EECE.5130 Control Systems

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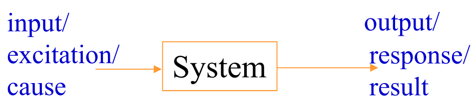
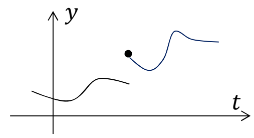
- Today:
  - Introduction
    - Motivation
    - Course Overview
    - Course project
    - My research projects
  - Matrix Operations -- Fundamental to Linear Algebra
    - Determinant
    - Matrix Multiplication
    - Eigenvalue
    - Rank

1. INTRODUCTION

1.1 Motivation

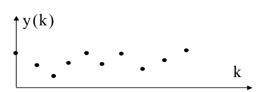
- What is a "system"?
  - A physical process or a mathematical model of a physical process that relates one set of signals to another set of signals





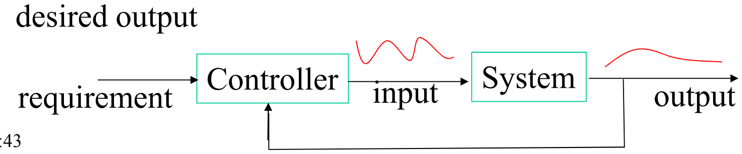
  - Examples: Air conditioner, cars, DC-DC converters
- Two general categories of signals/systems:
  - Continuous-time (CT) signals/systems
    - Examples: Speed/car, current/circuit, temperature/room
    - Described by differential eqs., e.g., dy/dt = ay(t) + bu(t)
    - Signals themselves could be discontinuous. But defined for each time instant.

  - Discrete-time (DT) signals/systems

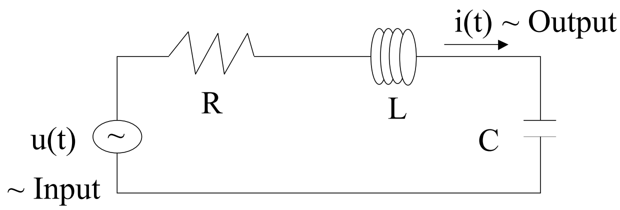


    - Examples: Money in a bank account, quarterly profit
    - Sequence of numbers
    - Input/output related by difference equations, e.g.,

      y[k+1] = ay[k] + bu[k], (on a daily or monthly base)
  - DT and CT are quite similar, and will be treated in parallel
- The goal of "System Theory":
  - Establish input/output relationship through models,
  - Predict output from input, know how to produce desired output
  - Alter input automatically (via controller) to produce desired output



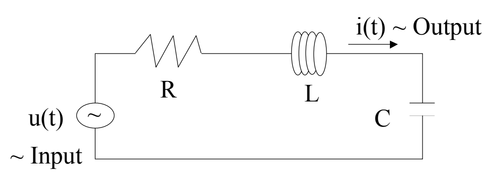
Example: A simple electric circuit



- Use physical laws to model/describe the behavior of the system:
  - What are the components? What properties do they have?

$$v_R = Ri_R, \quad v_L = L\frac{di_L}{dt}, \quad i_C = C\frac{dv_C}{dt}$$

  - Relationship among the variables by physical law:
    - KCL: Current to a node = 0, $i_R = i_C = i_L = i$.
    - KVL: Voltage across a loop = 0.



$$KVL: \quad Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau + v_0 = u(t) \quad L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{du(t)}{dt}$$

  - An integral-differential or differential equation
  - Input-output description or external description
- How to analyze the input-output relationship?
  - For example, find the output $i(t)$ given $u(t)$ and IC.
- We can use Laplace transform
  - Note: only effective for LTI systems

Laplace Transform, A Quick Review

$$f(t) \Leftrightarrow F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

- Key Properties
  - Linearity: $a_1 f_1(t) + a_2 f_2(t) \Leftrightarrow a_1 F_1(s) + a_2 F_2(s)$
  - Derivative theorem:

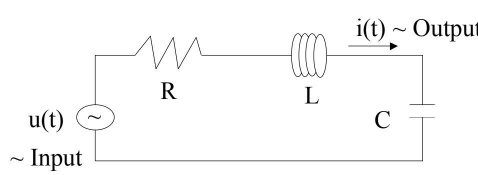
    $\dot{f}(t) \leftrightarrow \mathrm{sF(s)} - f(0^-), \qquad \int f(\tau)\, dt \leftrightarrow F(s)/s$

- Converting linear constant coefficient differential equations into algebraic equations
- Other properties
  - Differentiation in the frequency domain: $t \cdot f(t) \leftrightarrow (-1)F'(s)$
  - Convolution: $h(t) * f(t) \leftrightarrow H(s) \cdot F(s)$
  - Time and frequency shifting: $f(t-t_0)\ u(t-t_0) \leftrightarrow e^{st_0} F(s)$;

    $e^{s_0 t} f(t) \leftrightarrow F(s - s_0)$

- Time and frequency scaling: $f(at) \leftrightarrow 1/a\ F(s/a)$ for $a > 0$
- Initial Value Theorem: $f(0^+) = \lim_{s \to \infty} sF(s)$
- Final Value Theorem: $f(\infty) = \lim_{s \to 0} sF(s)$ if all the poles of $sF(s)$ have strictly negative real parts

Example (Continued)



$$KVL: \quad Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau) d\tau + v_0 = u(t)$$

$$R\hat{i}(s) + L\left[s\hat{i}(s) - i_0\right] + \frac{\hat{i}(s)}{Cs} + \frac{v_0}{s} = \hat{u}(s)$$

- An algebraic equation vs integral-differential equation. Solution:

$$\left(Ls + R + \frac{1}{cs}\right)\hat{i}(s) = \hat{u}(s) + \left[Li_0 - \frac{v_0}{s}\right]$$

$$\hat{i}(s) = \frac{cs}{LCs^2 + RCs + 1}\hat{u}(s) + \frac{LCsi_0 - cv_0}{LCs^2 + RCs + 1}$$

- Is there any pattern with the equation?
  - It has two components, one caused by input, and the other by IC
- How about the voltage across the capacitor?

$$\hat{v}(s) = \frac{\hat{i}(s)}{Cs} + \frac{v_0}{s} = \frac{1}{LCs^2 + RCs + 1}\hat{u}(s) + \frac{Li_0 + (LCs + RC)v_0}{LCs^2 + RCs + 1}$$

- What is the system's transfer function?

