#1

$$A = \begin{bmatrix} 1 & -2 \\ 5 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \quad x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$$

$$\lambda_1 = -2 - j \quad \lambda_2 = -2 + j$$

$$\bar{A} = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \quad Q = \begin{bmatrix} -.6 & -.2 \\ 1 & 0 \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} 0 & 1 \\ -5 & 3 \end{bmatrix}$$

$$e^{\bar{A}t} = e^{-2t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \qquad e^{At} = Q\, e^{\bar{A}t} Q^{-1}$$

$$e^{At} = \begin{bmatrix} .6 & -.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -5 & 3 \end{bmatrix} e^{-2t}$$

$$= e^{-2t} \begin{bmatrix} .6\cos t - .2\sin t & -.6\sin t - .2\cos t \\ \cos t & -\sin t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -5 & 3 \end{bmatrix}$$

$$= e^{-2t} \begin{bmatrix} 3\sin t + \cos t & .6\cos t - .2\sin t - 1.8\sin t - .6\cos t \\ 5\sin t & \cos t - 3\sin t \end{bmatrix}$$

$$e^{At} = e^{-2t} \begin{bmatrix} 3\sin t + \cos t & -2\sin t \\ 5\sin t & -3\sin t + \cos t \end{bmatrix}$$

#1.2

$$y(t) = Ce^{At}x_0$$

$$y(t) = [1 \ 1] e^{-2t} \begin{bmatrix} 3\sin t + \cos t & -2\sin t \\ 5\sin t & -3\sin t + \cos t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \ 1] e^{-2t} \begin{bmatrix} -2\sin t \\ -3\sin t + \cos t \end{bmatrix}$$

$$= e^{-2t} (-2\sin t - 3\sin t + \cos t)$$

For $t > 0$

$$y(t) = e^{-2t} [-5\sin t + \cos t]$$

Problem 2

$$\dot{x}=\begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 1\\ 0 & 0 & 0 & \lambda_2\end{bmatrix}x+\begin{bmatrix}1 & a\\ 1 & -a\\ b & 1\\ -b & 1\end{bmatrix}u \qquad y=\begin{bmatrix}1 & a & b & 1\\ a & 1 & 1 & a\end{bmatrix}x$$

The system is controllable if $M(\lambda)=\begin{bmatrix}A-\lambda I & B\end{bmatrix}$ has full row rank (4) at every eigenvalue of A.

$$M(\lambda)=\left[\left(\begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 1\\ 0 & 0 & 0 & \lambda_2\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0 & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & \lambda\end{bmatrix}\right)\begin{bmatrix}1 & a\\ 1 & -a\\ b & 1\\ -b & 1\end{bmatrix}\right]$$

$$M(\lambda)=\begin{bmatrix}\lambda_1-\lambda & 0 & 0 & 0 & 1 & a\\ 0 & \lambda_1-\lambda & 0 & 0 & 1 & -a\\ 0 & 0 & \lambda_2-\lambda & 1 & b & 1\\ 0 & 0 & 0 & \lambda_2-\lambda & -b & 1\end{bmatrix}$$

To test this we need to find the eigenvalues of A.

$$\det(sI-A)=\det\left(\begin{bmatrix}s & 0 & 0 & 0\\ 0 & s & 0 & 0\\ 0 & 0 & s & 0\\ 0 & 0 & 0 & s\end{bmatrix}-\begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 1\\ 0 & 0 & 0 & \lambda_2\end{bmatrix}\right)=\det\left(\begin{bmatrix}s-\lambda_1 & 0 & 0 & 0\\ 0 & s-\lambda_1 & 0 & 0\\ 0 & 0 & s-\lambda_2 & -1\\ 0 & 0 & 0 & s-\lambda_2\end{bmatrix}\right)$$

$$\det(sI-A)=(s-\lambda_1)^2(s-\lambda_2)^2$$

Obviously the eigenvalues are just λ_1 (multiplicity 2) and λ_2 (multiplicity 2). So we only need to check the row rank for $M(\lambda_1)$ and $M(\lambda_2)$.

$$M(\lambda_1)=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & a\\ 0 & 0 & 0 & 0 & 1 & -a\\ 0 & 0 & \lambda_2-\lambda_1 & 1 & b & 1\\ 0 & 0 & 0 & \lambda_2-\lambda_1 & -b & 1\end{bmatrix}, M(\lambda_2)=\begin{bmatrix}\lambda_1-\lambda_2 & 0 & 0 & 0 & 1 & a\\ 0 & \lambda_1-\lambda_2 & 0 & 0 & 1 & -a\\ 0 & 0 & 0 & 1 & b & 1\\ 0 & 0 & 0 & 0 & -b & 1\end{bmatrix}$$

We can make a list of conditions that will prevent $M(\lambda)$ from full row rank and thus preventing the system from being controllable:

$$\lambda_1=\lambda_2 \text{ or } a=0$$

$$\{A,B\}\textit{controllable} \text{ iff } \lambda_1\neq\lambda_2 \text{ and } a\neq 0$$

$$\dot{x}=\begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 1\\ 0 & 0 & 0 & \lambda_2\end{bmatrix}x+\begin{bmatrix}1 & a\\ 1 & -a\\ b & 1\\ -b & 1\end{bmatrix}u \qquad y=\begin{bmatrix}1 & a & b & 1\\ a & 1 & 1 & a\end{bmatrix}x$$

The system is observable if $M(\lambda)=\begin{bmatrix}A-\lambda I\\ C\end{bmatrix}$ has full column rank (4) at every eigenvalue of A.

$$M(\lambda)=\begin{bmatrix}\begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 1\\ 0 & 0 & 0 & \lambda_2\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0 & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & \lambda\end{bmatrix}\\ \begin{bmatrix}1 & a & b & 1\\ a & 1 & 1 & a\end{bmatrix}\end{bmatrix}$$

$$M(\lambda)=\begin{bmatrix}\lambda_1-\lambda & 0 & 0 & 0\\ 0 & \lambda_1-\lambda & 0 & 0\\ 0 & 0 & \lambda_2-\lambda & 0\\ 0 & 0 & 0 & \lambda_2-\lambda\\ 1 & a & b & 1\\ a & 1 & 1 & a\end{bmatrix}$$

