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Brief paper

Analysis of linear systems in the presence of actuator saturation and $\mathscr{L}_2$-disturbances☆

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Abstract

This paper presents a method for the analysis and control design of linear systems in the presence of actuator saturation and $\mathscr{L}_2$-disturbances. A simple condition is derived under which trajectories starting from an ellipsoid will remain inside an outer ellipsoid. The stability and disturbance tolerance/rejection ability of the closed-loop system under a given feedback law is measured by the size of these two ellipsoids and the difference between them. Based on the above mentioned condition, the problem of estimating the largest inner ellipsoid and/or the smallest difference between the two ellipsoid is then formulated as a constrained optimization problem. All the constraints are shown to be equivalent to LMIs. In addition, disturbance rejection ability in terms of $\mathscr{L}_2$ gain is also determined by the solution of an LMI optimization problem. By viewing the feedback gain as an additional free parameter, the optimization problem can easily be adapted for controller design. Numerical examples show that the proposed analysis and design methods significantly improve recent results on the same problems.

Keywords: Actuator saturation; Disturbance rejection; Disturbance tolerance

1. Introduction and problem statement

As a natural research topic beyond stabilization, the problem of disturbance rejection for linear systems subject to actuator saturation has been addressed by many authors. The results on this topic can be divided into two categories according to the way the disturbances enter the system. Examples of works on systems with input additive disturbances include Chitour, Liu, and Sontag (1995), Hu and Lin (2001b), Lin (1997), Lin, Saberi, and Teel (1996), and Liu, Chitour, and Sontag (1996). Because of the input additive nature of the disturbances, very strong results can be established. For neutrally stable open-loop systems, it was shown that a simple linear feedback law render the closed-loop system finite gain $\mathscr{L}_p$-stable (Liu et al., 1996) . Various continuity and incremental-gain properties of the closed-loop system were discussed in detail in Chitour et al. (1995). For a general open-loop system, it was shown that the $\mathscr{L}_p$ gain from the disturbance to the state can be made arbitrarily small by linear feedback if the disturbances are assumed to be bounded in magnitude (Lin et al., 1996). This boundedness assumption on the disturbances can be removed if nonlinear feedback is allowed (Lin, 1997). Also under the boundedness assumption on the magnitude of the disturbances, semi-global practical stabilization on the null controllable region is possible (Hu & Lin, 2001b). Here, the null controllable region is the set of all states that can be driven to the origin by the bounded control from the saturating actuators (Hu & Lin, 2001a). Semi-global practical stabilization is the design of a feedback law, for any (arbitrarily large) compact subset of the null controllable region and any (arbitrarily small) neighborhood of the origin, such that every closed-loop system trajectory that starts from the given set will enter the specified neighborhood in a finite time and remain in it thereafter.

The second category of the works are those on systems where disturbances are not input additive (see, for example,

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Hindi & Boyd, 1998; Hu, Lin, & Chen, 2002; Megretski, 1996; Nguyen & Jabbari, 1997, 1999; Paim, Tarbouriech, Gomes da Silva, & Castelan, 2002; Scherer, Chen, & Allgöwer, 2002; Suarez, Alvarez-Ramirez, Sznaier, & Ibarra-Valdez, 1997). As the disturbances enter the system independently from the bounded control inputs, strong results as in the situation of input additive disturbances cannot be expected. What can be expected is a certain degree of disturbance tolerance of the closed-loop system. Under the boundedness assumption on the magnitude of the disturbances and in the absence of initial condition, the $\mathscr{L}_2$ gain analysis and minimization in the context of both state and output feedback were carried out in Nguyen and Jabbari (1997, 1999). The work of (Hu et al., 2002) proposed a method for analysis and maximization of an ellipsoid which is invariant under magnitude bounded, but persistent disturbances. The works of Hindi and Boyd (1998), Megretski (1996), Paim et al. (2002), Scherer et al. (2002) and Suarez et al. (1997) all consider the situation where disturbances are bounded in energy. In particular, (Suarez et al., 1997) takes an ARE based approach to minimizing the $\mathscr{L}_2$ gain while achieving global stabilization. The work of Megretski (1996) leads to a gain scheduled feedback law that guarantees both closed-loop stability and bounded $\mathscr{L}_2$ gain from the disturbance to the state. The works of (Hindi & Boyd, 1998; Paim et al., 2002; Scherer et al., 2002) formulated and solved the problem of stability analysis and design as optimization problems with LMI or BMI constraints, with the former two papers considering linear feedback laws and the latter using hybrid state feedback laws.

This paper revisits the problem of analysis and control design for linear systems in the presence of actuator saturation and $\mathscr{L}_2$-disturbances. We will consider linear feedback laws. Here no boundedness assumption is made on the magnitude of the disturbances and the system initial conditions are not necessarily zero. Thus the situation considered in this paper is most closely related to the work of Hindi and Boyd (1998) and Paim et al. (2002). More specifically, we consider the following system subject to actuator saturation and $\mathscr{L}_2$-disturbances,

$$\dot{x} = Ax + B\mathrm{sat}(u) + Ew,$$
$$z = Cx, \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{R}^q$ is the disturbance, and $\mathrm{sat}(\cdot)$ is the standard saturation function with unity saturation level. We note that non-unity saturation level can be absorbed into the matrix B and the control u.

For a linear system, the disturbance rejection capability can be measured by the $\mathscr{L}_2$ gain, the largest ratio between the $\mathscr{L}_2$ norms of the output and the disturbance. However, this gain may not be well defined for the closed-loop system of (1) and the state feedback, since a sufficiently large disturbance may drive the state and the output of the system unbounded. For this reason, we need to restrict our attention to the class of disturbances whose energy is bounded by a given value, i.e.,

$$\mathscr{W}_\alpha := \left\{ w : \mathbf{R}_+ \to \mathbf{R}^q : \int_0^\infty w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t \leqslant \alpha \right\}, \tag{2}$$

for some positive number α.

