rho_2$. Let $H=\lambda H_1+(1-\lambda)H_2$. Clearly

$$\begin{bmatrix}\frac{1}{\rho} & h_i\\ h_i^{\mathrm{T}} & P\end{bmatrix}\geqslant 0.$$

From (20) and (21), and by convexity, we have (22). □

In view of Theorem 3, to solve Problem 4, we only need to construct two invariant ellipsoids $\mathscr{E}(P,\rho_1)$ and $\mathscr{E}(P,\rho_2)$ satisfying the condition of Theorem 3 such that $X_0\subset\mathscr{E}(P,\rho_2)$ and $\mathscr{E}(P,\rho_1)\subset\alpha_4X_\infty$ with α_4 minimized. Since ρ_2 can be absorbed into other parameters, we assume for simplicity that $\rho_2=1$ and $\rho_1<1$. Problem 4 can then be formulated as

$$\begin{aligned}\inf_{P>0,0<\rho_1<1,\eta>0,F,H_1,H_2,}\quad & \alpha_4\\ \text{s.t.}\quad & \text{(a) } X_0\subset\mathscr{E}(P,1),\\ & \qquad \mathscr{E}(P,\rho_1)\subset\alpha_4X_\infty,\\ & \text{(b) (20), (21)}\\ & \text{(c) } \mathscr{E}(P,\rho_1)\subset\mathscr{L}(H_1),\\ & \text{(d) } \mathscr{E}(P,1)\subset\mathscr{L}(H_2).\end{aligned} \tag{23}$$

If we fix ρ_1 and η, then (23) can also be transformed into a convex optimization problem with LMI constraints. To obtain the global infimum, we may vary ρ_1 from 0 to 1 and η from 0 to ∞.

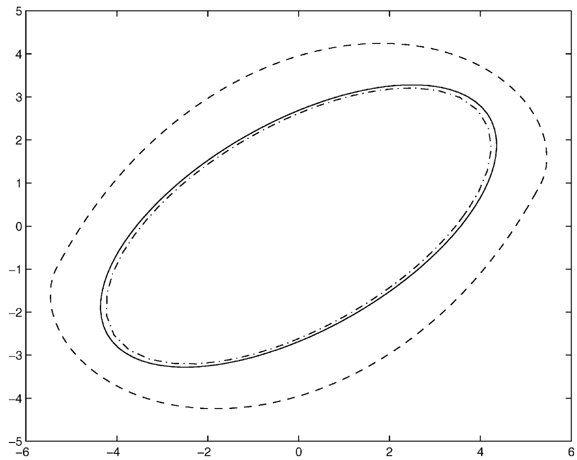

Fig. 2. The invariant ellipsoids and the null controllable region.

Example 2. The open-loop system is described by (15) with

$$A=\begin{bmatrix}0.6 & -0.8\\ 0.8 & 0.6\end{bmatrix},\quad B=\begin{bmatrix}2\\ 4\end{bmatrix},\quad E=\begin{bmatrix}0.1\\ 0.1\end{bmatrix}.$$

The system has a pair of unstable complex poles. We first ignore the disturbance and solve (13) for a feedback with the objective of maximizing the domain of attraction with respect to the unit ball, $X_{\mathrm{R}}=\mathscr{E}(I,1)$. The result is,

$$\alpha_1^*=1/(\gamma^*)^{1/2}=2.4417,$$

$$P_1^*=\begin{bmatrix}0.0752 & -0.0566\\ -0.0566 & 0.1331\end{bmatrix},$$

$$F_1^*=[-0.0025\quad -0.2987]$$

and the invariant ellipsoid is $\mathscr{E}(P_1^*,1)$ (see the larger ellipsoid in Fig. 2). As a comparison, we also plotted the boundary of the null controllable region (Hu & Lin, 2001a) of the open-loop system (see the dashed outer curve).

We next deal with Problem 2. By solving (18) with X_0 being a unit ball, we obtain $\alpha_2^*=2.3195$, with $\eta_2^*=0.019$. The resulting invariant ellipsoid is $\mathscr{E}(P_2^*,1)$, with

$$P_2^*=\begin{bmatrix}0.0835 & -0.0639\\ -0.0639 & 0.1460\end{bmatrix}$$

(see the inner dash–dotted ellipsoid in Fig. 2).

To deal with Problem 3, we solve (19), with X_∞ also being a unit ball. We obtain $\alpha_3^*=0.0606$, which shows that the disturbance can be rejected to a very small level. Now we turn to Problem 4. The optimization result by solving Problem 2 gives us some guide in choosing X_0. Here we choose $X_0=\mathscr{E}(I,2^2)$, $X_\infty=\mathscr{E}(I,1)$. The optimal solution is $\alpha_4^*=0.9725, \eta^*=0.006, \rho_1^*=0.0489$,

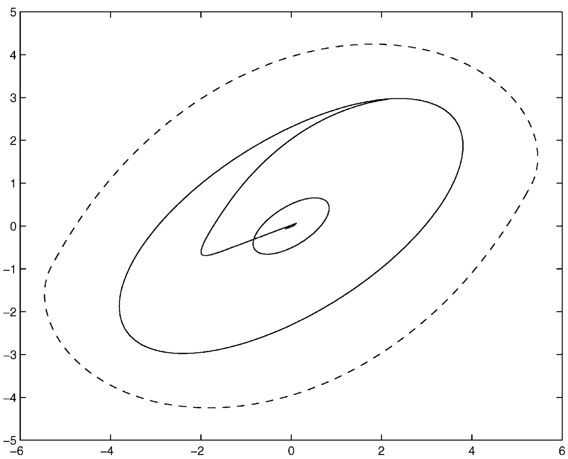

Fig. 3. The invariant ellipsoids and a trajectory.

$F_4^* = [0.2844 \quad -1.4430]$, and

$$P_4^* = \begin{bmatrix} 0.1145 & -0.0922 \\ -0.0922 & 0.1872 \end{bmatrix}.$$

In Fig. 3, the larger ellipsoid is $\mathscr{E}(P_4^*, 1)$, the smaller ellipsoid is $\mathscr{E}(P_4^*, \rho_1)$ and the outermost dashed closed curve is the boundary of the null controllable region. A trajectory is plotted with $x_0 \in \partial\mathscr{E}(P_4^*, 1)$ and $w = \text{sign}(\sin(0.3t))$.

4. Conclusions

We considered linear systems subject to actuator saturation and disturbance. A condition for determining if a given ellipsoid is contractively invariant was derived and shown to be less conservative than the existing conditions that are based on the circle criterion or the vertex analysis. With the aid of this condition, we developed analysis and design methods, both for closed-loop stability and disturbance rejection. Examples were used to demonstrate the effectiveness of these methods.

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