Conjugate convex Lyapunov functions for dual linear differential inclusions*

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Abstract

Tools from convex analysis are used to show how stability properties and Lyapunov inequalities translate when passing from a linear differential inclusion (LDI) to its dual. In particular, it is proved that a convex, positive definite function is a Lyapunov function for an LDI if and only if its convex conjugate is a Lyapunov function for the LDI's dual. Examples show how such duality effectively doubles the number of tools available for assessing stability of LDIs.

I. Introduction

Duality is a firmly established concept in linear systems theory. For example, a linear system $\dot{x}(t) = Ax(t)$ is exponentially stable if and only if its dual system $\dot{\xi}(t) = A^T\xi(t)$ is. One way to verify this relationship is through Lyapunov inequalities. A symmetric and positive definite matrix P verifies the stability of the first system if $A^TP + PA < 0$. This is equivalent to $AP^{-1} + P^{-1}A^T < 0$, which shows that P^{-1} establishes stability of the second system.

Not coincidentally, the function $\xi \mapsto \frac{1}{2}\xi \cdot P^{-1}\xi$ is the convex conjugate (in the sense of convex analysis) of the function $x \mapsto \frac{1}{2}x \cdot Px$. In this note, we use the convex conjugacy of general, not

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necessarily quadratic, convex functions to study the relationship between and Lyapunov functions for a linear differential inclusion

$$\dot{x}(t) \in \text{co}\,\{A_i\}_{i=1}^m\, x(t) \tag{1}$$

and its dual

$$\dot{\xi}(t) \in \text{co}\,\left\{A_i^T\right\}_{i=1}^m\, \xi(t)\,. \tag{2}$$

In (1), (2), "co" stands for the convex hull (of the finite set of matrices). By a solution to (1) on $[0,\infty)$ we mean a locally absolutely continuous function $x(\cdot)$ such that $\dot{x}(t)$ is an element of $\text{co}\{A_i\}_{i=1}^m x(t)$ for almost all t; similarly for (2).

Since the codification of the absolute stability problem (see, for example, [1]), researchers have looked for Lyapunov functions to guarantee exponential stability and/or input-output properties for systems that can be modeled as switching among a (possibly infinite) family of linear systems. The classical circle criterion gives necessary and sufficient condition for the existence of a quadratic Lyapunov function that certifies exponential stability in the absolute stability problem. However, it is well known that a system can be absolutely stable without the existence of a quadratic Lyapunov function; see, for example, [15]. In [14] it is noted that a convex, homogeneous of degree two Lyapunov function always exists for an exponentially stable LDI. In [6] it is shown furthermore that a Lyapunov function can always be taken to be smooth. Identifying favorable classes of potential Lyapunov functions is a key step towards computationally tractable stability analysis. In the papers [18], [13], and [5], the authors consider homogeneous polynomial Lyapunov functions and provide linear matrix inequality (LMI) conditions for exponential stability. In [4], a bilinear matrix condition verifying whether a pointwise maximum of a family of quadratic functions forms a Lyapunov function is outlined; see also [17].

In this note, we apply several well-established (in convex analysis) techniques to shed new light on stability of LDIs. Our main result, Theorem 4.1, shows that a convex function is a Lyapuov function for the LDI (1) if and only if the convex conjugate of that function is a Lyapunov function for the dual LDI (2). In particular, we recover a result by Barabanov [2] that (1) is exponentially stable if and only if (2) is exponentially stable. Based on the conjugacy of Lyapunov functions, we formulate in Corollary 4.5 a sufficient condition for stability of (1). This condition relies on the class of functions conjugate to those given by a maximum of quadratic functions. In this sense, it is "dual" to the sufficient condition proposed in [4], which we state for completeness in Corollary 4.4. The two conditions may lead to different stability estimates. In such case, one then chooses the better one; see Example 5.3. A more direct benefit of the equivalence of stability of the dual LDIs is that any previously designed method to verify stability of an LDI, say that in [5], can be applied to its dual LDI, in order to establish stability of the original one. This may lead to surprising improvements; see Example 5.2. Further benefits of duality are illustrated in [12], where stability regions of saturated linear systems are estimated.

II. Convex Analysis Preliminaries

Convex conjugacy and some other tools of convex analysis we use here seem not to have been used in stability analysis, except possibly for situations when operations on quadratic functions are considered (see, for example, [4], [11]). To make these tools more accessible, we informally present the basic background material, and conclude the section by showing how considering the pointwise maximum of several not necessarily quadratic functions or the convex hull of a nonconvex Lyapunov function can be used in constructing convex Lyapunov functions. The standard reference for the convex analysis material we summarize here is [16].