- Assume that the ICs are zero, then

$$\hat{u}(s) \longrightarrow \boxed{\hat{g}(s)} \longrightarrow \hat{i}(s) = \hat{g}(s)\,\hat{u}(s)$$

$$\hat{i}(s) = \boxed{\frac{Cs}{LCs^2 + RCs + 1}}\hat{u}(s) \qquad \hat{g}(s) = \frac{Cs}{LCs^2 + RCs + 1}$$

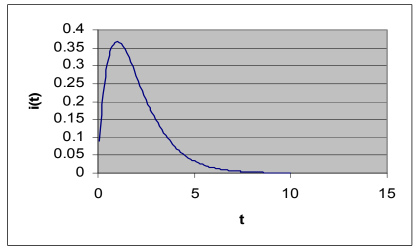
  - Frequency domain analysis
- How to obtain the response in time domain?

$$i(t) = L^{-1}\left\{\hat{i}(s)\right\}$$

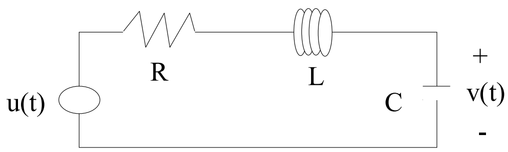
  - Suppose that L = C = 1, R = 2, $v_0 = i_0 = 0$, and u(t) = U(t) (unit step function). Then

$$\hat{u}(s) = \frac{1}{s} \qquad \hat{i}(s) = \frac{Cs\,\hat{u}(s)}{LCs^2 + RCs + 1} = \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}$$

$$i(t) = te^{-t}$$



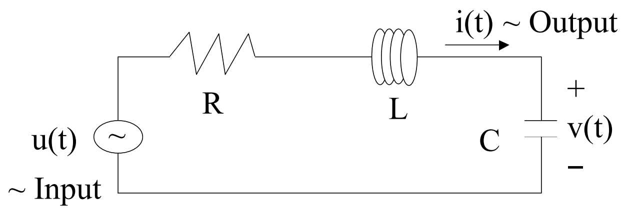
- Does this make sense for the circuit?



- Limitation of Laplace transform: not effective for time varying/nonlinear systems such as

$$\ddot{y}(t) + a_1(y,t)\,\dot{y}(t) + a_0(y,t)y(t) = b(y,t)u(t)$$

- The **state space description** to be studied in this course will be able to handle more general systems
  - How can we do it?
  - Properties can be characterized without solving for the exact output
- To get some general idea about state space description, we consider the same circuit system.



- **State-Space Description**
  – State variables: Voltage across C and current through L
  – state equation:

$$\frac{dv}{dt} = \frac{1}{C} i$$

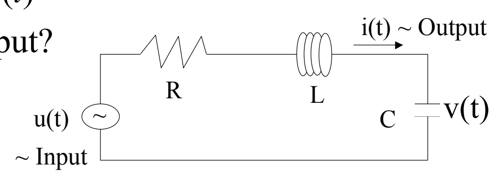
$$\frac{di}{dt} = -\frac{R}{L} i - \frac{1}{L} v + \frac{1}{L} u$$

$$\begin{bmatrix} \frac{dv}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

– A set of first-order differential equations

- It describes the behaviors inside the system by using the state variables $v(t)$ and $i(t)$
- How to describe the output?

$$y = i = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}$$



  – The output equation
- Combined with the state equation, we have the state-space description or internal description

$$\underbrace{\boxed{\begin{bmatrix} \dot{v} \\ i \end{bmatrix}}}_{\dot{x}} = \underbrace{\boxed{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}}_{A} \underbrace{\boxed{\begin{bmatrix} v \\ i \end{bmatrix}}}_{x} + \underbrace{\boxed{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}}_{B} u$$

$$y = \underbrace{\boxed{\begin{bmatrix} 0 & 1 \end{bmatrix}}}_{C} \begin{bmatrix} v \\ i \end{bmatrix}$$

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

- A general form:

$$\begin{cases} \dot{x} = f(x, u, t) \\ y = h(x, u, t) \end{cases}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathrm{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \mathrm{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}$$

- Main features of the state-space approach
  - ✓ It describes the behaviors inside the system
  - ✓ Characterizes stability and performances without solving the differential equations
  - ✓ Applicable to more general systems, nonlinear, time-varying, uncertain, hybrid
  - ✓ Most recent advancements in control theory are developed via state-space approaches

1.2 Course Overview

- **Textbook:**
  - Chi-Tsong Chen, *Linear System Theory and Design*, 3rd Edition, Oxford University Press, 1999 (Required)

  https://www.amazon.com/s/ref=nb_sb_ss_rsis_1_0?url=search-alias%3Daps&field-keywords=%22linear+system+theory+and+design%22&sprefix=%2Cstripbooks%2C283

- **Goals:** To achieve a thorough understanding about systems theory and multivariable system design

- **Tentative Outline** (12 lectures)**:**
  - Introduction
  - The fundamentals of linear algebra
  - Modeling: Use diff. equ. to describe a physical system
  - Analysis:
    - Quantitative: How to derive response for a given input
    - Qualitative: How to analyze controllability, observability, stability and robustness without knowing the exact solution?

