Problem Set #10:

I(3). Is the following state equation controllable? Observable?

\[
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 
\end{bmatrix}
+ \begin{bmatrix}
1 & 2 & 1 
\end{bmatrix}u
\]

\[
y = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 
\end{bmatrix}x
\]

Solution: Since this is a Jordan form equation, \( \lambda_1 = -1, \lambda_2 = 1, \)

\[
A = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0 
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & * & 0 & 1 & 0 & * & 0 \\
1 & * & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 
\end{bmatrix}
\]

\( \{b_{11}, b_{12}\} \) are LI, but \( \{b_{21}, b_{22}, b_{23}\} \) are LD since \( b_{21} = b_{22} + b_{23} \), thus, \((A, B)\) is uncontrollable;

\( \{c_{21}, c_{22}, c_{23}\} \) are LI, but \( \{c_{11}, c_{12}\} \) are LD since \( c_{11} = c_{12} \), thus, \((A, C)\) is unobservable.

2(4). For the following state equation

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 
\end{bmatrix}x + \begin{bmatrix}
0 \\
0 \\
0 
\end{bmatrix}u, \quad y = \begin{bmatrix}
1 & 1 
\end{bmatrix}x
\]

1) Find a state feedback \( u = r - kx \) to place the poles at \(-1, -2, -3\). Use both methods (via controllable canonical form, via solving matrix equation, show all steps) and compare the results.

2) Find a state feedback \( u = r - fx \) to place the poles at \(-2 + j2, -2 - j2, -6\). Use both methods and compare the results.
3) Use simulink to simulate the closed-loop systems resulting from 1) and 2), respectively, under initial condition \(x(0) = [1 \ 2 \ 3]'\) and \(r(t) = \text{unit step}\). Plot \(y(t)\) for the two cases in the same figure.

Solution: Given \(A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ C = [1 \ 1 \ 1],\)

\[
G^c_{n-p+1} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \rho(G^c_{n-p+1}) = 3 = n, \quad (A, B) \text{ is controllable.}
\]

\[
\det(\lambda I - A) = \lambda^3 \quad \Rightarrow \quad \text{Eigenvalues} \quad \lambda_1 = \lambda_2 = \lambda_3 = 0.
\]

1) Via controllable canonical form:

Step 1: The desired eigenvalues are \(-1, \ -2, \ -3,\)

\[
\Delta_d(s) = (s + 1)(s + 2)(s + 3) = s^3 + 6s^2 + 11s + 6 \quad \Rightarrow \quad \overline{\alpha}_1 = 6, \quad \overline{\alpha}_2 = 11, \quad \overline{\alpha}_3 = 6.
\]

Step 2: Characteristic polynomial of \(A,\)

\[
\det(sI - A) = \det\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix} = s^3 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.
\]

Let \(Q = P^{-1} = G^c_{n-p+1}\)

\[
\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

Step 3: \(k_1 = \overline{\alpha}_1 - \alpha_1 = 6, \quad k_2 = \overline{\alpha}_2 - \alpha_2 = 11, \quad k_3 = \overline{\alpha}_3 - \alpha_3 = 6 \quad \Rightarrow \quad k = [6 \ 11 \ 6]\)

Step 4: \(k = kP = [6 \ 11 \ 6] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = [6 \ 11 \ 6]\)

Step 5: Verify:

\[
A - Bk = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ -6 & -11 & -6 \end{bmatrix},
\]

\[
\det(sI - A + Bk) = \det\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s + 6 \end{bmatrix} = s^3 + 6s^2 + 11s + 6 = (s + 1)(s + 2)(s + 3)
\]

Eigenvalues of \(A - bk:\) \(-1, \ -2, \ -3.\)

Via solving matrix equation:

Choose \(F = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \text{and} \quad K_0 = [1 \ 1 \ 1], \quad \text{so that}\)

\[
G^o = \begin{bmatrix} K_0 \\ K_0F \\ K_0F^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}, \quad \rho(G^o) = 3, \quad (F, K_0) \text{ is observable.}
Use $T = \text{lyap}(A, F, -B*K0)$; $K = K0*\text{inv}(T)$

$$A = 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
$$

$$B = 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
$$

$$F = \text{diag}([-1, -2, -3])
$$

$$K0 = [1 \ 1 \ 1]
$$

$$T = \text{lyap}(A, F, -B*K0)
$$

$$K = K0*\text{inv}(T)
$$

These two methods obtained the same result, $K = [6 \ 11 \ 6]$.