Given any function $f : \mathbb{R}^n \to \mathbb{R}$, its conjugate function is defined, for $\xi \in \mathbb{R}^n$ by

$$f^*(\xi) = \sup_{x \in R^n} \{\xi \cdot x - f(x)\}.$$

This function is always convex and lower semicontinuous (but possibly infinite-valued). If f itself is convex, then the conjugate of f^* is the function f. That is, $(f^*)^*(x) = \sup_\xi \{x \cdot \xi - f^*(\xi)\} = f(x)$. This fact is fundamental to many arguments involving duality. Basic examples are:

- ⋄ For a positive definite matrix P,

$$f(x) = \frac{1}{2} x \cdot Px \quad \Longleftrightarrow \quad f^*(\xi) = \frac{1}{2} \xi \cdot P^{-1} \xi. \tag{3}$$

- ⋄ For any $p > 1$, $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$f(x) = \frac{1}{p} (\|x\|_p)^p \quad \Longleftrightarrow \quad f^*(\xi) = \frac{1}{q} (\|\xi\|_q)^q. \tag{4}$$

A more elaborate example is presented in Section III. If f is convex, positive definite, and positively homogeneous of degree $p > 1$ (that is, $f(\lambda x) = \lambda^p f(x)$ for $\lambda \geq 0$), then

(i) $f^*(\xi)$ is finite for every $\xi \in \mathbb{R}^n$;

(ii) f^* is convex, positive definite, and positively homogeneous of degree $q > 1$ where $1/p + 1/q = 1$;

(iii) if

$$\frac{\alpha}{p}\left(\|x\|_p\right)^p \le f(x) \le \frac{\beta}{p}\left(\|x\|_p\right)^p$$

for some $\alpha > 0$, $\beta > 0$ (such constants exist for any continuous, positively homogeneous of degree p and positive definite function), then

$$\frac{\beta^{1-q}}{q}\left(\|\xi\|_q\right)^q \le f^*(\xi) \le \frac{\alpha^{1-q}}{q}\left(\|\xi\|_q\right)^q .$$

For example, positive homogeneity of f^* can be verified directly from the definition:

$$\begin{aligned} f^*(\lambda\xi) &= \sup_x \{(\lambda\xi)\cdot x - f(x)\} = \lambda^q \sup_x \left\{\xi\cdot(x/\lambda^{q-1}) - f(x)/\lambda^q\right\} \\ &= \lambda^q \sup_x \left\{\xi\cdot(x/\lambda^{q-1}) - f(x/\lambda^{q/p})\right\} = \lambda^q \sup_x \{\xi\cdot x - f(x)\} = \lambda^q f^*(\xi). \end{aligned}$$

The bounds for $f^*(\xi)$ follow from (4) and the fact that conjugacy reverses inequalities. Note that, since $(f^*)^* = f$, the property of f^* described in (ii) above is in fact equivalent to the same property of f. Similarly, the bounds on f in (iii) are equivalent to those on f^*.

A *subgradient* of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at x is a vector $v \in \mathbb{R}^n$ such that

$$f(x') \ge f(x) + v\cdot(x'-x) \quad \forall x' \in \mathbb{R}^n,$$

and the subdifferential $\partial f(x)$ is the set of all subgradients at x. This set is always nonempty, and consists of one point if and only if f is differentiable at x (and then the unique point is $\nabla f(x)$). A key relationship between ∂f and ∂f^* is:

$$\xi \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^*(\xi).$$

This immediately leads to the following observation: for any positive definite f, f^*,

$$\partial f(x)\cdot Ax < 0 \;\forall x \ne 0 \quad \Longleftrightarrow \quad \partial f^*(\xi)\cdot A^T\xi < 0 \;\forall \xi \ne 0.$$

(The inequality $\partial f(x)\cdot Ax < 0$ should be understood as $\xi\cdot Ax < 0$ for all $\xi \in \partial f(x)$.) Indeed, suppose the condition on the left holds. Pick any $\xi \ne 0$, and any $x \in \partial f^*(\xi)$. Then $x \ne 0$, since $0 \in \partial f^*(\xi)$ would imply that ξ minimizes f^*. Thus $x\cdot A^T\xi = \xi\cdot Ax < 0$, since $x \in \partial f^*(\xi)$ is equivalent to $\xi \in \partial f(x)$.