We need to check the column rank for $M(\lambda_1)$ and $M(\lambda_2)$.

$$M(\lambda_1)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \lambda_2-\lambda_1 & 0\\ 0 & 0 & 0 & \lambda_2-\lambda_1\\ 1 & a & b & 1\\ a & 1 & 1 & a\end{bmatrix}, M(\lambda_2)=\begin{bmatrix}\lambda_1-\lambda_2 & 0 & 0 & 0\\ 0 & \lambda_1-\lambda_2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & a & b & 1\\ a & 1 & 1 & a\end{bmatrix}$$

We can make a list of conditions that will prevent $M(\lambda)$ from full row rank and thus preventing the system from being observable:

$$\lambda_1=\lambda_2 \text{ or } |a|=1$$

$$\{A,C\}\,observable \text{ iff } \lambda_1\neq\lambda_2 \text{ and } |a|\neq 1$$

Problem 3

$$\dot{x} = \begin{bmatrix} -1 & -2 & -2 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} -2 & -1 \\ 1 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 2 & 3 & 0 & 1 \end{bmatrix} x$$

$$G_3{}^C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} -2 & -1 & 6 & 2 & -8 & 0 \\ 1 & 1 & -2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & 0 & 8 & 2 \end{bmatrix} \qquad \rho\left(G_3{}^C\right) = 3 < 4 \text{ un-controllable}$$

Let Q be the first three columns of $G_3{}^C$ in the red box plus another linearly independent row.

$$Q = \begin{bmatrix} -2 & -1 & 6 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & -2 & 0 \end{bmatrix} \qquad \det(Q) \neq 0 \qquad \text{Q non-singular, } \rho(Q) = 4$$

$$\overline{A} = Q^{-1} \cdot A \cdot Q = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0.5 & 2 & 0 & -0.5 \\ 0.25 & 0 & 0 & 0.25 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 & -2 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} -2 & -1 & 6 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 & 0 \\ 0 & -1 & -4 & 2 \\ 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{B} = Q^{-1} \cdot B = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0.5 & 2 & 0 & -0.5 \\ 0.25 & 0 & 0 & 0.25 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\overline{C} = C \cdot Q = \begin{bmatrix} 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 6 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 0 \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} \overline{A}_C & \overline{A}_{12} \\ 0 & \overline{A}_{\overline{C}} \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & -1 & -4 & 2 \\ 1 & 0.5 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \overline{B} = \begin{bmatrix} \overline{B}_C \\ 0 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \overline{C} = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \end{array}\right]$$

Since $(\overline{A}_C, \overline{B}_C)$ is controllable, the eigenvalues using state feedback, specifically $eig(\overline{A}_C - \overline{B}_C \overline{K}_1)$ can be arbitrarily assigned. So this part of the system is stabilizable. The eigenvalues of $\overline{A}_{\overline{C}}$ (the uncontrollable mode) cannot be changed by any state feedback. For the system as a whole to be stabilizable, the uncontrollable mode has to be stable. So lets check.

$$eig(\overline{A}_{\overline{C}}) = eig([1]) = 1$$

This eigenvalue is on the right half plane, there is an unstable eigenvalue located in an uncontrollable mode, therefore the system is not stabilizable.

Problem 4

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \qquad y = Cx = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} x$$

First, we should check the observability:

$$G_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \qquad \rho(G_O) = 4 \therefore \text{observable}$$

Select F to match the desired eigenvalues:

$$F = \begin{bmatrix} -5 & -5 & 0 & 0 \\ 5 & -5 & 0 & 0 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & 10 & -10 \end{bmatrix}$$

Select L_0 such that $\{F, L_0\}$ is controllable.

$$L_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \rho\left(\begin{bmatrix} L_0 & F \cdot L_0 & F^2 \cdot L_0 & F^3 \cdot L_0 \end{bmatrix}\right) = 4 \therefore \text{Controllable}$$

$$T = lyap(-F, A, -L_0 \cdot C) = \begin{bmatrix} -0.0080 & 0.0318 & 0 & -0.0130 \\ 0.1991 & -0.0186 & -0.2000 & 0.0171 \\ -0.0010 & 0.0065 & 0 & -0.0039 \\ 0.1000 & -0.0050 & -0.1000 & 0.0047 \end{bmatrix}$$

$$L = T^{-1} \cdot L_O = 10^3 \cdot \begin{bmatrix} -2.1499 \\ -1.2543 \\ -2.1829 \\ -1.8103 \end{bmatrix}$$

Performing a sanity check by checking the eigenvalues of (A-LC) proves successful.

Next step is to design state feedback to place the eigenvalues. First, make sure that the system is controllable.

$$G_C = \begin{bmatrix} B & A \cdot B & A^2 \cdot B & A^3 \cdot B \end{bmatrix} \qquad \rho(G_C) = 4 \therefore \text{controllable}$$

Select F to get the desired eigenvalues:

$$F = \begin{bmatrix} -2 & -2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 4 & -4 \end{bmatrix}$$

Choose K_0:

$$K_0 = \begin{bmatrix} 0 & 1 & 0 & 0.1 \\ 0.1 & 0 & 0.1 & 0 \end{bmatrix}$$

Check that $\{F, K_0\}$ observable:

$$G_O = \begin{bmatrix} F & K_0 \cdot F & K_0 \cdot F^2 & K_0 \cdot F^3 \end{bmatrix} \qquad \rho(G_0) = 4 \therefore \text{observable}$$