The first question need to be answered is, what is the maximal value of α such that the state will be bounded for all $w \in \mathscr{W}_\alpha$? Here we have two situations, nonzero initial state and zero initial state. The problem related to this question is referred to as disturbance tolerance.

After the maximal α has been determined, say $\alpha_{\max}$, we can move on to study the disturbance rejection capability for $\mathscr{W}_\alpha$, with $\alpha < \alpha_{\max}$. The disturbance rejection capability can be measured by the restricted $\mathscr{L}_2$ gain over a given $\mathscr{W}_\alpha$ or by the largeness of the bound on the state trajectories.

We will approach these problems by establishing a simple condition under which trajectories starting from an ellipsoid will remain inside an outer ellipsoid. The stability and disturbance rejection ability of the closed-loop system under a given feedback law is measured by the sizes of these two ellipsoids and the difference between them. The disturbance tolerance, on the other hand, can be measured by the largest α for which the above two ellipsoid exist. Based on the above mentioned condition, the problem of assessing various stability and disturbance tolerance/rejection ability can be formulated as constrained optimization problems. We will show that all these constraints are equivalent to LMIs and hence the optimization problems can be readily solved. Furthermore, disturbance rejection ability in terms of $\mathscr{L}_2$ gain will also be determined by the solution of an LMI optimization problem. By viewing the feedback gain as an additional free parameter, the optimization problems can easily be adapted for controller design. Numerical examples show that the proposed analysis and design methods significantly improve recent results on these problems.

In developing our results in this paper, we follow the idea of placing the saturated linear feedback law $\mathrm{sat}(Fx)$ in the convex hull of a group of linear controls (Hu & Lin, 2001a). A similar idea was originally used in Hu et al. (2002) to establish a set invariance condition for system (1) under magnitude bounded disturbances. Thus, the current paper is an extension of the work Hu et al. (2002) to systems with $\mathscr{L}_2$-disturbances, under which set invariance cannot be established.

We note that our analysis is based on ellipsoids. There exist alternative approaches, such as the one based on positive invariance of the polyhedron formed by the states for which the actuator does not saturate (Benzaouia & Hmamed, 1993) Our method however results in ellipsoids that extend beyond the linear region of the actuator (see Fig. 2 in Section 4).

The remainder of the paper is organized as follows. Section 2 deals with Problem 1: the assessment of stability and disturbance tolerance/rejection ability of the closed-loop system under a given feedback law. Section 3 briefly explains how the analysis results of Section 2 can be adapted

to solve Problem 2: the design of a feedback law that maximizes the closed-loop stability and disturbance tolerance/rejection capability. Section 4 presents numerical examples to demonstrate the effectiveness of the proposed methods in comparison with the existing methods. Section 5 completes the paper with some concluding remarks.

2. Stability and disturbance tolerance/rejection

Consider the closed-loop system of (1) under the state feedback $u = Fx$. In the presence of disturbance, the basic requirement for the closed-loop system is the boundedness of state trajectories. We usually use an ellipsoid to bound the state trajectories. In the case where the disturbance is bounded by the $\mathscr{L}_\infty$ norm, we may use an invariant ellipsoid to bound the state trajectory (see, e.g., Hu et al., 2002). However, with disturbances bounded by energy rather than by magnitude, there exists no bounded invariant set. What we can do is to use two nested sets, specifically, an inner ellipsoid and an outer ellipsoid, such that all the trajectories starting from the inner ellipsoid will remain in the outer ellipsoid under all $w \in \mathscr{W}_\alpha$. In this section, we will first present conditions under which a pair of ellipsoids possess such property. Then we will use this condition to study various disturbance tolerance/rejection problems.

2.1. Two nested ellipsoids

First we introduce some notation. For a positive definite matrix $P \in \mathbf{R}^{n\times n}$ and a positive number ρ, we define an ellipsoid as

$$\varepsilon(P,\rho) = \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \leqslant \rho\}.$$

Also, for a feedback gain matrix $F \in \mathbf{R}^{m\times n}$, define the set of states for which saturation does not occur as

$$\mathscr{L}(F) = \{x \in \mathbf{R}^n : |F_i x| \leqslant 1,\ i \in [1,m]\},$$

where F_i is the ith row of F.

Let $\mathscr{D}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathscr{D}$ and we denote its elements as $D_i, i = 1, 2^m$. Denote $D_i^- = I - D_i$. It is easy to see that $D_i^- \in \mathscr{D}$. The following lemma from Hu and Lin (2001a) (main idea originated from Hu et al., 2002) will be useful for the development of the main results of this paper.

Lemma 1. *Let $u, v \in \mathbf{R}^m$ with $u = [u_1\ u_2, \ldots, u_m]^{\mathrm{T}}$ and $v = [v_1\ v_2, \ldots, v_m]^{\mathrm{T}}$. Suppose that $|v_i| \leqslant 1$ for all $i \in [1,m]$. Then,*

$$\mathrm{sat}(u) \in \mathrm{co}\{D_i u + D_i^- v : i \in [1, 2^m]\}, \tag{3}$$

where co *denotes the convex hull.*

Theorem 1. *Consider system (1) under a given state feedback law $u=Fx$. Let the positive definite matrix P be given. Then,*

(a) *if there exist an $H \in \mathfrak{R}^{m\times n}$ and a positive number η such that*

$$(A + B(D_iF + D_i^- H))^{\mathrm{T}}P + P(A + B(D_iF + D_i^- H)) + \frac{1}{\eta}PEE^{\mathrm{T}}P \leqslant 0, \quad \forall i \in [1, 2^m] \tag{4}$$

and $\varepsilon(P, 1+\alpha\eta) \subset \mathscr{L}(H)$, then every trajectory of the closed-loop system that starts from inside of $\varepsilon(P,1)$ will remain inside of $\varepsilon(P, 1+\alpha\eta)$ for every $w \in \mathscr{W}_\alpha$.