- Design:
  - How to realize a system given its mathematical description
  - How to design a control law so that desired output response is produced
  - How to design an observer to estimate the state of the system
  - How to design optimal control laws
- Continuous-time and discrete-time systems will be treated in parallel

- Prerequisites:
  - 1EECE4130 Linear Feedback Systems
  - Background on
    - **Linear algebra:** Matrices, vectors, determinant, eigenvalue, solving a system of equations
    - Ordinary differential equations
    - Laplace transform, and
    - Modeling of electrical and mechanical systems

- **Grading:**

Homework 15%

Mid Term 30%

(All by hand, computer/ calculator not allowed)

Project 25%

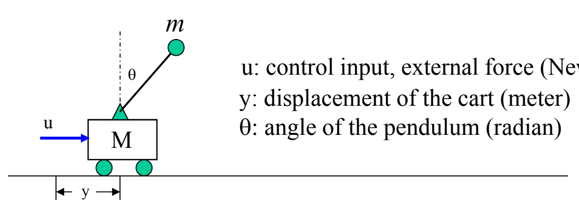
Final Examination 30%

All exams are open book, open notes

- **General Rules:**
  - Homework solutions will be provided together with lecture notes, before submitted.
  - Make sure you understand everything you write down.
  - Homework due next class.
  - Project should be done independently.
- **Attendance**: will be taken occasionally. Positive attitude is a key to success.
- If you decide to do something, use your heart and do it well. Otherwise it will be a waste of time.

Course Project

A cart with an inverted pendulum (page 22, Chen's book)



u: control input, external force (Newton)
y: displacement of the cart (meter)
θ: angle of the pendulum (radian)

The control problems are

1: Stabilization: bring the pendulum to the inverted position and keep it there. Assume the angle is initially small enough.

2: Assume the pendulum is initially downward. Design a control algorithm to bring it upward and keep inverted.

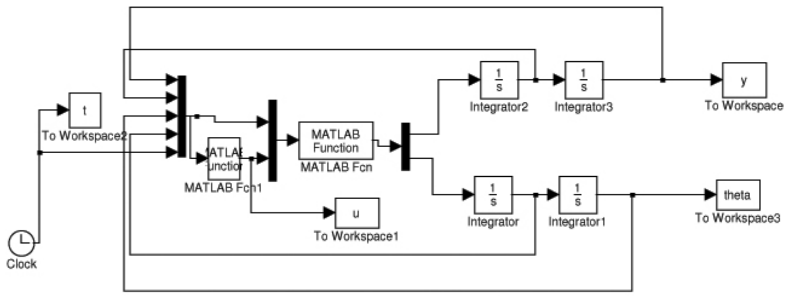
Assume that there is no friction or damping. The nonlinear model is as follows.

$$\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix}\begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u - ml\dot{\theta}^2\sin\theta \\ g\sin\theta \end{bmatrix}$$

m: mass of the pendulum
l: length of the pendulum
M: mass of the cart, $g = 9.8$

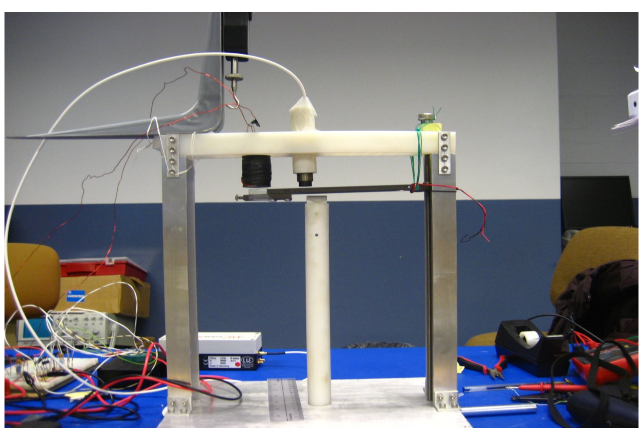
Problems:

1. Derive a linear model for the system.
2. Design feedback laws using Matlab.
3. Validate your designs with Simulink and animation.



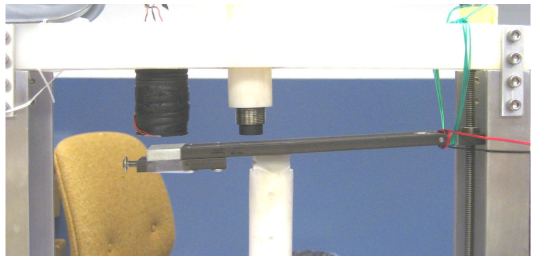
Next, I will describe some control systems in the lab (BL406).

A one –dimensional magnetic suspension test rig

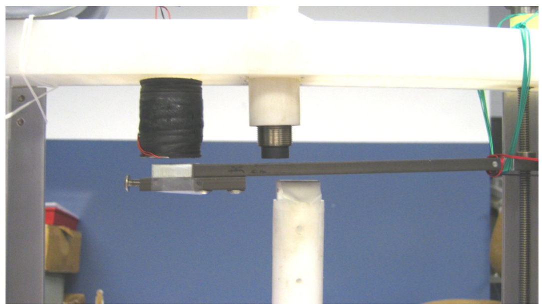


This experiment is part of the NSF project: (Sept. 06 – Aug. 10). The control objective is to keep the free end of the beam suspended. The gap between the beam and the electromagnet follows any set value via a nonlinear controller, which is implemented by a microprocessor, or the Labview. An eddie-current sensor converts the gap into a voltage signal which is fed into the microprocessor; The controller in the microprocessor computes the desired currents and output it to a power amplifier.

The beam rests on the stator when the controller is turned off



The beam suspended when a nonlinear feedback control is applied.