Get T

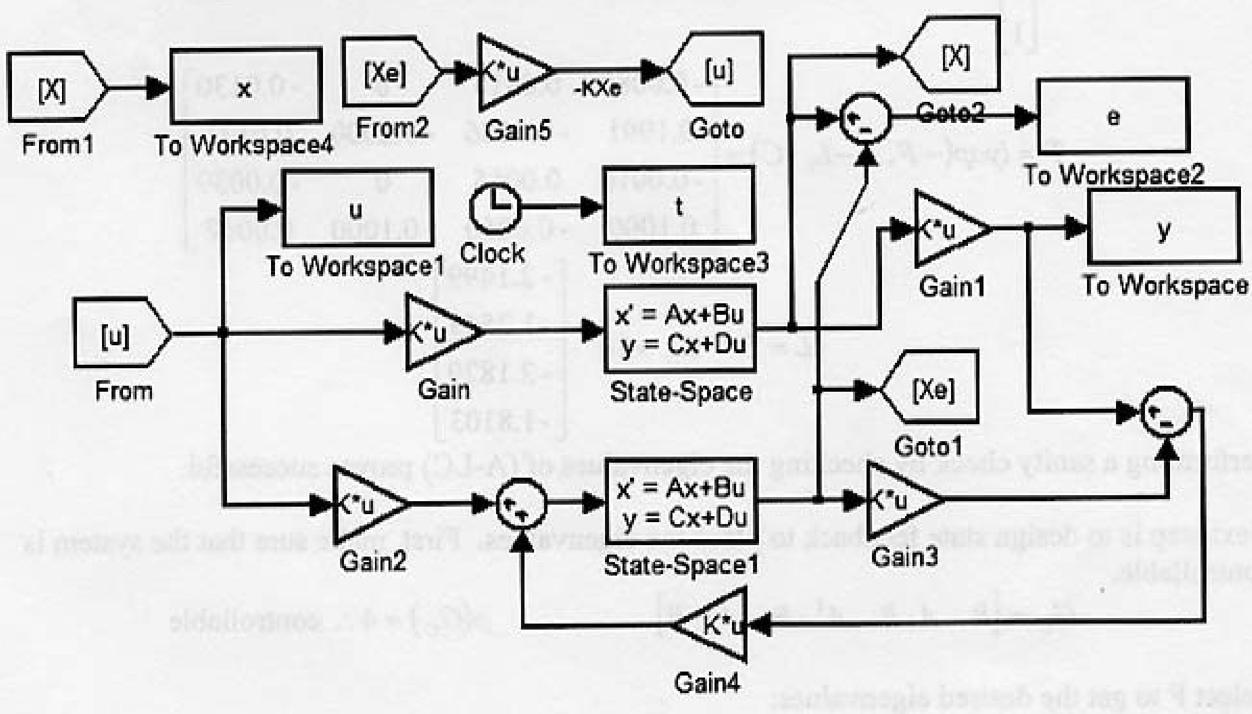
$$T = lyap(A, -F, -B \cdot K_0) = \begin{bmatrix} -0.1207 & 0.0817 & -0.0037 & 0.0005 \\ 0.4049 & 0.0780 & 0.0167 & 0.0128 \\ 0.0245 & -0.0397 & 0.0001 & 0.0013 \\ -0.1283 & 0.0304 & 0.0050 & -0.0057 \end{bmatrix}$$