(b) *if there exist an $H \in R^{m\times n}$ and an $\eta > 0$ such that (4) is satisfied and $\varepsilon(P,\alpha\eta) \subset \mathscr{L}(H)$, then the trajectories of the closed-loop system that start from the origin will remain inside the ellipsoid $\varepsilon(P,\alpha\eta)$ for every $w \in \mathscr{W}_\alpha$.*

Remark 1. We note that, in item (b) of Theorem 1, it is without loss of generality to assume that $\eta = 1$. Otherwise, we can multiply the left-hand side of (4) with $1/\eta$ and obtain

$$(A + B(D_iF + D_i^- H))^{\mathrm{T}}\frac{P}{\eta} + \frac{P}{\eta}(A + B(D_iF + D_i^- H)) + \frac{P}{\eta}EE^{\mathrm{T}}\frac{P}{\eta} \leqslant 0, \quad \forall i \in [1, 2^m]. \tag{5}$$

Let $P_1 = P/\eta$, then P_1 satisfies (4) with $\eta = 1$ and $\varepsilon(P_1,\alpha) = \varepsilon(P,\alpha\eta)$.

Proof of Theorem 1. Select $V(x) = x^{\mathrm{T}}Px$ as the Lyapunov function for the closed-loop system, then, the derivative of V along the trajectories of the closed-loop system can be evaluated as

$$\dot{V} = 2x^{\mathrm{T}}P[Ax + B\mathrm{sat}(Fx) + Ew]. \tag{6}$$

Let $\varepsilon(P,\rho)$ be an ellipsoid and $H \in \mathbf{R}^{m\times n}$ be such that $\varepsilon(P,\rho) \subset \mathscr{L}(H)$, then, by Lemma 1,

$$2x^{\mathrm{T}}P[Ax + B\mathrm{sat}(Fx) + Ew] \leqslant \max_{i\in[1,2^m]} 2x^{\mathrm{T}}P[Ax + B(D_iF + D_i^- H)x + Ew], \quad \forall x \in \varepsilon(P,\rho). \tag{7}$$

Noting that,

$$2x^{\mathrm{T}}PEw \leqslant \frac{1}{\eta}x^{\mathrm{T}}PEE^{\mathrm{T}}Px + \eta w^{\mathrm{T}}w, \quad \forall \eta > 0, \tag{8}$$

we have

$$\dot{V} \leqslant \max_{i\in[1,2^m]} 2x^{\mathrm{T}}P[A + B(D_iF + D_i^- H)]x + \frac{1}{\eta}x^{\mathrm{T}}PEE^{\mathrm{T}}Px + \eta w^{\mathrm{T}}w, \quad \forall x \in \varepsilon(P,\rho). \tag{9}$$

We are now ready to show both items a and b of the theorem. To show item a, set $\rho = 1 + \alpha\eta$. Then, by (9) and the conditions of item (a), we have

$$\dot{V} \leqslant \eta w^{\mathrm{T}}w, \quad \forall x \in \varepsilon(P, 1+\alpha\eta). \tag{10}$$

Integrating both sides of the above inequality from 0 to t results in

$$V(x(t)) \leqslant V(x(0)) + \eta \int_0^t w(\tau)^{\mathrm{T}} w(\tau)\,\mathrm{d}\tau \leqslant V(x(0)) + \alpha\eta. \tag{11}$$

This shows that if $V(x(0)) \leqslant 1$, i.e., $x(0) \in \varepsilon(P,1)$, then $V(x(t)) \leqslant 1+\alpha\eta$ and hence $x(t) \in \varepsilon(P, 1+\alpha\eta)$ for all $t \geqslant 0$.

To show item b, set $\rho = \alpha\eta$. Then, inequality (9), the conditions of item b and $x(0) = 0$ imply that

$$V(x(t)) \leqslant \eta \int_0^t w(\tau)^{\mathrm{T}} w(\tau)\,\mathrm{d}\tau \leqslant \alpha\eta, \tag{12}$$

which, in turn, implies that the trajectories of the closed-loop system that start from the origin will remain inside $\varepsilon(P, \alpha\eta)$ for all $w \in \mathscr{W}_\alpha$. □

Remark 2. Let $\leqslant$ in (4) be replaced with $<$. From the proof of item a of the theorem, we note that, in the absence of the disturbances, $\dot{V} < 0$ for all $x \in \varepsilon(P, 1+\alpha\eta)\backslash\{0\}$. Hence the ellipsoid $\varepsilon(P, 1+\alpha\eta)$ is inside the domain of attraction of the origin.

In what follows, we will use Theorem 1 to assess the stability and disturbance tolerance/rejection capabilities of the closed-loop system.

2.2. Disturbance tolerance

Problem 1 (Disturbance tolerance with non-zero initial condition). Let S be a given $n \times n$ positive definite matrix. Suppose that the initial conditions are inside the ellipsoid $\varepsilon(S,1)$. A basic problem is to determine/estimate the largest α such that all the state trajectories starting from $\varepsilon(S,1)$ will be bounded for all $w \in \mathscr{W}_\alpha$. To do so, we may try to determine the largest α such that there exist two ellipsoids $\varepsilon(P,1)$ and $\varepsilon(P, 1+\alpha\eta)$ satisfying the condition of Theorem 1(a). Moreover, $\varepsilon(S,1) \subset \varepsilon(P,1)$. This problem can be stated as the following optimization problem:

$$\sup_{P>0,\eta>0,H} \alpha \tag{13}$$

$$\begin{aligned} \text{s.t.(a) } & \varepsilon(S,1) \subset \varepsilon(P,1), \\ \text{(b) } & \text{inequalities (4)}, \\ \text{(c) } & \varepsilon(P, 1+\alpha\eta) \subset \mathscr{L}(H). \end{aligned} \tag{14}$$