Given any function $g : \mathbb{R}^n \to \mathbb{R}$, its convex hull $\operatorname{co} g$ is the greatest convex function bounded above by g. Under mild assumptions, for example when g^* is finite everywhere (this always holds if g is positively homogeneous of degree $p > 1$ and positive definite), we have

$$\operatorname{co} g(x) = \min\left\{\sum_{k=1}^{n+1}\lambda_k g(x_k) \;\middle|\; \sum_{k=1}^{n+1}\lambda_k x_k = x\right\}, \tag{5}$$

where the minimum is taken over all x_k's and λ_k such that $(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \in \Delta_{n+1}$. Here and in what follows, for any natural number r, $\Delta_r = \{(\lambda_1, \lambda_2, \ldots, \lambda_r) \mid \sum_{k=1}^r \lambda_k = 1, \lambda_k \geq 0\}$. If $\operatorname{co} g(x) = \sum_{k=1}^{n+1} \lambda_k g(x_k)$ then $\operatorname{co} g(x_k) = g(x_k)$ at each x_k with nonzero λ_k. Furthermore, if g is differentiable at each such x_k, then $\nabla \operatorname{co} g(x) = \nabla g(x_k)$ for each such k (in particular, $\operatorname{co} g$ is differentiable at x).

Now consider convex functions $h_j : I\!\!R^n \to I\!\!R$, $j = 1, 2, \ldots, l$, and define

$$h(x) = \max_{j=1,2,\ldots,l} h_j(x). \tag{6}$$

The level sets of h are intersections of the level sets of all h_i's. The conjugate function h^* is the convex hull of the function $g(\xi) = \min_{j=1,2,\ldots,l} h_j^*(\xi)$. If each h_j (and then also h_j^*) is positively homogeneous of degree greater than 1, the the level sets of h^* are the convex hulls of (the smallest convex sets containing) the level sets of all h_j^*'s.

Lemma 2.1: Consider a positive definite function h given by (6). Then, the implication

$$h(x) = h_j(x) \ \Rightarrow\ \partial h_j(x) \cdot Ax \leq -\gamma h_j(x) \tag{7}$$

holds at each $x \in I\!\!R^n$ if and only if $\partial h(x) \cdot Ax \leq -\gamma h(x)$ for all $x \in I\!\!R^n$.

Proof: Fix x and let $j_1, j_2, \ldots, j_s$ be the set of all indices for which $h_{j_k}(x) = h(x)$. In particular, the assumption implies that $\partial h_{j_k}(x) \cdot Ax \leq -\gamma h(x)$ for $k = 1, 2, \ldots, s$. The subdifferential $\partial h(x)$ is the convex hull of the union of $\partial h_{j_k}(x)$. More precisely, for $r = \min\{n+1, s\}$, given any $\xi \in \partial h(x)$, there exist $\xi_1, \xi_2, \ldots, \xi_r$ with $\xi_k \in \partial h_{j_k}(x)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Delta_r$ such that $\sum_{k=1}^r \lambda_k \xi_{j_k} = \xi$. Since, for each k, we have $\xi_k \cdot Ax \leq -\gamma h(x)$, multiplying these inequalities by λ_k and summing them over $k = 1, 2, \ldots, s$ yields $\xi \cdot Ax \leq -\gamma h(x)$. ■

Finally, we note that "convexifying" any Lyapunov function (for a linear system or an LDI) leads to another Lyapunov function. In fact, the "convexified" function can be a Lyapunov function even if the Lyapunov inequality involving the original function fails at many points. In particular, the convex hull of two quadratic functions can be a Lyapunov function even if the minimum of these quadratics is not. Consequently, using the minimum as a Lyapunov function may lead to suboptimal stability estimates. This is verified to an extent by the examples in [17].

Proposition 2.2: For a function $W : I\!\!R^n \to I\!\!R$, let $V = \operatorname{co} W$. Suppose W is differentiable at every point x with $W(x) = V(x)$ and at such points $\nabla W(x) \cdot Ax \leq -\gamma W(x)$ for a given matrix A and $\gamma \geq 0$. Suppose that W^* is finite everywhere. Then V is differentiable and, for all x, $\nabla V(x) \cdot Ax \leq -\gamma V(x)$.