2) Via controllable canonical form:

Step 1: The desired eigenvalues are $-2 + j2, -2 - j2, -6$,

$$\Delta_d(s) = (s + 2 - j2)(s + 2 + j2)(s + 6) = s^3 + 10s^2 + 32s + 48 \Rightarrow \bar{\alpha}_1 = 10, \bar{\alpha}_2 = 32, \bar{\alpha}_3 = 48.$$  

Step 2: Same as 1), Characteristic polynomial of $A$,

$$\det(sI - A) = \det\begin{bmatrix}s & -1 & 0 \\
0 & s & -1 \\
0 & 0 & s\end{bmatrix} = s^3 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$$  

Let $Q = P^{-1} = G_c^{n-p-1}$, 

$$\begin{bmatrix}1 & \alpha_1 & \alpha_2 \\
0 & 1 & \alpha_1 \\
0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1\end{bmatrix}$, 

$$P = \begin{bmatrix}0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0\end{bmatrix}, P = \begin{bmatrix}0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0\end{bmatrix}$$  

Step 3: $\bar{f}_1 = \bar{\alpha}_1 - \alpha_1 = 10, \bar{f}_2 = \bar{\alpha}_2 - \alpha_2 = 32, \bar{f}_3 = \bar{\alpha}_3 - \alpha_3 = 48 \Rightarrow \bar{f} = [10 \ 32 \ 48]$
Step 4: \( \bar{f}P = \begin{bmatrix} 10 & 32 & 48 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 48 & 32 & 10 \end{bmatrix} \)

Step 5: Verify:

\[
A - Bf = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -48 & -32 & -10 \end{bmatrix},
\]

\[
\det(sI - A + Bf) = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 48 & 32 & s+10 \end{bmatrix} = s^3 + 10s^2 + 32s + 48
\]

\[
= (s + 2 - j2)(s + 2 + j2)(s + 6)
\]

Eigenvalues of \( A - bf \): \(-2 + j2, -2 - j2, -6\).

Via solving matrix equation:

Choose \( F = \begin{bmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -6 \end{bmatrix}, \) and \( f_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \), so that

\[
G^o = \begin{bmatrix} f_0 \\ f_0F \\ f_0F^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -4 & 0 & -6 \\ 8 & -8 & 36 \end{bmatrix}, \quad \rho(G^o) = 3, \quad (F, f_0) \) is observable.

Use \( T = \text{lyap}(A, -F, -B*f0); \) \( f = f0*\text{inv}(T) \)

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T
\]

\[
F = \begin{bmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -6 \end{bmatrix}
\]

\[
f_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
\]
\[
\begin{bmatrix}
-0.0625 & 0 & 0.0046 \\
0.1250 & -0.1250 & -0.0278 \\
0 & 0.5000 & 0.1667
\end{bmatrix}
\]

These two methods obtained the same result, \( f = [48 \ 32 \ 10] \).

3) The simulink model and output responses are,
3(3). For the following state equation
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix}
x + \begin{bmatrix}
1 & 0 \\
1 & 0 \\
1 & 1
\end{bmatrix} u, \quad y = [1 \ 1 \ 1] x
\]
1) Find two different state feedbacks \( u = r - K_1 x \) and \( u = r - K_2 x \) to place the poles at \(-2 + j2\), \(-2 - j2\), \(-4\). Try to find \( K_1 \) and \( K_2 \) such that one has relatively larger elements and the other one has relatively small elements.

2) Use simulink to simulate the closed-loop systems resulting from \( K_1 \) and \( K_2 \), respectively, under initial condition \( x(0) = [1 \ 2 \ 3]' \) and \( r(t) = 0 \).

Plot \( y(t) \) for the two cases in the same figure.
Plot \( u_1(t) \) for the two cases in the same figure.
Plot \( u_2(t) \) for the two cases in the same figure.

Solution: Given \( A = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1
\end{bmatrix}, \quad C = [1 \ 1 \ 1],
\)
\[
G^{e}_{n-p+1} = \begin{bmatrix}
b_1 & b_2 & Ab_1 & Ab_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 & -1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 2 & 2
\end{bmatrix}, \quad \rho(G^{e}_{n-p+1}) = 3 = n, \quad (A, B) \text{ is controllable.}
\]
\[
\det(\lambda I - A) = (\lambda - 1)[(\lambda - 1)^2 + 1] \Rightarrow \text{Eigenvalues } \lambda_1 = 1 + j, \lambda_2 = 1 - j, \lambda_3 = 1, \text{ unstable.}
\]

1) From the new poles \(-2 + j2\), \(-2 - j2\), \(-4\),
Choose \( F = \begin{bmatrix}
-2 & 2 & 0 \\
-2 & -2 & 0 \\
0 & 0 & -4
\end{bmatrix} \), Use \( T = \text{lyap}(A,-F,-B*K0); \quad K = K0*\text{inv}(T), \)
\[
K_0 = \begin{bmatrix}
3 & -1 & 3 \\
3 & -2 & 1 \\
1 & -1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 \ 1.1 \ -1
\end{bmatrix},
K_0 = \begin{bmatrix}
3 & -1 & 3 \\
3 & -2 & 1 \\
1 & -1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 \ 1.1 \ -1
\end{bmatrix},
K_1 = \begin{bmatrix}
-4.0455 & 2.5469 & 4.5156 \\
-1.3810 & -3.0445 & 5.3185
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
101.5385 & 106.5385 & -166.1538 \\
-104.5385 & -109.5385 & 175.1538
\end{bmatrix}
\]
The simulink model is,

The input and output responses are,