Name: ____________________ March 22, 2007

16.513 Midterm Exam (Spring 2007)

There are 5 problems and two bonus problems.

1. (16) For each of the following sets of vectors, determine if it is linearly dependent or independent:

$$S_1=\left\{\begin{bmatrix}\lambda\\-1\\0\end{bmatrix},\begin{bmatrix}-1\\-2\\1\end{bmatrix},\begin{bmatrix}1\\\lambda-2\\0\end{bmatrix}\right\},\quad S_2=\left\{\begin{bmatrix}1\\0\\2\\3\end{bmatrix},\begin{bmatrix}2\\0\\3\\1\end{bmatrix},\begin{bmatrix}3\\0\\1\\2\end{bmatrix}\right\},\quad S_3=\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\a\end{bmatrix},\begin{bmatrix}-a\\1\end{bmatrix}\right\},\quad S_4=\left\{\begin{bmatrix}\cos t\\\sin t\\1\end{bmatrix},\begin{bmatrix}\sin t\\\cos t\\0\end{bmatrix}\right\}$$

(1) $\det S_1 = -\lambda(\lambda-2)+1 = -(\lambda^2-2\lambda+1) = -(\lambda-1)^2 \begin{cases}=0 \text{ if } \lambda=1 & \to LD\\ \neq 0 \text{ elsewhere} & \to LI\end{cases}$

(2) $\det\begin{vmatrix}1&2&3\\2&3&1\\3&1&2\end{vmatrix} = 6+6+6-9\times3-1-8 = 18-27-1-8=-18\neq0 \to LI$

(3) LD

(4) $\det\begin{bmatrix}\cos t & \sin t\\ \sin t & \cos t\end{bmatrix} = \cos^2 t-\sin^2 t = 0$ if $t=\frac{\pi}{4}\pm k\pi$ two determinants not zero at the same time.

$\det\begin{bmatrix}\sin t & \cos t\\ 1 & 0\end{bmatrix} = -\cos t = 0$ only if $t=\frac{\pi}{2}+k\pi \Rightarrow LI$

2. (24) Find the general solution to the following equations:

(1) $\underbrace{\begin{bmatrix}1&0&-1\\0&-1&1\\-1&1&0\end{bmatrix}}_{A}x=\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{Y}$;

$\text{rank}(A)=2 < \text{rank}([A\ y])=3.$

No solution

(2) $\begin{array}{c}\begin{array}{cccc}a_1&a_2&a_3&a_4\end{array}\\ \begin{bmatrix}1&0&2&1\\0&1&2&2\\-1&1&0&1\end{bmatrix}\end{array}x=\begin{array}{c}b\\ \begin{bmatrix}-2\\-1\\1\end{bmatrix}\end{array}$,

$\text{rank}(A)=2,\quad \nu=4-2=2$

$a_3=2a_1+2a_2 \to n_1=[2\ \ 2\ \ -1\ \ 0]^T$

$2a_4=2a_2+a_3 \to n_2=[0\ \ 2\ \ 1\ \ -2]^T$

$b=-2a_1-a_2 \to$ a particular solution $x_p=[-2\ \ -1\ \ 0\ \ 0]$

General solution $= x_p+k_1n_1+k_2n_2$,

(3) $\begin{bmatrix}1&2&0\\-3&-2&0\end{bmatrix}x=\begin{bmatrix}0\\1\end{bmatrix}$

$\text{rank}(A)=2,\quad \text{rank}([A\cdot b])=2,\quad \exists$ solutions.

$\nu=3-\text{rank}(2)=1,\quad n_1=[0\ \ 0\ \ 1]^T$

a particular solution $x_p=[-0.5\ \ 0.25\ \ 0]^T$

General solution. $x=x_p+kn_1$

3. (15) Let the subspace S be spanned by, $s_1 = \begin{bmatrix} 0 \\ 0.4 \\ 0 \\ 0.3 \end{bmatrix}$, $s_2 = \begin{bmatrix} 1.2 \\ 0.8 \\ -1.6 \\ 0.6 \end{bmatrix}$,

1). Use Gram-Schmidt procedure to obtain orthonormal basis $\{v_1,v_2\}$ for the subspace S;
2). Express s_1,s_2 in terms of v_1, and v_2;
3). Express $x = 2v_1+3v_2$ in terms of s_1 and s_2.