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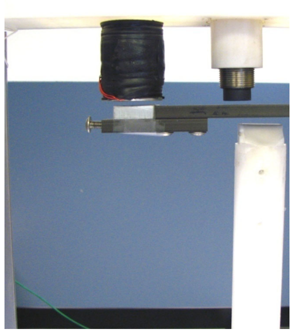
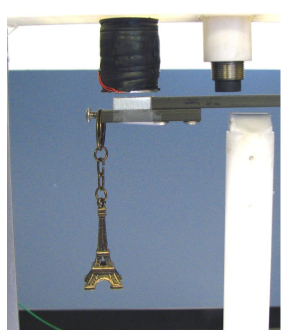
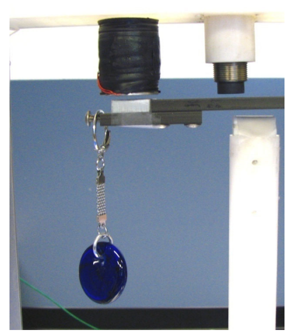
The robust controller adjusts the current of the electromagnet
so that the gap is maintained at the same set value under different load







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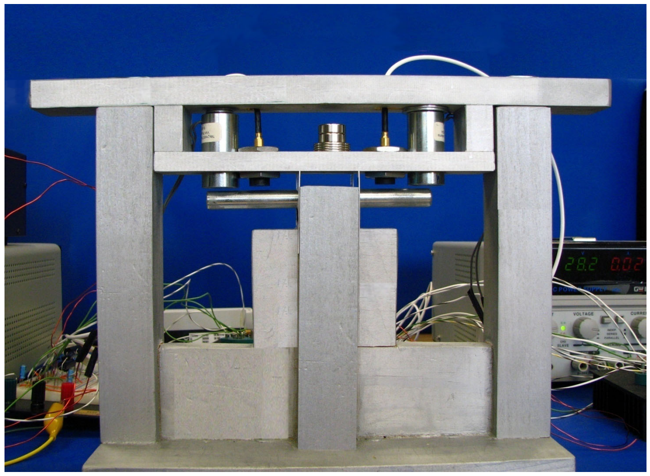
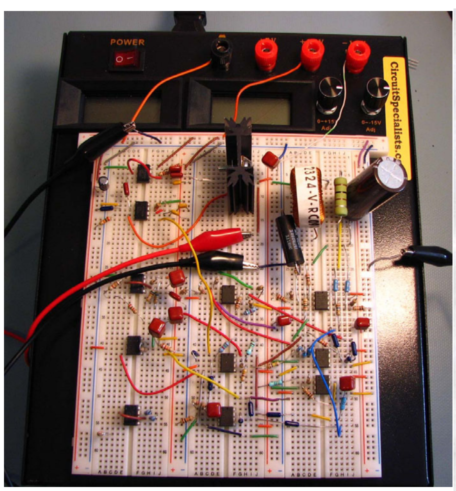
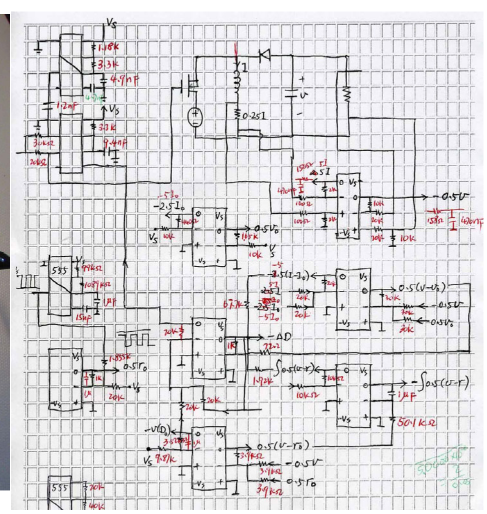
A two dimensional suspension system

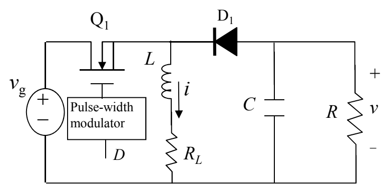


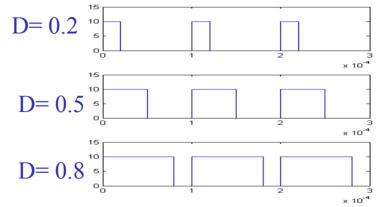
A power electronic converter: buck-boost DC-DC converter



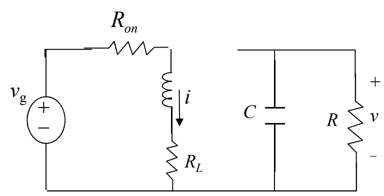


Supported by NSF Sept. 2009 – Aug. 2012

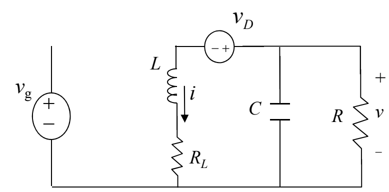




When MOSFET is on



When MOSFET is off

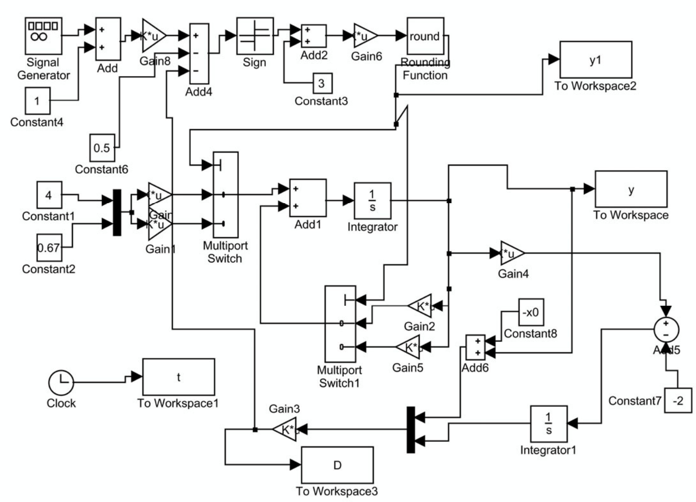


$$\begin{bmatrix} \dfrac{di}{dt} \\ \dfrac{dv}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_{on}+R_L}{L} & 0 \\ 0 & -\dfrac{1}{RC} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_g \\ v_D \end{bmatrix}$$