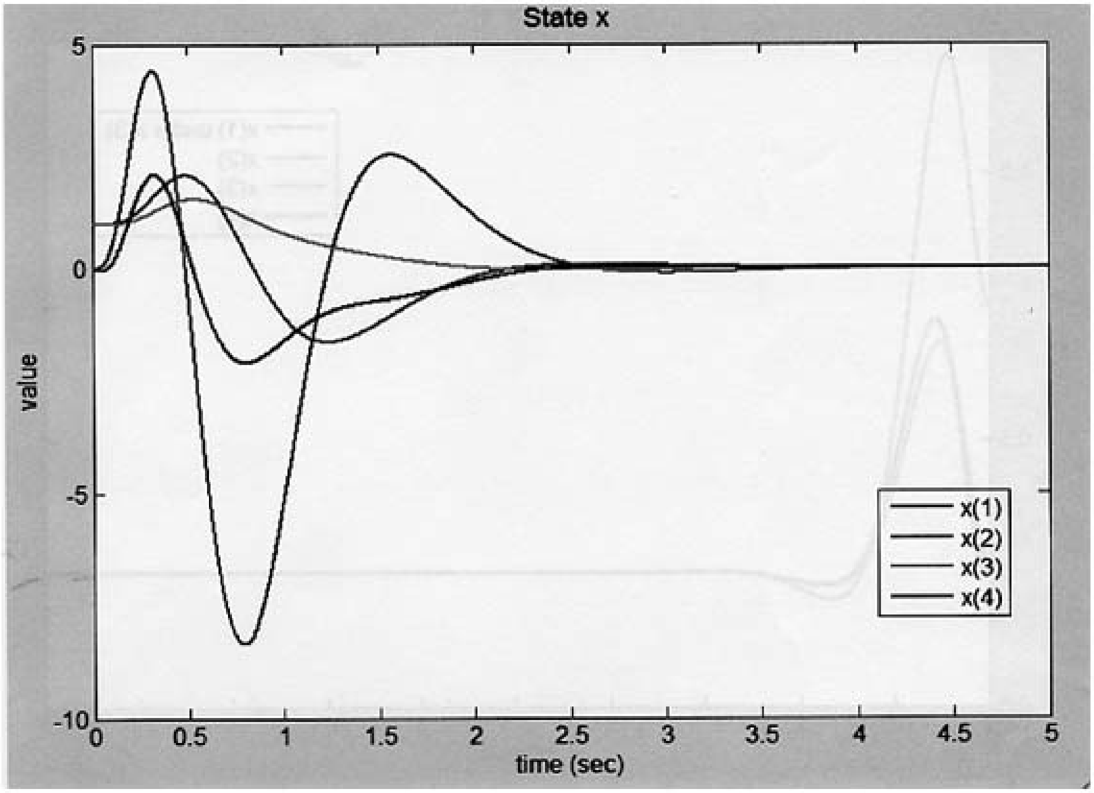
Get K

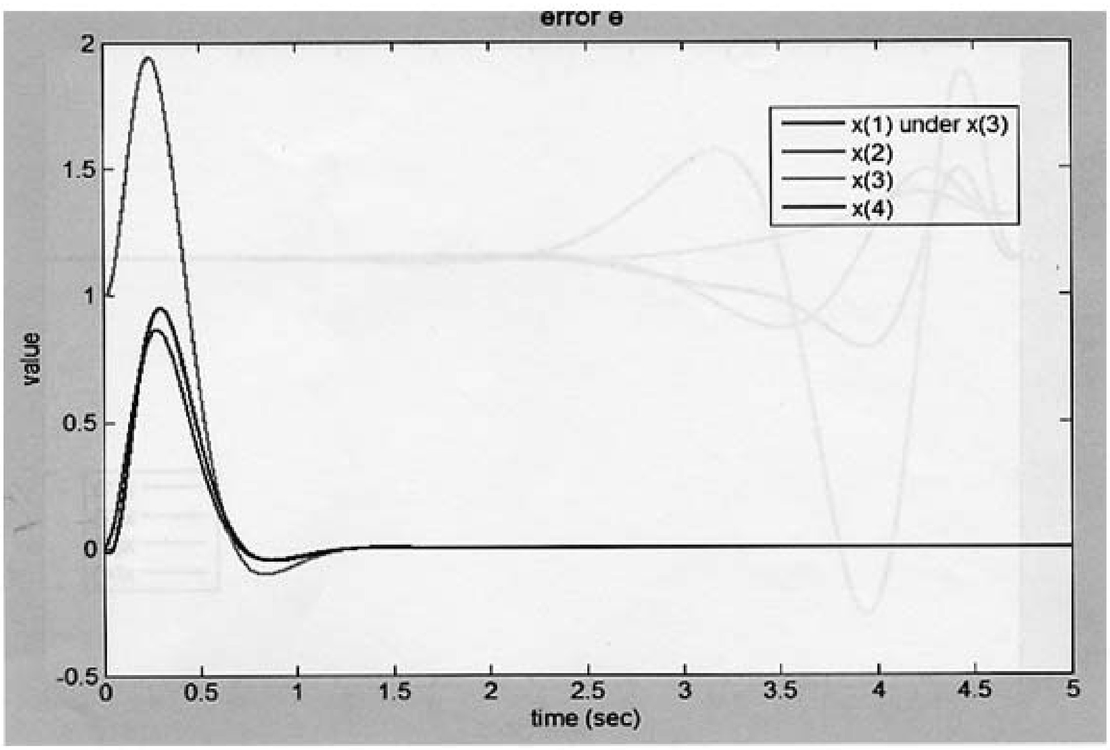
$$K = \begin{bmatrix} 19.9430 & 4.3860 & 24.2803 & -0.2909 \\ 1.0942 & 2.9571 & 16.1985 & 10.6140 \end{bmatrix}$$

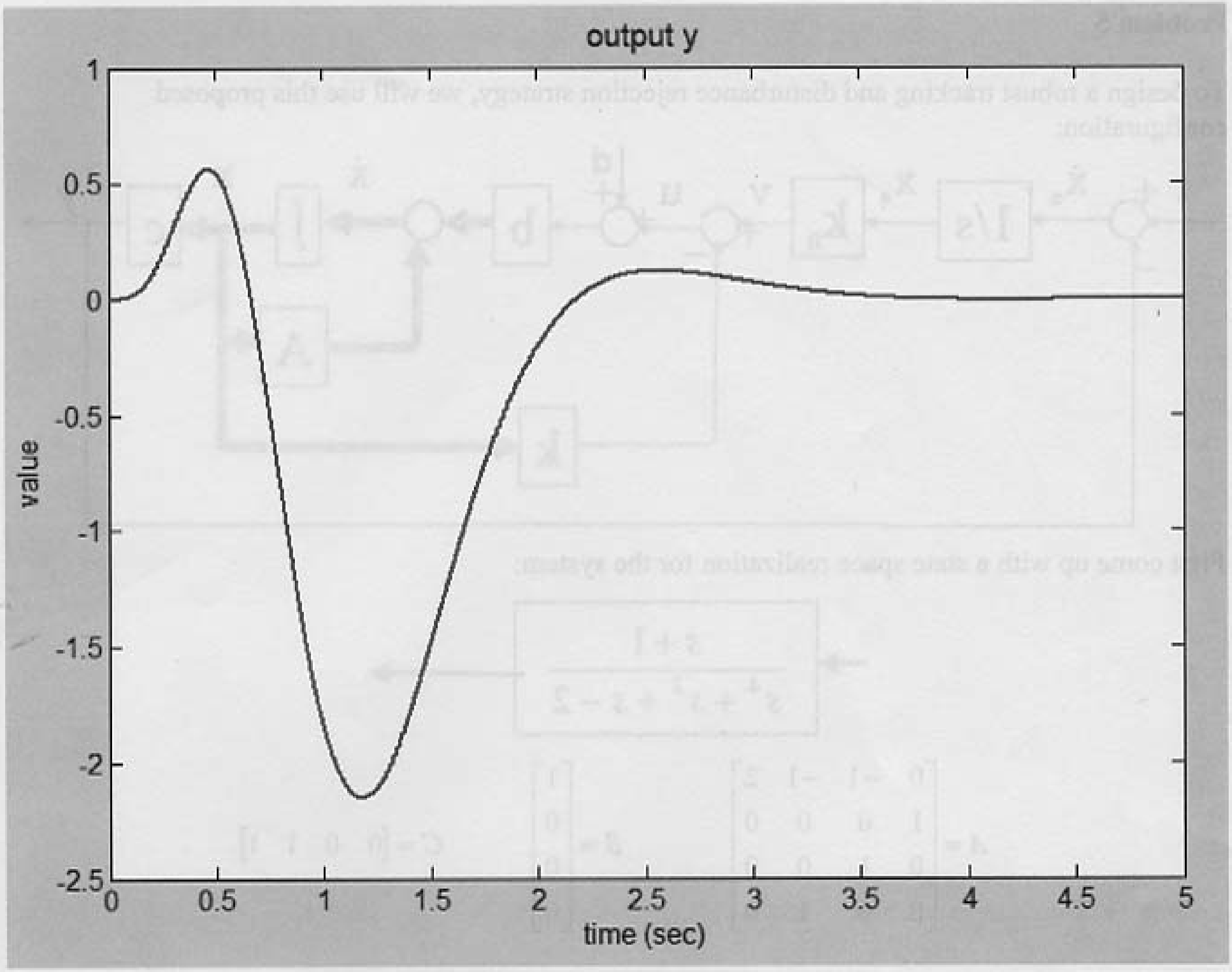
Checking the eigenvalues of (A-B·K) proves successful.