1) Let $e_1 = s_1$, $\quad e_2 = s_2 - \dfrac{\langle s_1, s_2\rangle}{\langle s_1, s_1\rangle} s_1 = \begin{bmatrix} 1.2 \\ 0.8 \\ -1.6 \\ 0.6 \end{bmatrix} - \dfrac{0.5}{0.25}\begin{bmatrix} 0 \\ 0.4 \\ 0 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 0 \\ -1.6 \\ 0 \end{bmatrix} = s_2 - 2s_1$

Normalize $\quad v_1 = \dfrac{e_1}{\|e_1\|} = 2s_1 = \begin{bmatrix} 0 \\ 0.8 \\ 0 \\ 0.6 \end{bmatrix}$, $v_2 = \dfrac{e_2}{\|e_2\|} = \frac{1}{2}e_2 = \frac{1}{2}s_2 - s_1 = \begin{bmatrix} 0.6 \\ 0 \\ -0.8 \\ 0 \end{bmatrix}$

2) $[v_1 \ v_2] = [s_1 \ s_2]\begin{bmatrix} 2 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\quad [s_1 \ s_2] = [v_1 \ v_2]\begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 2 \end{bmatrix}$

3). $x = [v_1 \ v_2]\begin{bmatrix} 2 \\ 3 \end{bmatrix} = [s_1 \ s_2]\begin{bmatrix} 2 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = [s_1 \ s_2]\begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$

4. (30) For each of the following matrices, $A_1 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 & -4 & 0 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$,

20 1) Find the diagonal form (or Jordan form) $\overline{A}_i$ and Q_i such that $\overline{A}_i = Q_i^{-1}A_iQ_i$;

2) Compute $e^{A_1 t}$, $e^{A_2 t}$

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1). For A_1, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -1$

$S = (A-\lambda_1 I) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ rank $S = 2$, $\nu(S) = 1$, generalized eigenvector.

$(A-\lambda_1 I)^2 = \begin{bmatrix} 2 & -2 & -2 \\ 0 & 0 & 0 \\ -2 & 2 & 2 \end{bmatrix}$ Pick $v_2 = \begin{bmatrix} 1 \\ +1 \\ 0 \end{bmatrix}$, then $(A-\lambda_1 I)v_2 \neq 0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$v_1 = (A-\lambda_1 I)v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$(A-\lambda_3 I) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & +2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$\bar{A}_1 = Q^{-1}A_1Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\qquad A_1 = Q\bar{A}_1Q^{-1}$

$e^{A_1 t} = Qe^{\bar{A}_1 t}Q^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix}\frac{1}{2}$

$$A_2 = \begin{bmatrix} 3 & -4 & 0 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = -1, \quad \lambda_2 = 2.$$

$$S = (A_2 - \lambda_1 I) = \begin{bmatrix} 4 & -4 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad \text{rank } S = 1, \quad \nu(S) = 2. \qquad V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(A_2 - \lambda_2 I) = \begin{array}{c} \begin{matrix} 4 & \quad 1 & \quad 2 \end{matrix} \\ \begin{bmatrix} 1 & -4 & 0 \\ 1 & -4 & 0 \\ 2 & -2 & -3 \end{bmatrix} \end{array} \qquad V_3 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} -1 & 4 & 0 \\ -2 & 2 & 4 \\ 1 & -1 & 0 \end{bmatrix} \frac{1}{3}$$