Proof: The finiteness of W^* guarantees that V can be described through (5). Thus, given any x and any representation $V(x) = \sum_{k=1}^{n+1} \lambda_k W(x_k)$, we have $\nabla V(x) = \nabla W(x_k)$ and $V(x_k) = W(x_k)$ for any k with nonzero λ_k. For such k's, by assumption we have $W(x_k) \leq -\gamma W(x_k)$. It follows that (with the sum taken over k's with nonzero λ_k):

$$\begin{aligned} \nabla V(x) \cdot Ax &= \nabla V(x) \cdot A\left(\Sigma \lambda_k x_k\right) = \Sigma \lambda_k \nabla V(x) \cdot Ax_k \\ &= \Sigma \lambda_k \nabla W(x_k) \cdot Ax_k \leq \Sigma \lambda_k \left(-\gamma W(x_k)\right) = -\gamma V(x). \end{aligned}$$

■

III. Composite Quadratic / Convex Hull Function

For positive definite symmetric Q_j, $j = 1, 2, \ldots, l$, consider

$$q(x) = \max_{j=1,2,\ldots,l} \frac{1}{2} x \cdot Q_j x. \tag{8}$$

It turns out that the conjugate of q, which is the convex hull of the functions $\xi \mapsto \frac{1}{2}\xi \cdot Q_j^{-1}\xi$, is the same as the composite quadratic function used in [11] for stability analysis. Indeed,

$$\max_{\lambda \in \Delta_l} \sum_{j=1}^{l} \lambda_j \frac{1}{2} x \cdot Q_j x = \max_{\lambda \in \Delta_l} \frac{1}{2} x \cdot \left(\sum_{j=1}^{l} \lambda_j Q_j\right) x$$

since the maximum of a linear function of λ over a simplex is attained at one of the vertices. Consequently,

$$\begin{aligned} q^*(\xi) &= \sup_{x \in R^n} \left\{ \xi \cdot x - \max_{\lambda \in \Delta_l} \frac{1}{2} x \cdot \left(\sum_{j=1}^{l} \lambda_j Q_j\right) x \right\} = \sup_{x \in R^n} \min_{\lambda \in \Delta_l} \left\{ \xi \cdot x - \frac{1}{2} x \cdot \left(\sum_{j=1}^{l} \lambda_j Q_j\right) x \right\} \\ &= \min_{\lambda \in \Delta_l} \sup_{x \in R^n} \left\{ \xi \cdot x - \frac{1}{2} x \cdot \left(\sum_{j=1}^{l} \lambda_j Q_j\right) x \right\}. \end{aligned}$$

Switching $\sup$ and $\min$ is possible, since the function in the brackets above is concave in x, convex in γ, and the minimum is taken over a compact set; see, for example, Corollary 37.3.2 in [16]. Now, calculating the conjugate of a quadratic function yields

$$q^*(\xi) = \min_{\lambda \in \Delta_l} \frac{1}{2} \xi \cdot \left(\sum_{j=1}^{l} \lambda_j Q_j\right)^{-1} \xi. \tag{9}$$

This is exactly the composite quadratic function of [11]. An alternate expression for q^* can be derived from (5).

The dual description of (9) leads to an alternate way to study its properties. For example, the function q is strongly convex with constant ρ, where $\rho > 0$ is any constant smaller than every eigenvalue of Q_j, $j = 1, 2, \ldots, l$. (Strong convexity means that $q(x) - \frac{1}{2}\rho\|x\|^2$ is convex.) This is equivalent to q^* being differentiable and ∇q^* being Lipschitz continuous with constant $1/\rho$.

Numerical examples in Section V illustrate the use of both q and q^* in stability analysis.

IV. Lyapunov Inequalities

The subdifferential mappings of a pair of conjugate convex functions are inverses of one another. A more precise relationship exists for positively homogeneous functions. We use it to show that the conjugate of a Lyapunov function for a linear system is a Lyapunov function for the dual system.