Constraint (a) is equivalent to

$$\begin{bmatrix} S & I \\ I & P^{-1} \end{bmatrix} \geqslant 0.$$

Constraint (c) is equivalent to (Hu & Lin, 2001a),

$$(1+\alpha\eta) h_i P^{-1} h_i^{\mathrm{T}} \leqslant 1, \quad i \in [1,m],$$

where h_i is the ith row of H. Let $\mu = 1/(1+\alpha\eta)$, $Q = P^{-1}$, $Y = HQ$. Then, $\mu \in (0,1)$ and constraints (a) and (c) are further equivalent to

$$\begin{bmatrix} S & I \\ I & Q \end{bmatrix} \geqslant 0 \tag{15}$$

and

$$\begin{bmatrix} \mu & y_i \\ y_i^{\mathrm{T}} & Q \end{bmatrix} \geqslant 0, \quad \forall i \in [1,m], \tag{16}$$

respectively, where y_i is the ith row of Y. Meanwhile, constraint (b) is equivalent to

$$Q(A+BD_iF)^{\mathrm{T}} + (A+BD_iF)Q + (BD_i^- Y)^{\mathrm{T}} + BD_i^- Y + \frac{\alpha\mu}{1-\mu} EE^{\mathrm{T}} \leqslant 0, \quad \forall i \in [1,2^m]. \tag{17}$$

Let $\bar{\alpha} = \sqrt{\alpha}$. By Schur complement, (17) is equivalent to

$$\begin{bmatrix} Q(A+BD_iF)^{\mathrm{T}} + (A+BD_iF)Q + (BD_i^- Y)^{\mathrm{T}} + BD_i^- Y & \bar{\alpha}E \\ \bar{\alpha}E^{\mathrm{T}} & \dfrac{\mu-1}{\mu} I \end{bmatrix} \leqslant 0, \quad \forall i \in [1,2^m]. \tag{18}$$

Hence, the optimization problem (13) can be transformed into

$$\sup_{Q>0,Y,\mu\in(0,1)} \bar{\alpha} \quad \text{s.t. } (15), (18), (16), \tag{19}$$

where all the constraints are in LMIs for each fixed $\mu \in (0,1)$. Thus, the optimization problem (19) can be solved by sweeping μ over the interval $(0,1)$.

Problem 2 (Disturbance tolerance with zero initial condition). Here we would like to estimate the largest disturbance that can be tolerated by the closed-loop system at zero initial condition. This problem is fundamental to the determination of the restricted $\mathscr{L}_2$ gain, since this gain is meaningful only if the state trajectories starting from the origin are bounded. This problem can be described as follows:

$$\sup_{P>0,H} \alpha$$

$$\begin{aligned} \text{s.t. (a) } & \text{inequalities (4)}, \\ \text{(b) } & \varepsilon(P,\alpha) \subset \mathscr{L}(H). \end{aligned} \tag{20}$$

Let $P^{-1} = Q$ and $\upsilon = 1/\alpha$. Then, (20) is equivalent to

$$\inf_{Q>0,H} \upsilon$$

$$\begin{aligned} \text{s.t. (a) } & Q(A+B(D_iF+D_i^- H))^{\mathrm{T}} \\ & +(A+B(D_iF+D_i^- H))Q \\ & +EE^{\mathrm{T}} \leqslant 0, \quad \forall i \in [1,2^m], \\ \text{(b) } & h_i Q h_i^{\mathrm{T}} \leqslant \upsilon, \quad \forall i \in [1,m], \end{aligned} \tag{21}$$

where h_i is the ith row of H.

By the change of variable $Y = HQ$ and Schur complement, (21) is further equivalent to the following LMI optimization problem:

$$\begin{aligned}
&\inf_{Q>0,Y} \ \upsilon \\
&\text{s.t. (a) } Q(A+BD_iF)^{\mathrm{T}} + (A+BD_iF)Q + (BD_i^{-}Y)^{\mathrm{T}} \\
&\qquad +(BD_i^{-}Y) + EE^{\mathrm{T}} \leqslant 0, \\
&\qquad \forall i \in [1, 2^m], \\
&\text{(b) } \begin{bmatrix} \upsilon & y_i \\ y_i^{\mathrm{T}} & Q \end{bmatrix} \geqslant 0, \quad \forall i \in [1, m],
\end{aligned} \tag{22}$$

where y_i is the ith row of Y.

We note that an algorithm for estimating such a largest α was earlier proposed in Hindi and Boyd (1998). In what follows, we examine the conservativeness of both (22) and the algorithm of Hindi and Boyd (1998). Let $H = R^{-1}F$, where $R = \mathrm{diag}\{r_1, r_2, \ldots, r_m\}$, $r_i > 1$, $\forall i \in [1, m]$. Then, (21) becomes

$$\begin{aligned}
&\inf_{Q>0,F,R} \ \upsilon \\
&\text{s.t. (a) } Q(A+B(D_i+D_i^{-}R^{-1})F)^{\mathrm{T}} \\
&\qquad +(A+B(D_i+D_i^{-}R^{-1})F)Q \\
&\qquad +EE^{\mathrm{T}} \leqslant 0, \ \forall i \in [1, 2^m], \\
&\text{(b) } f_iQf_i^{\mathrm{T}} \leqslant r_i^2\upsilon, \quad \forall i \in [1, m].
\end{aligned} \tag{23}$$

Let $\Gamma_r = \frac{1}{2}(I + R^{-1})$ and $\Pi_r = \frac{1}{2}(I - R^{-1})$. Then, constraint (a) of (23) can be written as

$$\begin{aligned}
&Q(A+B\Gamma_rF)^{\mathrm{T}} + (A+B\Gamma_rF)Q + EE^{\mathrm{T}} \\
&\quad +B(D_i+D_i^{-}R^{-1}-\Gamma_r)FQ \\
&\quad +QF^{\mathrm{T}}(D_i+D_i^{-}R^{-1}-\Gamma_r)^{\mathrm{T}}B^{\mathrm{T}} \leqslant 0, \quad \forall i \in [1, 2^m],
\end{aligned} \tag{24}$$

which holds if

$$\begin{aligned}
&Q(A+B\Gamma_rF)^{\mathrm{T}} + (A+B\Gamma_rF)Q + EE^{\mathrm{T}} \\
&\quad +B(D_i+D_i^{-}R^{-1}-\Gamma_r)S(D_i+D_i^{-}R^{-1}-\Gamma_r)^{\mathrm{T}}B^{\mathrm{T}} \\
&\quad +QF^{\mathrm{T}}S^{-1}FQ \leqslant 0, \quad \forall i \in [1, 2^m],
\end{aligned} \tag{25}$$

for some $S > 0$, $S = \mathrm{diag}\{s_1, s_2, \ldots, s_m\}$.