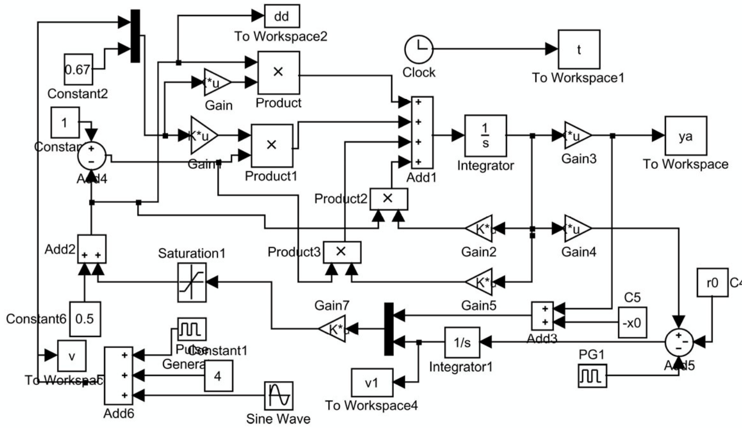
$$\begin{bmatrix} \dfrac{di}{dt} \\ \dfrac{dv}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_L}{L} & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{1}{L} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_g \\ v_D \end{bmatrix}$$

State space description

Simulink with switching model

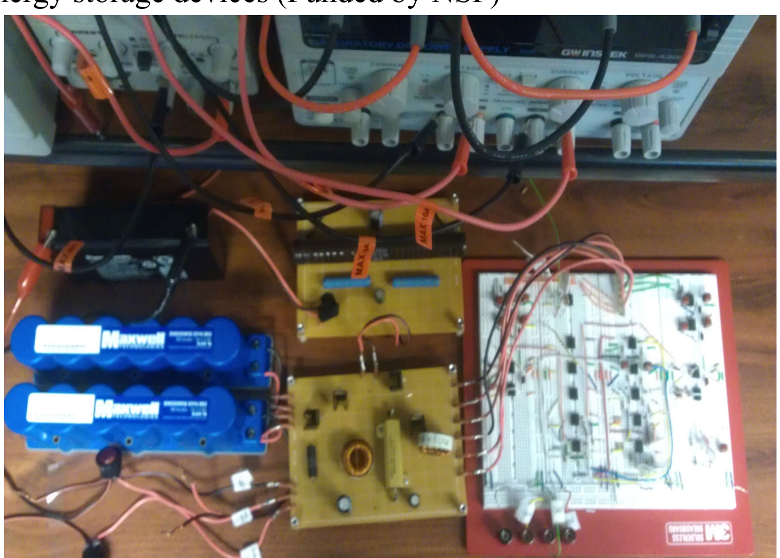


Simulink with averaged model

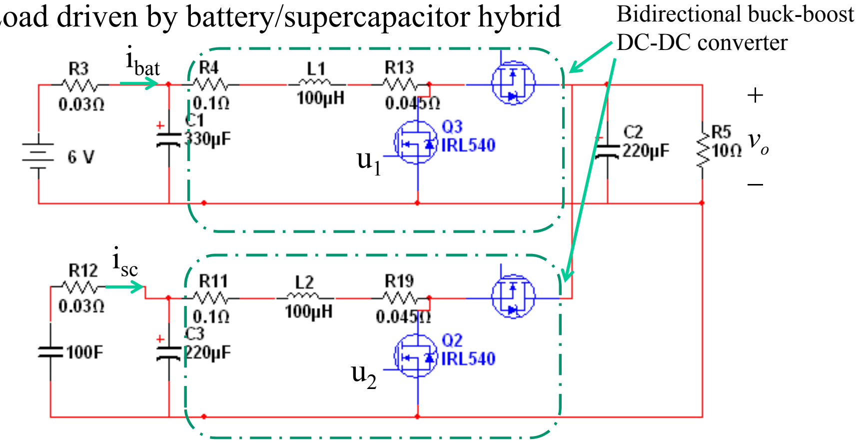


T. Hu, "A nonlinear system approach to analysis and design of power electronic converters with saturation and bilinear terms," *IEEE Trans. Power Electronics*, 26(2), pp.399-410, 2011.

Power systems driven by battery/supercapacitor hybrid energy storage devices (Funded by NSF)



Load driven by battery/supercapacitor hybrid



Control inputs: u_1,u_2 (duty cycle)

Output: i_{bat}, i_{sc}, v_o

Control objective: Given reference $i_{bat,ref}$, $v_{o,ref}$, design control law for u_1,u_2 so that i_{bat} follows $i_{bat,ref}$, v_o follows $v_{o,ref}$.

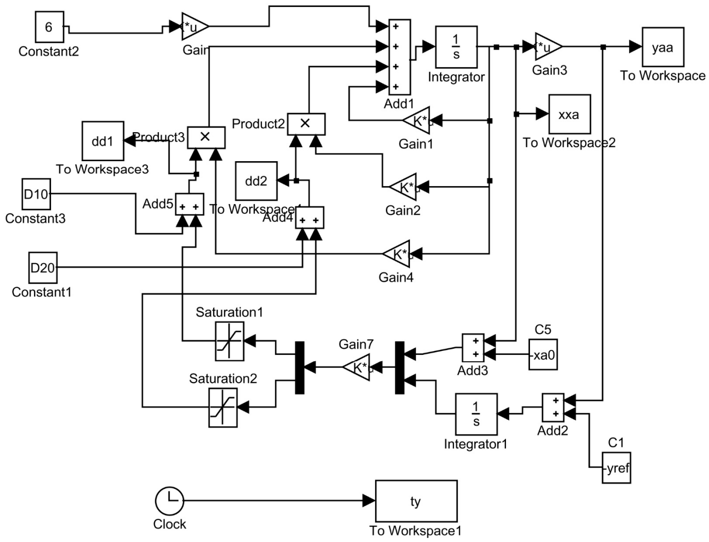
State space description:

Let x be state variable, $x=[v_1,v_2,v_{sc},i_{L1},i_{L2},v_o]'$. Then

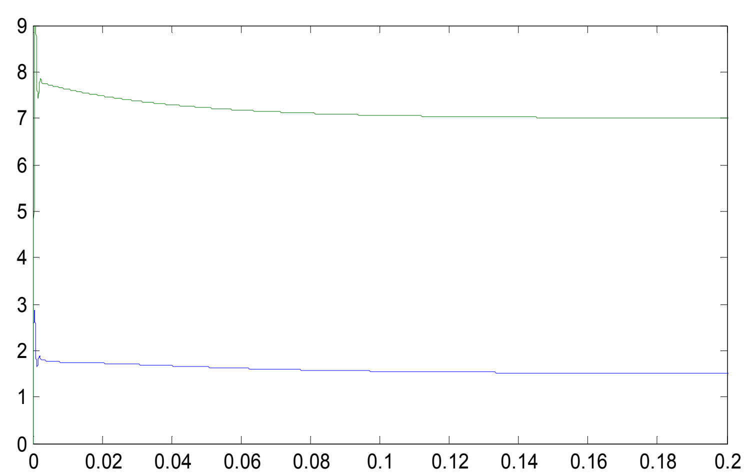
$$\dot{x} = A_0 x + A_1 x u_1 + A_2 x\, u_2$$

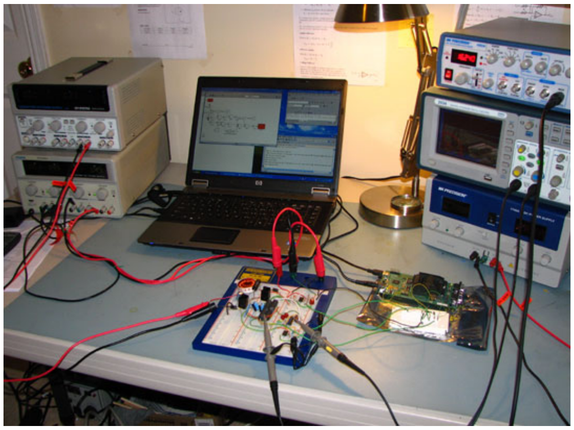
$$A_0=\begin{bmatrix} -1/R_e/C_1 & 0 & 0 & -1/C_1 & 0 & 0 \\ 0 & -1/R_u/C_2 & 1/R_u/C_2 & 0 & -1/C_2 & 0 \\ 0 & 1/R_u/C_u & -1/R_u/C_u & 0 & 0 & 0 \\ 1/L_1 & 0 & 0 & -(R_{L1}+R_{on})/L_1 & 0 & -1/L_1 \\ 0 & 1/L_2 & 0 & 0 & -(R_{L2}+R_{on})/L_2 & -1/L_2 \\ 0 & 0 & 0 & 1/Co & 1/C_o & -1/R/C_o \end{bmatrix}$$

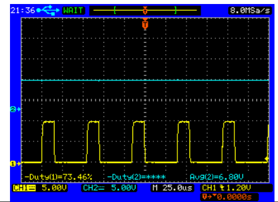
$$A_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/L_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/C_o & 0 & 0 \end{bmatrix} \qquad A_2=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/L_2 \\ 0 & 0 & 0 & 0 & -1/Co & 0 \end{bmatrix}$$

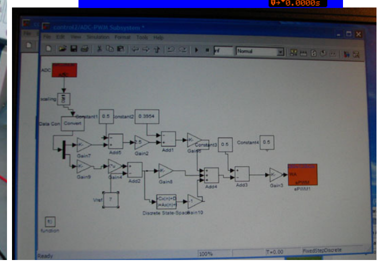


Tracking $i_{bat,ref}$=1.5A, $v_{0,ref}$= 7V

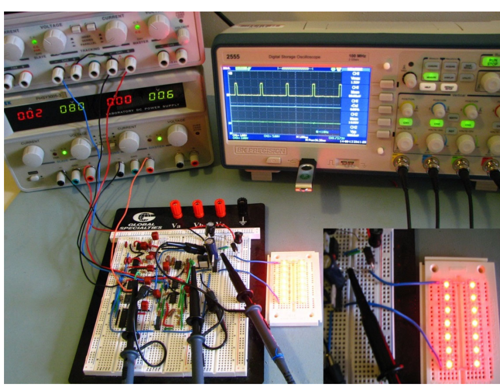


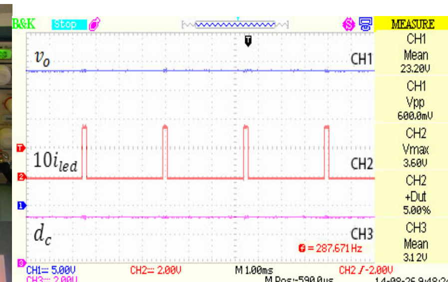


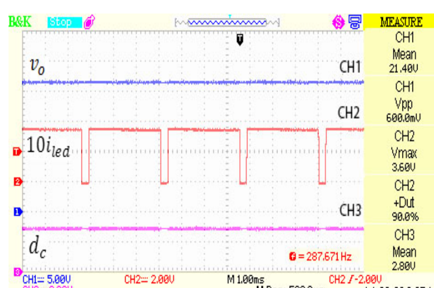




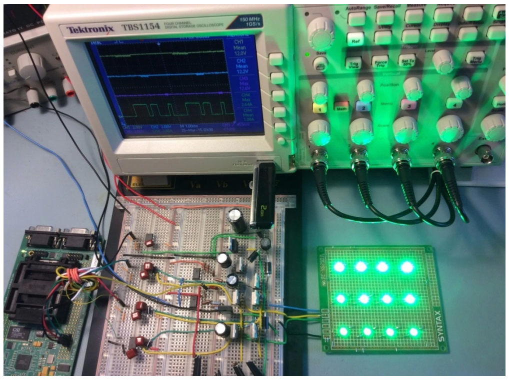
A boost converter controlled by a microcontroller
The controller is constructed using Matlab/Simulink,
Then written into the microprocessor
Funded by NSF.