Design is done, now we can plot the responses with the following model:



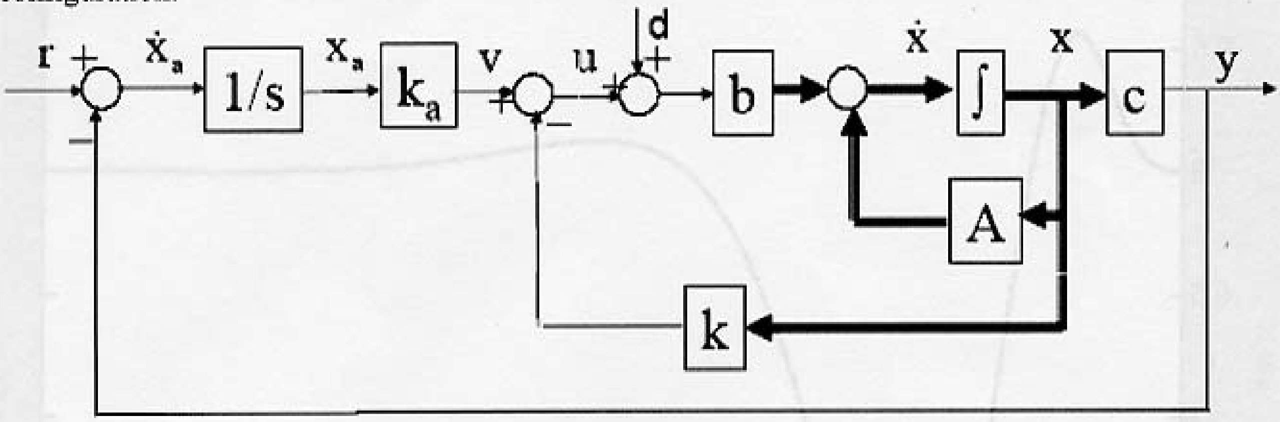

State x
value
time (sec)
x(1)
x(2)
x(3)
x(4)


error e
value
time (sec)
x(1) under x(3)
x(2)
x(3)
x(4)


output y
value
time (sec)
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Problem 5

To design a robust tracking and disturbance rejection strategy, we will use this proposed configuration:



First come up with a state space realization for the system:

$$\frac{s+1}{s^4+s^2+s-2}$$

$$A=\begin{bmatrix}0 & -1 & -1 & 2\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\qquad B=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\qquad C=\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}$$

Now for design we define:

$$A_L=\begin{bmatrix}A & 0\\ -c & 0\end{bmatrix}=\begin{bmatrix}0 & -1 & -1 & 2 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & -1 & 0\end{bmatrix}\qquad b_L=\begin{bmatrix}B\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

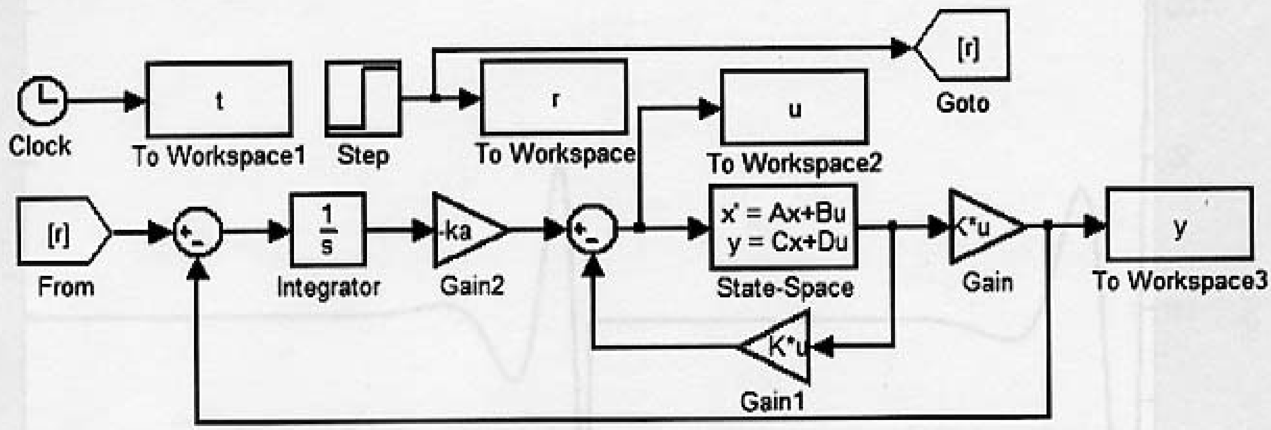
Using the method of pole assignment we now pick:

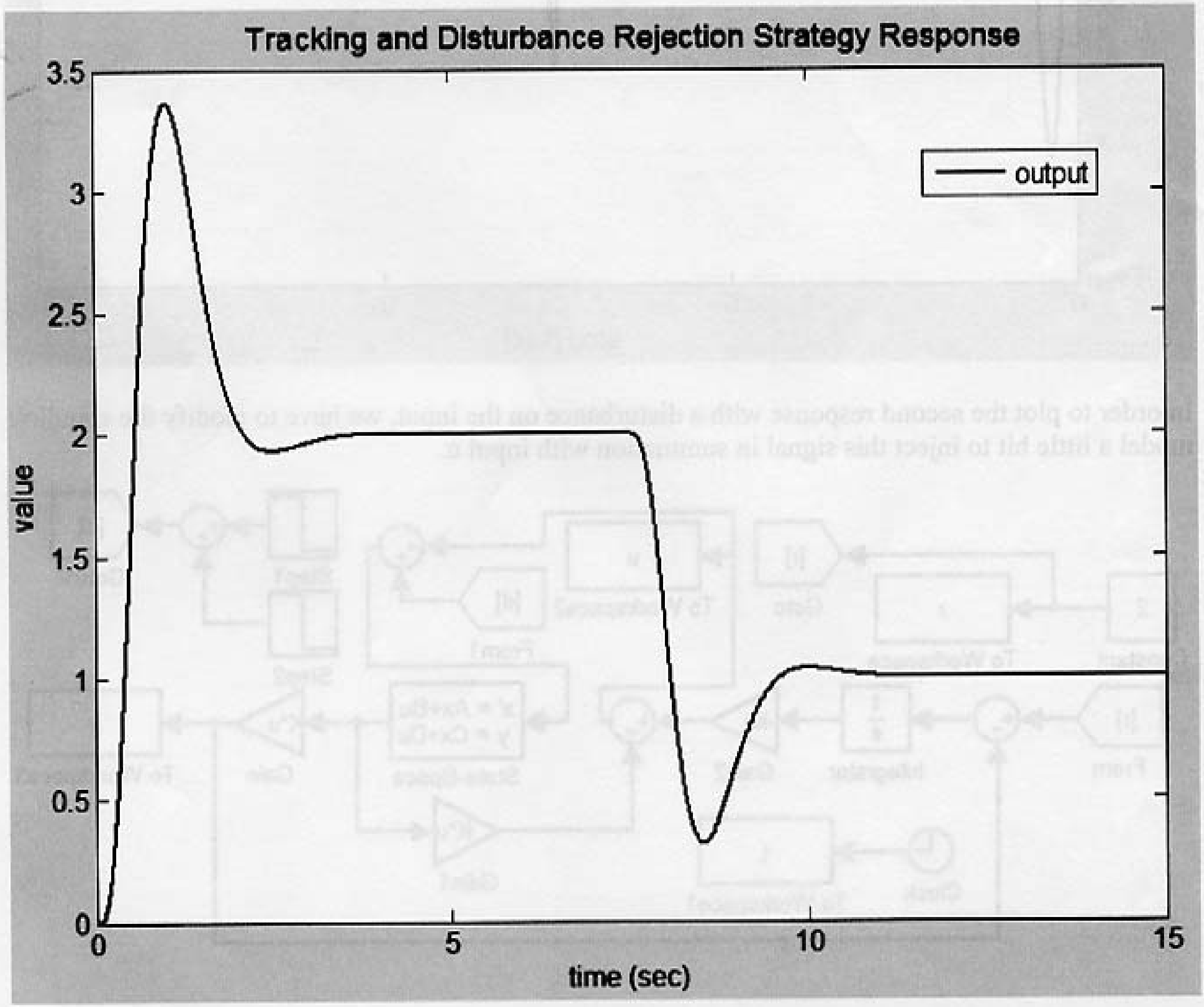
$$F=\begin{bmatrix}-2 & -2 & 0 & 0 & 0\\ 2 & -2 & 0 & 0 & 0\\ 0 & 0 & -4 & -4 & 0\\ 0 & 0 & 4 & -4 & 0\\ 0 & 0 & 0 & 0 & -8\end{bmatrix}\qquad k_0=\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}$$

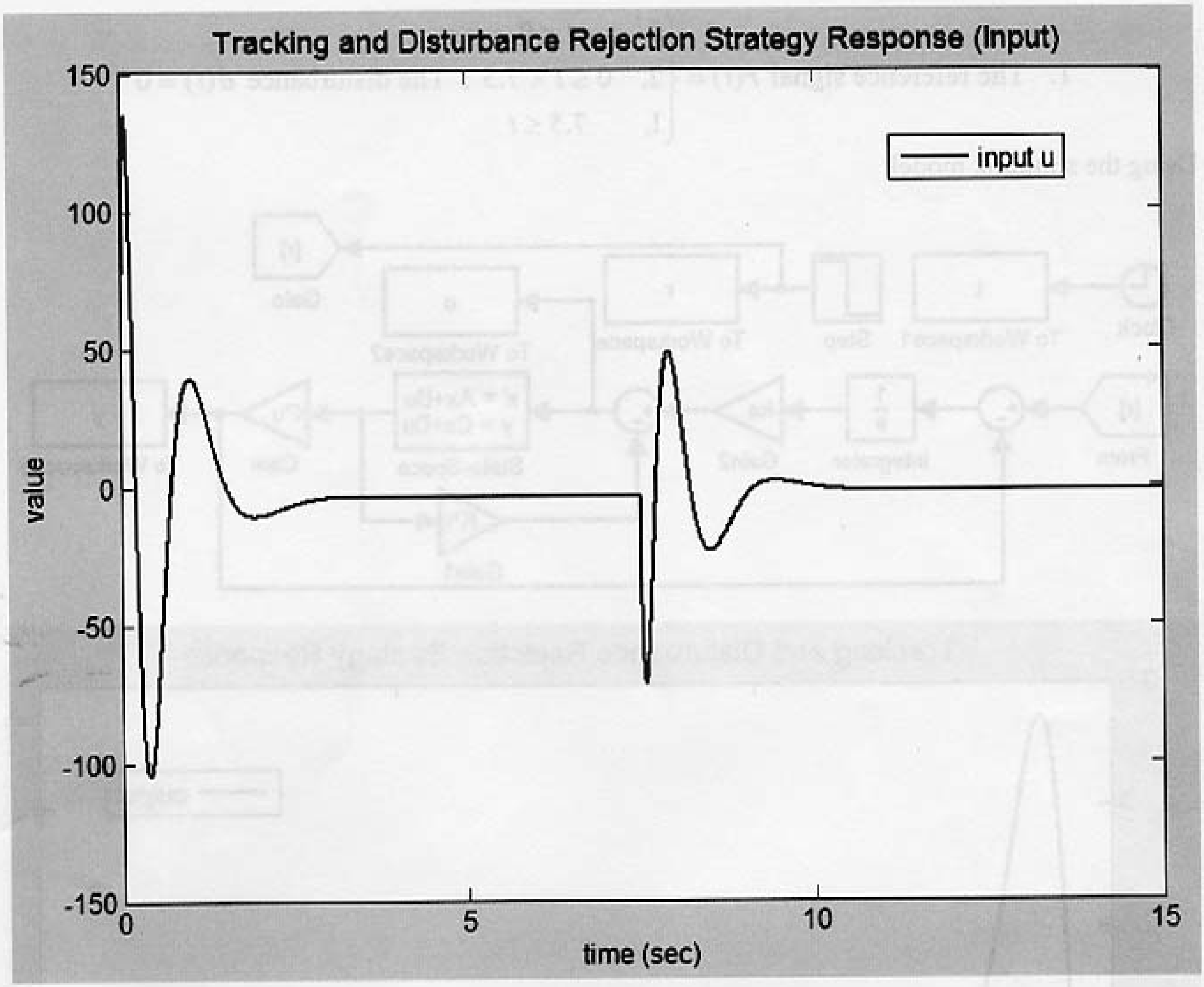
Now we can plot the response in the following case:

1. The reference signal $r(t) = \begin{cases} 0, & t < 0 \\ 2, & 0 \le t < 7.5 \\ 1, & 7.5 \le t \end{cases}$: The disturbance $d(t) = 0$

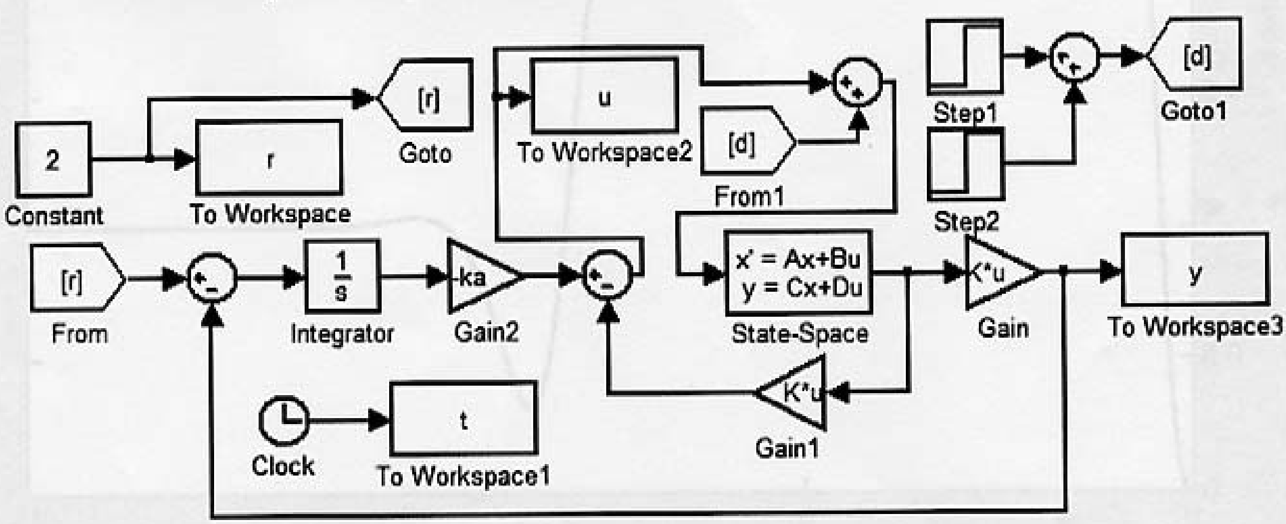
Using the simulink model:



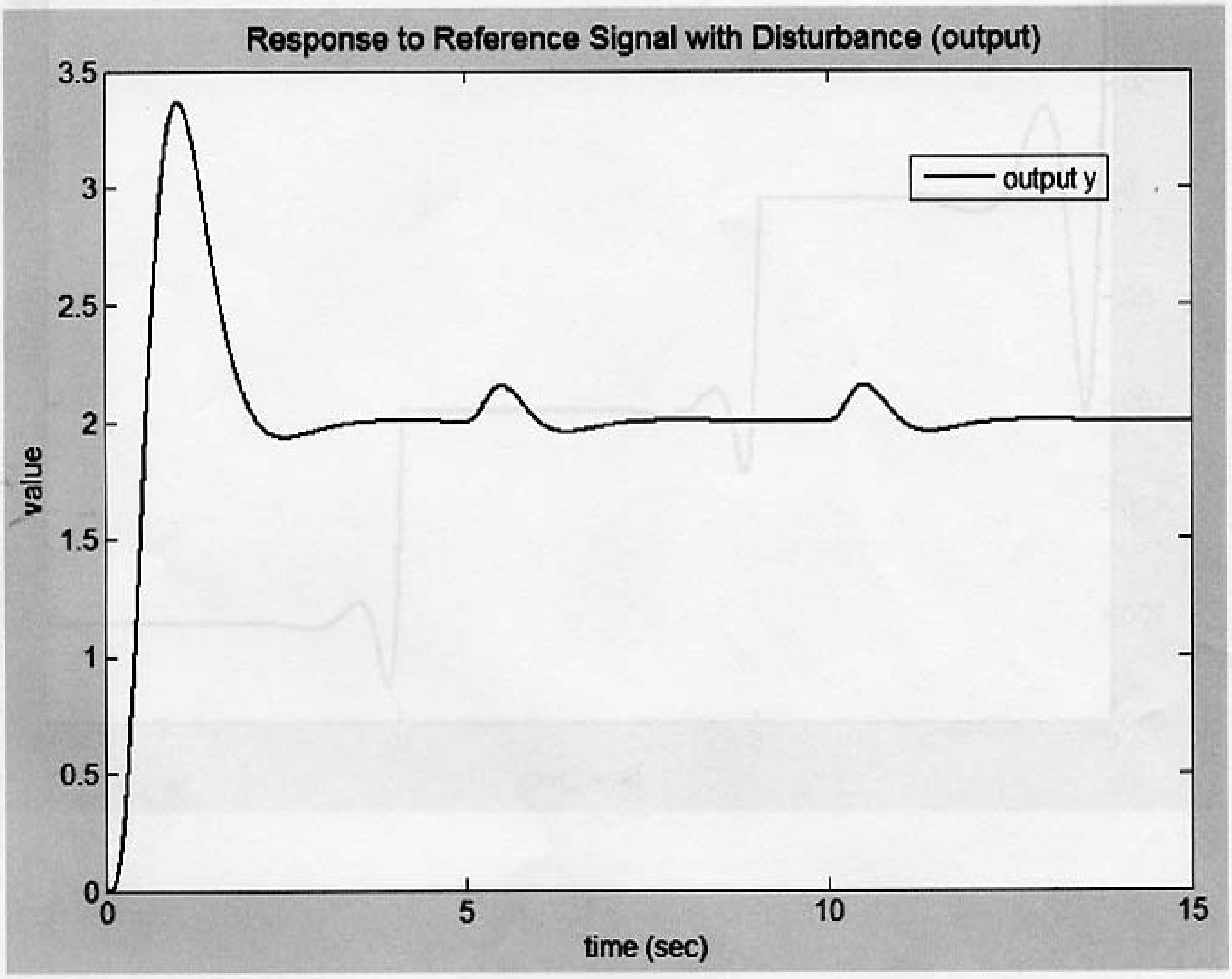


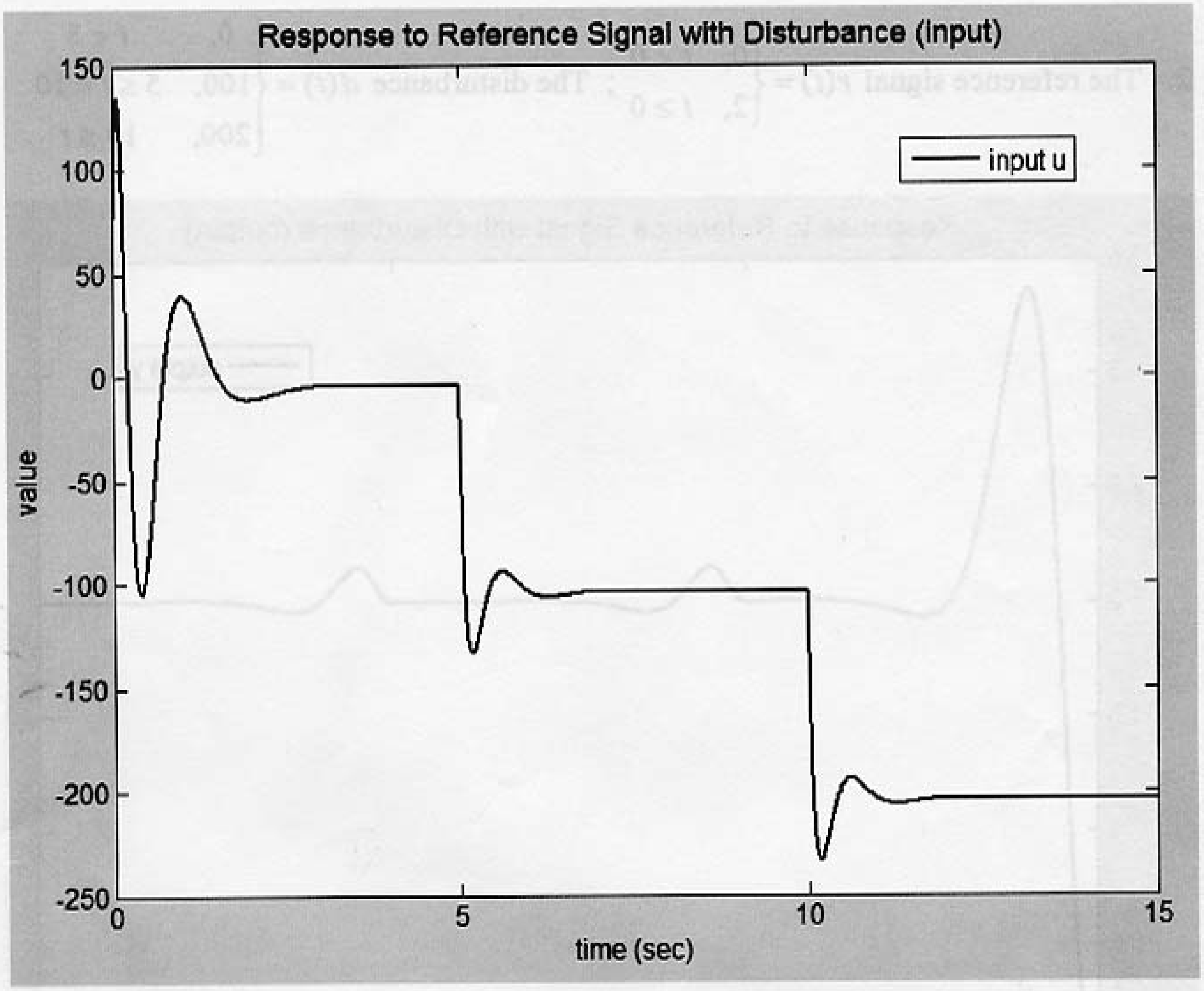


In order to plot the second response with a disturbance on the input, we have to modify the simulink model a little bit to inject this signal in summation with input u.



2. The reference signal $r(t) = \begin{cases} 0, & t < 0 \\ 2, & t \geq 0 \end{cases}$; The disturbance $d(t) = \begin{cases} 0, & t < 5 \\ 100, & 5 \leq t < 10 \\ 200, & 10 \leq t \end{cases}$





Response to Reference Signal with Disturbance (input)
input u
value
time (sec)
150
100
50
0
-50
-100
-150
-200
-250
0
5
10
15