Since the diagonal elements of the matrix $(D_i + D_i^{-}R^{-1} - \Gamma_r)$ are either $\frac{1}{2}(1 - 1/r_i)$ or $\frac{1}{2}(1/r_i - 1)$, (25) is equivalent to

$$\begin{bmatrix} Q(A+B\Gamma_rF)^{\mathrm{T}} + (A+B\Gamma_rF)Q + EE^{\mathrm{T}} + B\Pi_rS\Pi_r^{\mathrm{T}}B^{\mathrm{T}} & QF^{\mathrm{T}} \\ FQ & -S \end{bmatrix} \leqslant 0. \tag{26}$$

Using constraint (b) of (23) and (26) as constraints, we have the following optimization problem:

$$\begin{aligned}
&\inf_{Q>0,R} \ \upsilon \\
&\text{s.t. (a) LMI (26)} \\
&\qquad \text{(b) } f_iQf_i^{\mathrm{T}} \leqslant r_i^2\upsilon, \quad \forall i \in [1, m].
\end{aligned} \tag{27}$$

This optimization problem involves bilinear matrix inequality constraints. If R is fixed, then it becomes the optimization problem used in Hindi and Boyd (1998) to find the r-level disturbance rejection bound. We can see that the constraints in (27) are more conservative than those in (23). Meanwhile, the constraints in (23) are more conservative than those in (21). Hence, constraints in (21) are the least conservative.

2.3. Disturbance rejection

A traditional way to measure the disturbance rejection capability is to use the $\mathscr{L}_2$ gain (or restricted $\mathscr{L}_2$ gain for a nonlinear system). For a linear system, the effect of initial condition will vanish and can be ignored as time goes by. For a nonlinear system, the initial condition may affect the trajectory for all the future time. One way to measure the disturbance rejection capability is to compare the relative size between the set containing the initial condition and the set that eventually bounds the trajectories. It is desirable that the state trajectories will stay close to the set of initial conditions. Hence good disturbance rejection should imply a small difference (relative size) between $\varepsilon(P, 1)$ and $\varepsilon(P, 1 + \alpha\eta)$. For this reason, we use η to denote the disturbance rejection level for a given $\mathscr{W}_\alpha$. It is clear that small η implies good disturbance rejection.

Problem 3 (The disturbance rejection level). Given the set of initial condition $\varepsilon(S, 1)$. Let $\alpha_{\max}$ be the maximal energy of the tolerable disturbances determined in Problem 1. We now consider $\alpha \leqslant \alpha_{\max}$. The problem of minimizing the disturbance rejection level η can be formulated as the following optimization problem:

$$\begin{aligned}
&\inf_{P>0,H} \ \eta \\
&\text{s.t. (a) } \varepsilon(S, 1) \subset \varepsilon(P, 1), \\
&\qquad \text{(b) inequalities (4)}, \\
&\qquad \text{(c) } \varepsilon(P, 1 + \alpha\eta) \subset \mathscr{L}(H),
\end{aligned} \tag{28}$$

which, by using Schur complement in its constraints, is equivalent to the following LMI optimization problem:

$$\begin{aligned}
&\sup_{Y,Q>0} \ \tilde{\eta} \\
&\text{s.t. (a) } \begin{bmatrix} S & I \\ I & Q \end{bmatrix} \geqslant 0, \\
&\qquad \text{(b) } Q(A+BD_iF)^{\mathrm{T}} + (A+BD_iF)Q + (BD_i^{-}Y)^{\mathrm{T}} \\
&\qquad\quad +B(D_i^{-}Y) + \tilde{\eta}EE^{\mathrm{T}} \leqslant 0, \ \forall i \in [1, 2^m],
\end{aligned}$$

$$\text{(c)}\quad \begin{bmatrix} Q & y_i^{\mathrm{T}} & y_i^{\mathrm{T}} \\ y_i & 1 & 0 \\ y_i & 0 & \dfrac{\tilde{\eta}}{\alpha} \end{bmatrix} \geqslant 0, \quad \forall i \in [1, m], \tag{29}$$

where $\tilde{\eta} = \frac{1}{\eta}$, $Q = P^{-1}$, $Y = HQ$, and y_i is the ith row of Y.

Like in the problem of enlarging the domain of attraction in the absence of disturbance, it is meaningful to maximize the size of the set of initial conditions with a guaranteed level of disturbance rejection. This problem can be described by replacing the objective function of (29) with trace(Q) and letting η be fixed, i.e.,

$$\sup_{Y,Q>0} \ \operatorname{trace}(Q)$$
$$\text{s.t. (a), (b) and (c) of (29).} \tag{30}$$

Problem 4 (Estimation of the restricted $\mathscr{L}_2$ gain). We now consider the problem of estimating the upper bound on the $\mathscr{L}_2$-gain (restricted on $\mathscr{W}_\alpha$) for a given closed-loop system. We first establish the following theorem.