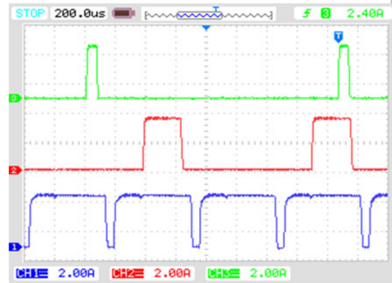






LED driver with PWM dimming control



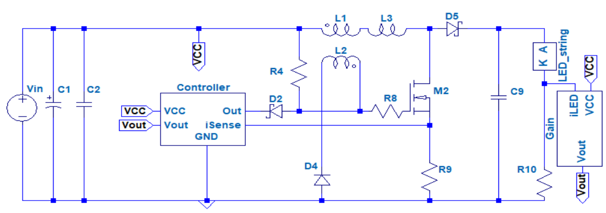


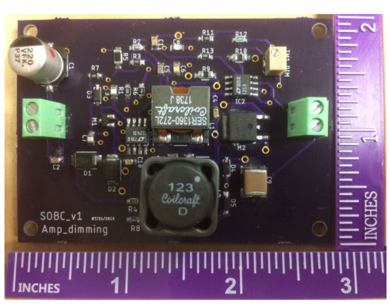
A recent project:

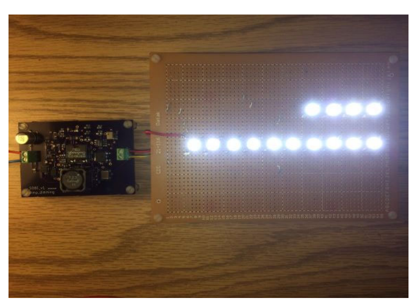
A high efficient high performance LED driver with PWM dimming

The paper by my PhD student and myself will be published by IEEE Transactions on Power Electronics.

Last year's project: A low cost but high performance LED driver based on a self oscillating boost converter







IEEE TRANSACTIONS ON POWER ELECTRONICS 1

Low Cost High Performance LED Driver Based on a Self-Oscillating Boost Converter

David O. Bamgboje, *Student Member, IEEE*, William Harmon, Mohammad Tahan, and Tingshu Hu, *Senior Member, IEEE*

***Abstract*—In this paper, a self-oscillating boost converter with a blocking diode is proposed to meet the desire for simple, cost-effective, high performance, and highly efficient LED drivers. As compared with traditional self-oscillating converters, the proposed converter demonstrates several appealing advantages including design simplicity, robustness, soft-switching characteristics (zero-voltage switching and zero-current switching), tight current regulation, and high efficiency over a wide line/load range. The control stage is implemented with a compact and low-cost industry standard controller, which assumes multiple roles in switching and LED current regulation. A type III compensator with anti-windup is designed to limit the maximum LED current at startup and to achieve tight LED current regulation at steady state. The efficiency and desired transient/steady-state performances are verified with SPICE simulation and a prototype circuit, which demonstrate a maximum efficiency of 95.9% and 2.3% ripple factor for the LED current. The robustness of the proposed driver is verified under a range of power supply voltage and different numbers of LEDs at the load side. In addition, the circuit is modified to implement high efficiency pulsewidth modulation dimming between 5% and 95%.**

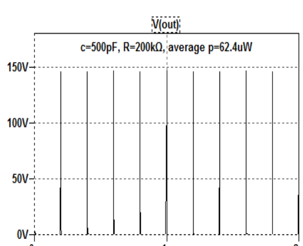
***Index Terms*—Current regulation, LED driver, pulsewidth modulation (PWM) dimming, self-oscillating boost converter**

For example, a self-oscillating soft-switching converter is developed for LED driving in [11] with reduced LED current change in the presence of voltage changes without using additional current feedback. A half-bridge self-oscillating converter is proposed in [12] to reduce LED current ripple due to input voltage ripple, without using electrolytic capacitors. Similarly, Juarez *et al.*, applied the self-oscillating half-bridge converter to LED driving with a focus on reducing the output capacitor size [13]. In the same vein, Mineiro *et al.*, took advantage of self-oscillation by using a bipolar junction transistor (BJT) half-bridge converter in LED driver application [14]. The authors reported minimal temperature drift, moderate efficiency of 81%, and an LED current ripple factor of 15%. In a work by Chen *et al.*, an ultralow input voltage self-oscillating boost converter (SOBC) was proposed for LED driver applications [15]. An input adaptive peak current and blanking time control was used to extend the input voltage range of operation and to ensure tight LED current regulation. The authors reported an efficiency of 72% and an LED current ripple factor of 7%.

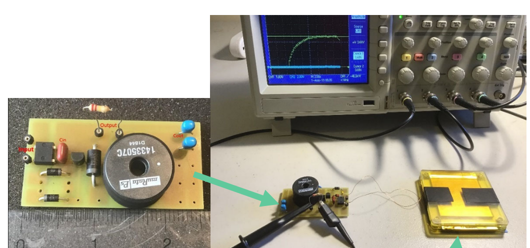
This year's new project:

Build a power management system for triboelectric nanogenerators (TENG)

- The most cutting edge technology in energy harvesting (invented in 2012)
- Harvest energy while walking, doing exercise, use the energy to power cell phone, health monitors, …, internet of things
- TENGs are more powerful and versatile, as compared to other harvesters
- However: The power generated is badly behaved
- A power management system converts the power to well-behaved form



TENG's output without power management



The power management circuit, by my PhD students

The TENG device, made by Georgia Tech

42

- Today:
  - Introduction
    - Motivation
    - Course Overview
    - Course project
    - Control systems in my research projects
  - Matrix Operations -- Fundamental to Linear Algebra
    - Determinant
    - Matrix Multiplication
    - Eigenvalue
    - Rank

Operations on Matrices

- The classical control theory is based on Laplace transform and z-transform
  - also called frequency-domain approach
- The modern control theory is established upon Linear Algebra
  - State-space approach, or time domain approach
- A linear time invariant system can be described as

$$\dot{x}(t) = A\,x(t) + Bu(t)$$
$$y(t) = C\,x(t) + Du(t)$$

Systems properties all characterized with the matrices A,B,C,D.

Matrices: Square or non-square

2 by 2 (or 2×2) matrices: $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

A 2 by 3 matrices: $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, A 3 by 2 matrix: $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

An n×m matrix: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$ a_{ij}: element at the i-th row, j-th column

n: the number of rows; m: the number of columns.

n×1: a column vector; 1×m : a row vector.

Addition and subtraction: element by element

Multiplication is not by element.