Theorem 4.1: Let $V : I\!R^n \to I\!R$ be a convex, positive definite, positively homogeneous of degree $p > 1$ function, and let A be any matrix. Then, the condition

$$\partial V(x) \cdot Ax \leq -\gamma p V(x) \quad \text{for all } x \in I\!R^n \tag{10}$$

is equivalent to

$$\partial V^*(\xi) \cdot A^T \xi \leq -\gamma q V^*(\xi) \quad \text{for all } \xi \in I\!R^n. \tag{11}$$

Proof: First, we argue that $V(x) = 1/p$ and $\xi \in \partial V(x)$ if and only if $V^*(\xi) = 1/q$ and $x \in \partial V^*(\xi)$. The subdifferential inclusions are equivalent; thus we only need to show that $V(x) = 1/p$ and $\xi \in \partial V(x)$ imply $V^*(\xi) = 1/q$. Since $\xi \in \partial V(x)$, it follows that x maximizes $x' \mapsto \xi \cdot x' - V(x')$, and so $V^*(\xi) = \xi \cdot x - V(x) = \xi \cdot x - 1/p$. Furthermore, $\lambda = 1$ maximizes the function $\lambda \mapsto \xi \cdot \lambda x - V(\lambda x) = \xi \cdot \lambda x - \lambda^p/p$ over $\lambda \geq 0$. The derivative being 0 at $\lambda = 1$ yields $\xi \cdot x = 1$. Thus $V^*(\xi) = 1 - 1/p$.

Now note that, by positive homogeneity of V, inequality (10) is equivalent to

$$\partial V(x) \cdot Ax \leq -\gamma \quad \text{for all } x \text{ s.t. } V(x) = 1/p. \tag{12}$$

Indeed, fix $x' \neq 0$, so $V(x') \neq 0$. Let $s = (pV(x'))^{1/p}$ and $x = x'/s$. Then $V(x) = 1/p$, while

$$\partial V(x) = \frac{1}{s^{p-1}} \partial V(x').$$

Thus (12) becomes $\dfrac{1}{s^{p-1}} \partial V(x') \cdot A \dfrac{x'}{s} \leq -\gamma$ for all $x' \neq 0$ which is exactly (10). Similarly, (11) is equivalent to

$$\partial V^*(\xi) \cdot A\xi \leq -\gamma \quad \text{for all } \xi \text{ s.t. } V(\xi) = 1/q. \tag{13}$$

Now, (12) means that $\xi \cdot Ax \leq -\gamma$ for any element ξ of $\partial V(x)$ with $V(x) = 1/p$. By what we have shown at the beginning of the proof, such x and ξ can be equivalently characterized by $x \in \partial V^*(\xi)$, $V^*(\xi) = 1/q$. Thus (12) is equivalent to (13). ■

When V (and automatically V^*) is positively homogeneous of degree 2, the coefficients $p\gamma$ and $q\gamma$ in (10) and (11) are the same. Such functions naturally appear in stability analysis of LDIs. Suppose that the origin of (1) is asymptotically stable (which is equivalent to exponential stability, that is, for some $c \geq 1$ and decay rate $\beta > 0$,

$$\|x(t)\| \leq c\|x(0)\|e^{-\beta t}$$

for all solutions to (1)). Then, as was shown in [14], then there exist $\gamma > 0$ and a convex, positive definite, and homogeneous of degree 2 function such that

$$\partial V(x) \cdot Ax \leq -2\gamma V(x) \quad \text{for all } x \in \mathbb{R}^n \tag{14}$$

for all $A \in \operatorname{co}\{A_i\}_{i=1}^m$. In Example 4.3 we write down one possible Lyapunov function, for which we actually have $\gamma = \beta$, that is, γ in (14) is exactly the decay rate.

Corollary 4.2: The origin of (1) is exponentially stable (with decay rate β) if and only if (2) is exponentially stable (with decay rate β).

An immediate practical consequence of this corollary is that to verify exponential stability of (1) with a particular computational method, one can also apply that same method to transpose matrices. This can dramatically improve the results, as we illustrate in Example 5.2.

Example 4.3: Suppose that the LDI (1) is exponentially stable with decay rate β. One way to construct a Lyapunov function verifying this is to consider

$$V(x_0) = \frac{1}{2} \sup \|x(t)\|^2 e^{2\beta\tau}, \tag{15}$$

where the supremum is taken over all solutions to (1) with $x(0) = x_0$ and all $t \geq 0$. It is a convex, positive definite, and homogeneous of degree 2 function. The conjugate function V^* turns out to be $V^*(\xi) = \frac{1}{2} \operatorname{co} \inf e^{-2\beta\tau}\|\xi(t)\|^2$ with the infimum taken over all $t \geq 0$ and all solutions to $\dot{\xi}(t) \in \operatorname{co}\left\{-A_i^T\right\}_{i=1}^m \xi$. Theorem 4.1 states that this is a Lyapunov function for the dual LDI (2).

Lemma 2.1 and its dual interpretation lead to practical conditions for stability of LDIs, with Lyapunov functions given by (8) or (9).