Theorem 2. *Let $\alpha_{\max}$ be the maximal tolerable disturbance level determined in Problem 2. Consider an $\alpha \leqslant \alpha_{\max}$. For a given constant $\gamma > 0$, if there exists an $H \in \Re^{m\times n}$ such that*

$$P(A + B(D_iF + D_i^- H)) + (A + B(D_iF + D_i^- H))^{\mathrm{T}} P + PEE^{\mathrm{T}}P + \frac{1}{\gamma^2} C^{\mathrm{T}} C \leqslant 0, \quad \forall i \in [1, 2^m] \tag{31}$$

and $\varepsilon(P, \alpha) \subset \mathscr{L}(H)$, then the $\mathscr{L}_2$ gain from w to z for $w \in \mathscr{W}_\alpha$ is less than or equal to γ.

Proof. We first note that the conditions of this theorem imply those of item (b) of Theorem 1. Hence, the trajectories starting from the origin will remain inside $\varepsilon(P, \alpha) \subset \mathscr{L}(H)$ for all time. Thus the $\mathscr{L}_2$ gain analysis can be carried out within $\varepsilon(P, \alpha)$. By Lemma 1, we have that

$$\operatorname{sat}(Fx) \in \operatorname{co}\{D_iFx + D_i^- Hx : \ i \in [1, 2^m]\}, \quad \forall x \in \varepsilon(P, \alpha) \subset \mathscr{L}(H). \tag{32}$$

Let $V(x) = x^{\mathrm{T}}Px$ be a Lyapunov function. Then, by (32),

$$\begin{aligned} \dot{V} &= x^{\mathrm{T}}P[Ax + B\operatorname{sat}(Fx) + Ew] \\ &\quad + [Ax + B\operatorname{sat}(Fx) + Ew]^{\mathrm{T}}Px \\ &\leqslant \max_{i\in[1,2^m]} x^{\mathrm{T}}[P(A + B(D_iF + D_i^- H)) \\ &\quad + (A + B(D_iF + D_i^- H))^{\mathrm{T}}P]x \\ &\quad + x^{\mathrm{T}}PEE^{\mathrm{T}}Px + w^{\mathrm{T}}w. \end{aligned} \tag{33}$$

By (31) and (33), we obtain

$$\dot{V} \leqslant -\frac{1}{\gamma^2} x^{\mathrm{T}}C^{\mathrm{T}}Cx + w^{\mathrm{T}}w = -\frac{1}{\gamma^2} z^{\mathrm{T}}z + w^{\mathrm{T}}w. \tag{34}$$

Integrating both sides of the above inequality from 0 to t results in

$$V(x(t)) \leqslant -\frac{1}{\gamma^2}\int_0^t z^{\mathrm{T}}(\tau)z(\tau)\,\mathrm{d}\tau + \int_0^t w^{\mathrm{T}}(\tau)w(\tau)\,\mathrm{d}\tau. \tag{35}$$

Noting that $V(x(t)) \geqslant 0$, we have

$$\int_0^t z^{\mathrm{T}}(\tau)z(\tau)\,\mathrm{d}\tau \leqslant \gamma^2 \int_0^t w^{\mathrm{T}}(\tau)w(\tau)\,\mathrm{d}\tau. \qquad \square \tag{36}$$

Based on Theorem 2, the $\mathscr{L}_2$ gain bound can be estimated by solving the following optimization problem,

$$\inf_{P>0,H} \quad \gamma^2 \tag{37}$$
$$\text{s.t. (a) inequalities in (31),}$$
$$\text{(b) } \varepsilon(P, \alpha) \subset \mathscr{L}(H). \tag{38}$$

By Schur complement, (31) is equivalent to

$$\begin{bmatrix} (A+B(D_iF+D_i^- H))^{\mathrm{T}}P + P(A+B(D_iF+D_i^- H)) & PE & C^{\mathrm{T}} \\ E^{\mathrm{T}}P & -I & 0 \\ C & 0 & -\gamma^2 I \end{bmatrix} \leqslant 0, \ \forall i \in [1, 2^m]. \tag{39}$$

Let $Q = P^{-1}$ and $Y = HQ$. Then the optimization problem (37) is equivalent to the following LMI problem:

$$\inf_{Q>0,Y} \ \gamma^2$$

$$\text{s.t. (a)} \begin{bmatrix} Q(A+BD_iF)^{\mathrm{T}} + (A+BD_iF)Q + BD_i^- Y + (BD_i^- Y)^{\mathrm{T}} & E & QC^{\mathrm{T}} \\ E & -I & 0 \\ CQ & 0 & -\gamma^2 I \end{bmatrix} \leqslant 0, \ \forall i \in [1, 2^m],$$

$$\text{(b)} \begin{bmatrix} \dfrac{1}{\alpha} & y_i \\ y_i^{\mathrm{T}} & Q \end{bmatrix} \geqslant 0, \quad \forall i \in [1, m], \tag{40}$$

where y_i is the ith row of Y.

3. Controller synthesis

By viewing F as an additional free parameter, all the optimization problems in the previous section (Problems 1–4) can be adapted for the design of feedback gain F. In particular, by setting $Z = FQ$, all those LMI optimization problems remain as LMI optimization problems. Once these new LMI problems are solved, the feedback gain can then be computed as $F = ZQ^{-1}$.

4. Numerical examples

In this section, we will demonstrate the effectiveness of our methods by some numerical examples.

Example 1. Consider system (1) with

$$A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad F = \begin{bmatrix} 1.2231 & -2.2486 \end{bmatrix}.$$

By specifying an allowable magnitude of the input to the actuator, r, the algorithm proposed in Hindi and Boyd (1998) (see (27)) determines the largest tolerable disturbance with zero initial conditions, called the r-level disturbance rejection bound $\alpha_{\max,r}$. Shown in Fig. 1 is the r-level disturbance rejection bound as a function of r for the system. The largest tolerable disturbance determined by this algorithm is

$$\max\{\alpha_{\max,r} : r > 1\} = 577.92.$$

For $r = 1$, we have $\alpha_{\max,1} = 516.3178$. This shows that if the energy of the disturbance is less that 516.3178, the state