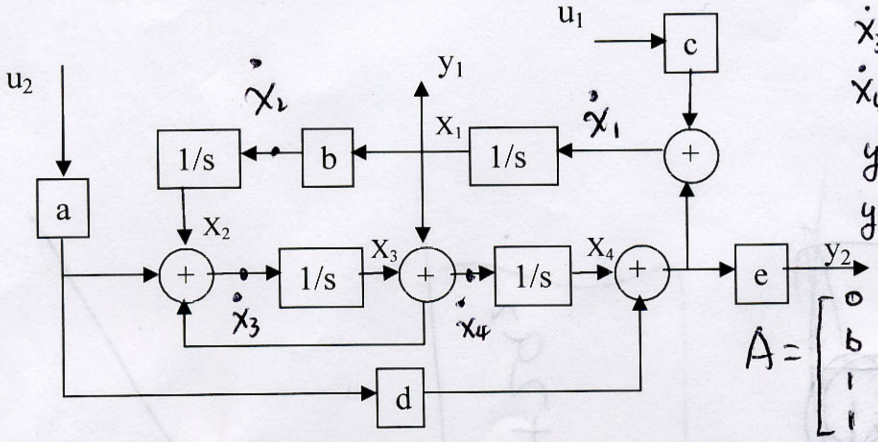
$$\bar{A}_2 = Q^{-1} A_2 Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$e^{A_2 t} = Q e^{\bar{A}_2 t} Q^{-1} = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 4 & 0 \\ -2 & 2 & 4 \\ 1 & -1 & 0 \end{bmatrix} \frac{1}{3}$$

A, B, C, D matrices for the

5. (15) Find the state-space description for the following system:

$$\dot{x} = Ax + Bu$$
$$\dot{y} = Cx + Du$$



$$\dot{x}_1 = x_4 + cu_1 + ad\,u_2$$
$$\dot{x}_2 = bx_1$$
$$\dot{x}_3 = x_1 + x_2 + x_3 + au_2$$
$$\dot{x}_4 = x_1 + x_3$$
$$y_1 = x_1$$
$$y_2 = \quad ex_4 + ade\,u_2$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} c & ad \\ 0 & 0 \\ 0 & a \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & ade \end{bmatrix}$$

Bonus problems

1. (20) : For the following matrix A, compute e^{At} and A^k.

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$(A+I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad (A+I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$(A+I)^3 = \begin{bmatrix} 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ -1 & 0 & \\ 0 & -1 & \end{bmatrix} \qquad (A+I)^4 = \begin{bmatrix} 0 & \\ 0 & \\ 0 & \\ 0 & \\ 1 & \end{bmatrix}$$

$\lambda(A) = -1$ has multiplicity 5.

For $f(\lambda) = e^{\lambda t}$, consider $g(\lambda) = \beta_0 + \beta_1(\lambda+1) + \beta_2(\lambda+1)^2 + \beta_3(\lambda+1)^3 + \beta_4(\lambda+1)^4$

$f(-1) = e^{-\lambda t}$ $\quad g(-1) = \beta_0$ $\quad \beta_0 = e^{-\lambda t}$

$f'(-1) = te^{-\lambda t}$ $\quad g'(-1) = \beta_1$ $\quad \beta_1 = te^{-\lambda t}$

$f''(-1) = t^2 e^{-\lambda t}$ $\quad g''(-1) = 2\beta_2$ $\quad \beta_2 = t^2 e^{-\lambda t}/2$

$f'''(-1) = t^3 e^{-\lambda t}$ $\quad g'''(-1) = 6\beta_3$ $\quad \beta_3 = t^3 e^{-\lambda t}/6$

$f''''(-1) = t^4 e^{-\lambda t}$ $\quad g^{(4)}(-1) = 24\beta_4$ $\quad \beta_4 = t^4 e^{-\lambda t}/24$

$$e^{At} = f(A) = g(A) = \begin{bmatrix} \beta_0 & 0 & 0 & 0 & 0 \\ -\beta_1 & \beta_0 & 0 & 0 & 0 \\ -\beta_2 & \beta_1 & \beta_0 & 0 & 0 \\ -\beta_3 & \beta_2 & \beta_1 & \beta_0 & 0 \\ \beta_4 & -\beta_3 & -\beta_2 & -\beta_1 & \beta_0 \end{bmatrix}$$