Corollary 4.4: Suppose that there exist positive definite and symmetric matrices $Q_1, Q_2, \ldots, Q_l$ and numbers $\lambda_{ijk} \geq 0$ for $i = 1, 2, \ldots, m,\ j, k = 1, 2, \ldots, l$ such that

$$A_i^T Q_j + Q_j A_i \leq \sum_{k=1}^{l} \lambda_{ijk}(Q_k - Q_j) - 2\gamma Q_j \tag{16}$$

for all $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, l$. Then

$$\partial V(x) \cdot Ax \leq -2\gamma V(x) \quad \forall x \in I\!R^n, A \in \mathrm{co}\{A_i\}_{i=1}^m, \tag{17}$$

where V is the maximum of quadratic functions $x \mapsto \frac{1}{2} x \cdot Q_j x$.

Proof: Since $\lambda_{ijk} \geq 0$, the inequality (16) implies that for any x with $x \cdot Q_k x \leq x \cdot Q_j x$ for all $k = 1, 2, ...l$, it holds that

$$x \cdot (A_i^T Q_j + Q_j A_i)x \leq -2\gamma x \cdot Q_j x.$$

Invoking Lemma 2.1 with $h_j(x) = \frac{1}{2} x \cdot Q_j x$ finishes the proof. ■

Corollary 4.5: Suppose that there exist positive definite and symmetric matrices $R_1, R_2, \ldots, R_l$ and numbers $\lambda_{ijk} \geq 0$ for $i = 1, 2, \ldots, m,\ j, k = 1, 2, \ldots, l$ such that

$$R_j^{-1} A_i^T + A_i R_j^{-1} \leq \sum_{k=1}^{l} \lambda_{ijk}(R_k^{-1} - R_j^{-1}) - 2\gamma R_j^{-1} \tag{18}$$

for all $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, l$. Then

$$\partial V(x) \cdot Ax \leq -2\gamma V(x) \quad \forall x \in I\!R^n, A \in \mathrm{co}\{A_i\}_{i=1}^m, \tag{19}$$

where V is the convex hull of quadratic functions $x \mapsto \frac{1}{2} x \cdot R_j x$.

Proof: By Corollary 4.4, (18) implies that

$$\partial V^*(\xi) \cdot A^T \xi \leq -2\gamma V^*(\xi) \quad \forall \xi \in I\!R^n, A \in \mathrm{co}\{A_i\}_{i=1}^m,$$

with V^* being the maximum of quadratic functions $\xi \mapsto \frac{1}{2} \xi \cdot R_j^{-1} \xi$. This is equivalent to the desired conclusion, by Theorem 4.1. ■

We note that the existence of solutions to the bilinear matrix inequalities in Corollary 4.4 is not equivalent to the existence of solutions to the inequalities in Corollary 4.5. This is expected, as the existence is only sufficient for the max function and the convex hull function to be Lyapunov functions. Existence of solutions is necessary only when $l = 2$, see [4], page 73. Even then, there may exist a Lyapunov function for a particular LDI, given by a convex hull (of two quadratics), but not one given by a maximum. Consequently, numerical tests based on Corollaries 4.4 and 4.5, carried out with the same l, may yield different results. See Example 5.3.

Now consider a control system

$$\dot{x} \in \mathrm{co}\left\{ \begin{bmatrix} A & B \end{bmatrix}_i \right\}_{i=1}^{m} \begin{bmatrix} x \\ u \end{bmatrix} \tag{20}$$

and its dual system with output

$$\begin{bmatrix} \dot{\xi} \\ z \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A^T \\ B^T \end{bmatrix}_i \right\}_{i=1}^{m} \xi \ . \tag{21}$$

We say the system (20) is stabilizable by linear feedback (switched linear feedback) if there exists K (m matrices K_i) such that the origin of the system

$$\dot{x} \in \mathrm{co}\left\{A_i + B_i K\right\}_{i=1}^{m} x$$

(respectively the origin of the system

$$\dot{x} \in \mathrm{co}\left\{A_i + B_i K_i\right\}_{i=1}^{m} x \ )$$

is exponentially stable. The system (21) is stabilizable by linear output injection (switched linear output injection) if there exists L (m matrices L_i) such that the origin for

$$\dot{\xi} \in \mathrm{co}\left\{A_i^T + L B_i^T\right\}_{i=1}^{m} \xi$$

(the origin for

$$\dot{\xi} \in \mathrm{co}\left\{A_i^T + L_i B_i^T\right\}_{i=1}^{m} \xi \ )$$

is exponentially stable.