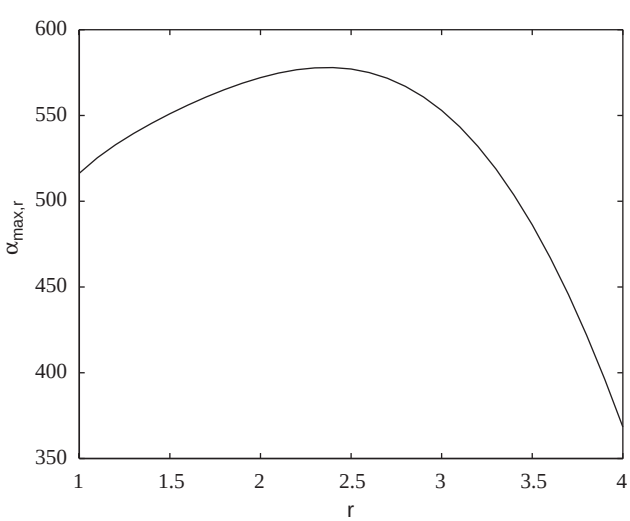

Fig. 1. r-Level disturbance rejection bounds determined by the algorithm of (27).

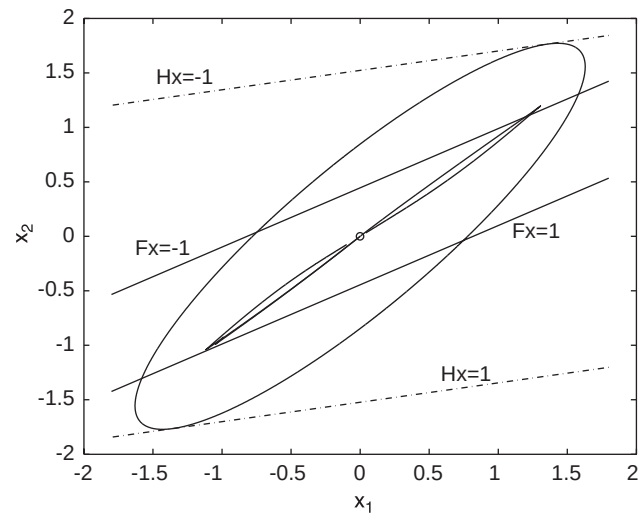

Fig. 2. Two trajectories under a disturbance with the maximal energy.

of the system will stay in an ellipsoids which is inside the linear region and the system will behave as a linear system. By solving the optimization problem (22), we can determine the maximal disturbance energy the system (with $x(0) = 0$) can tolerate as

$$\alpha_{\max} = 628.92.$$

This testifies the assertion that the constraints in (22) are less conservative than those in (27). We also note that, by our method, the $\mathscr{L}_2$ norm of the largest tolerable disturbance can be estimated by solving a single optimization problem. Fig. 2 plots two trajectories from zero initial condition, one under a ramp disturbance (of duration 1 s) and the other under a step disturbance (of duration 0.4 s). Both of the disturbances have energy $\alpha_{\max}$. Also plotted in Fig. 2 are the ellipsoid $\varepsilon(P, \alpha_{\max})$ and the straight lines $Fx = \pm 1$ and $Hx = \pm 1$. We note that one of the trajectories passes the line $Fx = -1$ before turning back to the origin after the disturbance disappears.

Example 2. In this example, we consider the problem of maximizing the volume of the inner ellipsoid with a guaranteed disturbance rejection level. The system is described by (1) with

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3.0 \end{bmatrix}, \quad B = \begin{bmatrix} 25 & 0 \\ 0 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is a system considered in Paim et al. (2002). The matrix B is scaled from the original B in Paim et al. (2002) to normalize the saturation bound for each input. For a given disturbance rejection level $\eta = 1$, Paim et al. (2002) designs a feedback gain that maximizes the ellipsoid $\varepsilon(P, 1)$ (or $\varepsilon(P, 1 + \alpha)$). The optimization problem (30) in this paper maximizes this ellipsoid for a given F. It can be easily turned into a design problem by considering F as an additional optimizing parameter. Shown in Table 1 are the volumes of the maximized $\varepsilon(P, 1)$ and $\varepsilon(P, 1 + \alpha)$ obtained by

Table 1
The volumes of the maximized ellipsoids by different methods

	Algorithm of Paim et al. (2002)		Our method	
α	$\mathrm{vol}(\varepsilon(P,1))$ $\eta=1$	$\mathrm{vol}(\varepsilon(P,1+\alpha\eta))$ $\eta=1$	$\mathrm{vol}(\varepsilon(P,1))$ $\eta=1$	$\mathrm{vol}(\varepsilon(P,1+\alpha\eta))$ $\eta=1$
1	25.0581	50.11	1.4714×10^{7}	2.9428×10^{7}
100	2.8743	290.30	1.4661×10^{6}	1.4808×10^{8}
625	1.6933	1060.00	6.1365×10^{5}	3.8415×10^{8}
2500	1.6848	4213.6	2.8942×10^{5}	7.2384×10^{8}

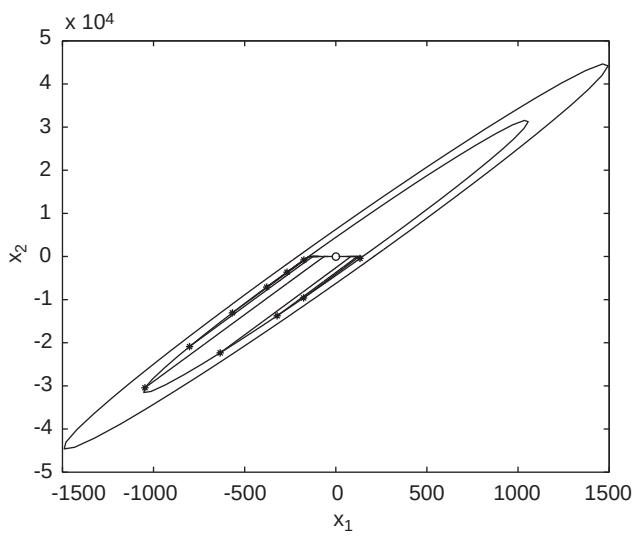

Fig. 3. Some trajectories under the unit energy disturbance.

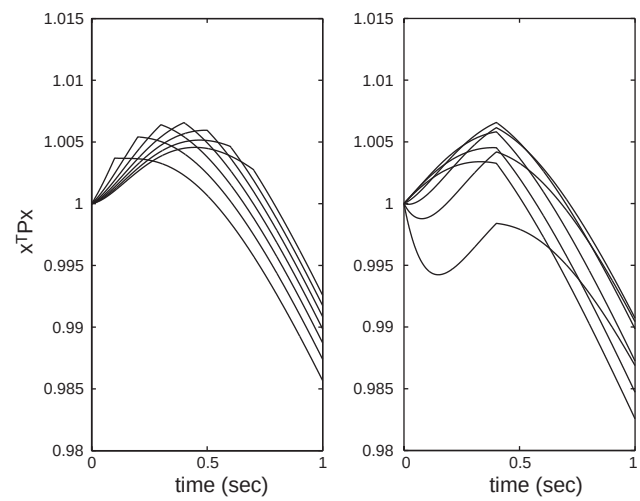

Fig. 4. Time history of $x^{\mathrm{T}}Px$ under different disturbances and initial conditions.

the algorithm of Paim et al. (2002) and those by the design version of (30) in this paper. These results show that the method proposed in this paper significantly improves that of Paim et al. (2002).

As α increases, the ratio between the short axis and the long axis of the ellipsoids decreases. For $\alpha = 1, 100, 625, 2500$, the ratios are 0.0047, 0.00047, 0.0002, 0.000092, respectively. If we plot the ellipsoid for $\alpha = 100$, it is very narrow and for $\alpha = 2500$, it appears to be a straight-line. Because of this, we only present the detailed results for the case $\alpha = 1$. In this case, the optimal F and P are

$$F = \begin{bmatrix} -0.4749 & 0.0155 \\ 0.2149 & -0.0070 \end{bmatrix} \times 10^{-2},$$

$$P = \begin{bmatrix} 0.455616 & -0.015074 \\ -0.015074 & 0.000508 \end{bmatrix} \times 10^{-4}.$$

Some trajectories under a unit energy disturbance $w_1(t) = \pm\sqrt{2.5}(1(t)-1(t-0.4)), w_2(t)=0$ are plotted in Fig. 3 with respect to the ellipsoids $\varepsilon(P,1)$ and $\varepsilon(P,1+\alpha)$, where $*$'s denote different initial conditions on the boundary of $\varepsilon(P,1)$. We set $w_2=0$ because its effect is much weaker than w_1. We see that all the trajectories are inside $\varepsilon(P,1+\alpha)$. Since the ellipsoid is very thin, it is not clearly shown how much the trajectories go outside of $\varepsilon(P,1)$. For this reason, we plotted the time history of $x^{\mathrm{T}}Px$ in Fig. 4 to demonstrate the effect of different unit energy disturbances (left plot) and different initial conditions (right plot).

Example 3. Consider the same system as in Example 1, with the output matrix $C = [1\ 1]$. Here we would like to estimate the $\mathscr{L}_2$-gain for disturbances with energy bounded by an α less than the maximal tolerable value. For a given α, a bound on this $\mathscr{L}_2$-gain can be obtained by solving (40). We may also use the algorithm in Hindi and Boyd (1998) to estimate this bound. To do so, we need to choose R (a scalar for this system) over some interval, computing a bound for each R and then take the minimal value of the bounds.

From Example 1, we know that if $\alpha \leqslant 516.3178$, the state will stay in the linear region and the system will behave as a linear system. So we are only interested in $\alpha > 516.3178$. In particular, we consider $\alpha > 550$ and compare the estimated $\mathscr{L}_2$-gain by our method and that by Hindi and Boyd (1998). In Fig. 5, the dashed curve is the bound on the $\mathscr{L}_2$-gain (as a function of α) by the method of Hindi and Boyd (1998) and the solid curve is the bound by our method. As expected, the dashed curve diverges to infinity as α approaches

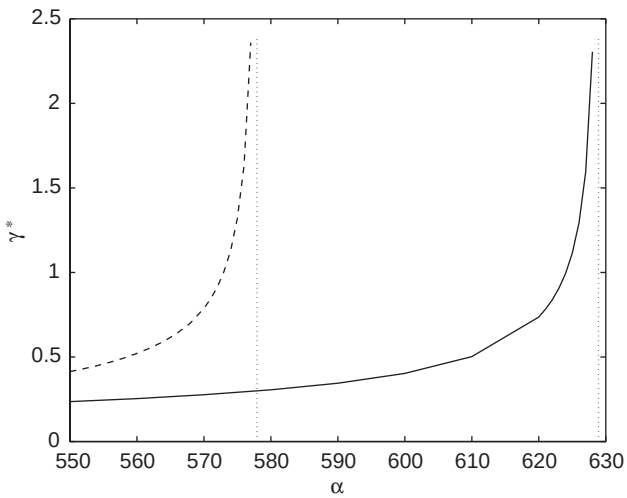

Fig. 5. Restricted $\mathscr{L}_2$ gain estimated by different methods.

577.92, the maximal tolerable disturbance level by Hindi and Boyd (1998) and the solid curve diverges to infinity as α approaches 628.92, the maximal tolerable disturbance level by our method.

5. Conclusions

In this paper, we present a simple condition under which any trajectories starting from an ellipsoid will remain inside an outer ellipsoid for linear systems subject to actuator saturation and $\mathscr{L}_2$-disturbances. Based on this condition, the assessment of system stability and disturbance tolerance/rejection can be formulated as optimization problems with LMI constraints. Meanwhile, these optimization problems can be easily adapted for the controller design. Furthermore, it was proved and/or shown by examples that our methods significantly improve the existing methods.

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