Corollary 4.6: The system (20) is stabilizable by linear feedback (respectively, switched linear feedback) if and only if the system (21) is stabilizable by linear (respectively, switched linear) output injection.

V. Numerical Examples

In this section, we illustrate the main points of the paper through numerical examples. Example 5.1 is a reminder that the use of nonquadratic functions can improve stability estimates over those obtained with quadratic functions. We show this using the max function, but the same conclusion can be made based on homogeneous polynomial functions of Example 5.2. Furthermore, Example 5.1 suggests that considering a broad enough class of potential Lyapunov functions can improve stability estimates over those computed via specialized analytical approaches. Examples 5.2 and 5.3 illustrate the benefits of duality. Example 5.2 uses homogeneous polynomial functions, and implicitly (through conjugacy) finds a Lyapunov function homogeneous of degree $4/3$. Example 5.3 shows that using the maximum of quadratic functions and the convex hull of the same number of quadratics can lead to different stability estimates.

In Examples 5.1 and 5.3 we rely on the bilinear matrix inequalities of Corollaries 4.4 or 4.5 to show that the max function q given by (8) or the convex hull function q^* given by (9) verifies stability of certain LDIs. In solving the matrix inequalities, we rely on algorithms based on a path-following method in [10]. For related work on solution methods, see also [3], [9].

Example 5.1: In [6], an LDI given by $\mathrm{co}\{A_1, A_2\}$ with

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -a \\ 1/a & -1 \end{bmatrix}$$

and $a > 1$ was used to show that the existence of a common quadratic Lyapunov function is not necessary for exponential stability of the LDI. The maximal a ensuring existence of such a function was found to be $a_q = 3 + \sqrt{8} = 5.8284$, while the LDI was shown, via a phase plane method not leading to a Lyapunov function, to be stable for all $a \in [1, 10]$. (As pointed out in [6], the analytical method is highly unlikely to be feasible for general systems.)

With q^* formed by two quadratics ($l = 2$), the maximal a is 8.11. With $l = 3$, the maximal a is 8.95. The three matrices (corresponding to R_j^{-1} in Corollary 4.5) determined under $a = 8.95$ are as follows:

$$\begin{bmatrix} 26.1802 & -0.0273 \\ -0.0273 & 2.9146 \end{bmatrix}, \begin{bmatrix} 16.5961 & 3.0303 \\ 3.0303 & 3.6388 \end{bmatrix}, \begin{bmatrix} 32.5579 & -3.0335 \\ -3.0335 & 1.8518 \end{bmatrix}.$$

Corresponding ellipsoids (points x with $x \cdot R_j x = 1$) and the boundary of their convex hull (points x with $q^*(x) = 1$) are in the upper two plots of Fig. 1. Also plotted there are directions of $\dot{x} = A_1 x$ (left) and that of $\dot{x} = A_2 x$ (right) along the boundary of the convex hull. Lower plots of Fig. 1 are the ellipsoids $x \cdot R_j^{-1} x = 1$ and their intersection, along with the direction of $\dot{y} = A_1^T y$ and $\dot{y} = A_2^T(a) y$ along the boundary of the intersection.

We add that using seven quadratic functions ($l = 7$) verifies that the LDI is stable for a up to 10.108, and thus improves on the analytical estimate in [6]. For further details, see [7].

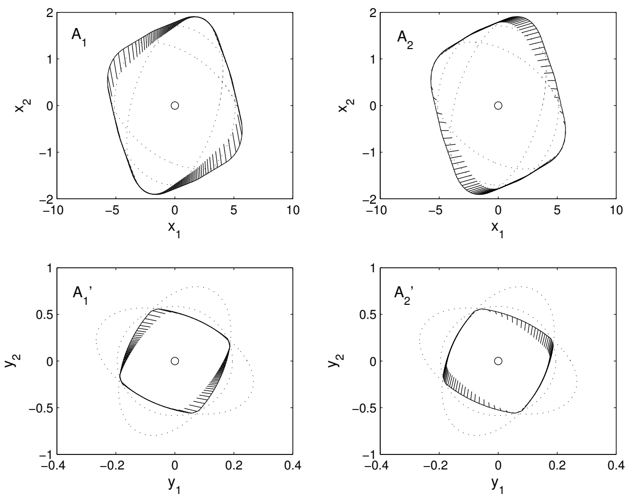

Fig. 1. Vector fields and invariant level sets

Example 5.2: The following third-order LDI was discussed in [5]. For the matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix}, \quad M = \begin{bmatrix} -2 & 0 & -1 \\ 1 & -10 & 3 \\ 3 & -4 & 2 \end{bmatrix},$$

let $A_2 = A_1 + aM$ with $a > 0$, and consider the LDI with the state matrix belonging to the set $\mathrm{co}\{A_1, A_2(a)\}$. The maximal a that ensures the existence of a common quadratic function is $a_q = 1.9042$. The maximal a that ensures the existence of a common fourth-order homogeneous Lyapunov function was found in [5] to be $a_h = 75.1071$.

By Corollary 4.2, exponential stability of the LDI is equivalent to that of the dual LDI described by $\mathrm{co}\{A_1^T, A_2(a)^T\}$. For this dual system, we used the method from [5] to determine a parameter range of a over which a common fourth-order homogeneous Lyapunov function exists. It turns out that there is no upper bound for a. Let A_{m1}^T be the augmented matrix for A_1^T and $A_{m2}(a)^T$ be the augmented matrix for $A_2(a)^T$. Let $L(\alpha)$ be the matrix containing auxiliary parameters (see, page 1032 of [5]). Then for each $a > 0$, there exist a symmetric positive definite matrix $Q = \mathbb{R}^{6\times 6}$ and parameters $\alpha, \beta \in \mathbb{R}^6$ such that

$$QA_{m1}^T + A_{m1}Q + L(\alpha) \leq -0.0606Q,$$

$$QA_{m2}(a)^T + A_{m2}(a)Q + L(\beta) \leq -0.0606Q.$$

No numerical problem arises even for $a = 10^{20}$.

We point out that the existence of a 4-th order homogeneous polynomial Lyapunov function for the dual LDI implies, by Theorem 4.1, the existence of a Lyapunov function for the original LDI, homogeneous of degree $4/3$.

Example 5.3: We analyzed the LDI of Example 5.2, and its dual LDI, using the max function q and the composite quadratic function q^* with $l = 2$. Stability of the original LDI can be verified with q^* for a up to 441. For $a = 441$, there exist $Q_1 > 0$ and $Q_2 > 0$ satisfying

$$
\begin{aligned}
&Q_1A_1^T + A_1Q_1 < 5.008(Q_2 - Q_1)\\
&Q_2A_1^T + A_1Q_2 < 0\\
&Q_1A_2^T + A_2Q_1 < 0\\
&Q_2A_2^T + A_2Q_2 < 2708.9(Q_1 - Q_2)
\end{aligned}
$$

The same algorithm used for the dual LDI (equivalently, relying on q for the original LDI) shows that there is no upper bound for a. Actually, for each $a > 0$, there exist $Q_1, Q_2 > 0$ and $\lambda_{ij} \geq 0$ such that

$$Q_jA_i + A_i^TQ_j < \lambda_{ij}(Q_k - Q_j) - 0.0530Q_j$$

for $i, j, k = 1, 2,\ j \neq k$. We tested a up to 10^{20} and no numerical issues occur. For $a = 10^8$, $\lambda_{11} = 3.5473$, $\lambda_{12} = 0$, $\lambda_{21} = 87614$, $\lambda_{22} = 7.4699 * 10^8$.

This suggests that, for the case of $l = 2$, q "performs better" than q^* in the stability analysis of the original LDI. However, duality implies that for the inclusion $\dot{\xi} \in \mathrm{co}\{A_1^T, A_2^T(a)\}$, the reverse is true: q^* "performs better" than q.

VI. Conclusions

In this note, we established a one-to-one relationship between convex positively homogeneous Lyapunov functions verifying the asymptotic stability of a linear differential inclusion and such Lyapunov functions verifying the asymptotic stability of a dual linear differential inclusion. As a consequence, the asymptotic stability of an LDI turns out to be equivalent to the asymptotic stability of the dual LDI. Based on this equivalence, and on the operations of pointwise maximization or forming a convex hull of a family of functions, we showed how Lyapunov functions for LDIs can be constructed. Through numerical examples, we illustrated how applying known numerical techniques to a dual LDI may improve stability estimates for the original LDI. Further examples, a method for verifying instability of an LDI, and a discussion of duality of dissipativity properties, can be found in [8], [7]. Similar results are possible in discrete time; see [7].

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