10:43 45

Matrix Multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & x \\ v & y \end{bmatrix} = \begin{bmatrix} au+bv & ax+by \\ cu+dv & cx+dy \end{bmatrix}$$

Product of a 1×n matrix a n×1 matrix is a scalar.
Product of a n×1 matrix and a 1×n matrix is an n×n matrix

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax+by+cz \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} xa & xb & xc \\ ya & yb & yc \\ za & zb & zc \end{bmatrix}$$

Generally, A B ≠ B A

- You cannot multiply any two matrices. They have to be compatible: to get AB, the number of columns of A must equal to the number of rows of B, e.g., A: k×n, B: n×m.

10:43 46

Let A be a 3×4 matrix, B be a 4×2 matrix

$$A=\begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{bmatrix}=\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix},\quad B=\begin{bmatrix}a&e\\b&f\\c&g\\d&h\end{bmatrix}$$

$$B=[B_1\quad B_2]$$

$$AB=A\begin{bmatrix}B_1&B_2\end{bmatrix}=\begin{bmatrix}AB_1&AB_2\end{bmatrix}$$

$$AB=\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix}B=\begin{bmatrix}A_1B\\A_2B\\A_3B\end{bmatrix}=\begin{bmatrix}A_1\begin{bmatrix}B_1&B_2\end{bmatrix}\\A_2\begin{bmatrix}B_1&B_2\end{bmatrix}\\A_3\begin{bmatrix}B_1&B_2\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}A_1B_1&A_1B_2\\A_2B_1&A_2B_2\\A_3B_1&A_3B_2\end{bmatrix}=\begin{bmatrix}a+2b+3c+4d&e+2f+3g+4h\\5a+6b+7c+8d&5e+6f+7g+8h\\9a+10b+11c+12d&9e+10f+11g+12h\end{bmatrix}$$

$$A=\begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{bmatrix}=\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix},\quad B=\begin{bmatrix}a&e\\b&f\\c&g\\d&h\end{bmatrix}$$

$$B=[B_1\quad B_2]$$

How about BA? Not defined at all for this case.

BA is defined only if the number of columns of B is the Same as the number of rows of A. Suppose

$$B=\begin{bmatrix}B_1&B_2&B_3\end{bmatrix},\quad A=\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix},\quad \text{then}$$

$$BA=\begin{bmatrix}B_1&B_2&B_3\end{bmatrix}\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix}=B_1A_1+B_2A_2+B_3A_3$$

But each B_1A_1, B_2A_2, B_3A_3 is a matrix.

How about

$$A\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} AB_1 \\ AB_2 \end{bmatrix}? \qquad \begin{bmatrix} A_1 & A_2 \end{bmatrix} B = \begin{bmatrix} A_1 B & A_2 B \end{bmatrix}?$$

If A and B are compatible, (AB defined), then AB_1 and AB_2 are not defined.

Be careful. Correct partition is important. Compatibility is essential.

Example: $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = ?$

Product of block partitioned matrices: Suppose that A and B are partitioned compatibly as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

Then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Compatibly partitioned means that the partition of the columns of A is the same as the partition of the rows of B

Determinant: A scalar defined for a <u>square matrix</u>

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc;$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + dhc + gbf - gec - ahf - dbi$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \begin{matrix} a & b & c \\ d & e & f \\ g & h & i \\ a & b & c \\ d & e & f \end{matrix}$$

Exercise: $\det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}$

- Determinant of a triangular matrix:

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = ? \qquad \det\begin{bmatrix} a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & h & c & 0 \\ g & i & j & d \end{bmatrix} = ?$$

Upper triangular Lower triangular

All zero below the diagonal, or all zero above the diagonal

→ det is the product of the diagonal elements.

The determinant can be simplified by making the matrix a diagonal one through elementary operations that preserve the determinant.

If an entire row or an entire column is 0, the determinant is 0.

- Elementary operations that preserve the determinant:

$$\det(A) = \det\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 4 & 5 & 6\end{bmatrix} = 12+45+8-24-5-36=0$$

1) Add one row scaled by a number to another row

$$A = \begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 4 & 5 & 6\end{bmatrix} \xrightarrow[\text{(keep row 1)}]{\text{Add row 1 to row 2}} \begin{bmatrix}1 & 2 & 3\\ 4 & 4 & 4\\ 4 & 5 & 6\end{bmatrix}$$

2) Add one column scaled by a number to another column

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 4 & 4\\ 4 & 5 & 6\end{bmatrix} \xrightarrow[\text{(keep column 2)}]{\text{Add column 2 ×(-1) to column 3}} \begin{bmatrix}1 & 2 & 1\\ 4 & 4 & 0\\ 4 & 5 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 1\\ 4 & 4 & 0\\ 4 & 5 & 1\end{bmatrix} \xrightarrow[\text{(keep row 1)}]{\text{Add row 1 × (-1) to row 3}} \begin{bmatrix}1 & 2 & 1\\ 4 & 4 & 0\\ 3 & 3 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 1\\ 4 & 4 & 0\\ 3 & 3 & 0\end{bmatrix} \quad \text{Row 3 minus row 2×(3/4)} \quad \begin{bmatrix}1 & 2 & 1\\ 4 & 4 & 0\\ 0 & 0 & 0\end{bmatrix}$$

10:43 53

- Determinant of the product of matrices:

$$\det(AB) = \det A \times \det B$$
$$\det(ABC) = \det A \times \det B \times \det C$$

How about det(A+B), det(A-B)?

- Determinant of a block triangular matrix:

$$\det\begin{bmatrix}A_1 & * & * & *\\ 0 & A_2 & * & *\\ 0 & 0 & A_3 & *\\ 0 & 0 & 0 & A_4\end{bmatrix} = ? \quad = \det A_1 \det A_2 \det A_3 \det A_4$$

Assume that A_1,A_2,A_3,A_4 are all square**.**

Examples: $\det\begin{bmatrix}-2 & 2 & 1\\ 0 & 3 & 5\\ 0 & 4 & 6\end{bmatrix} =$, $\det\begin{bmatrix}1 & 2 & 9 & 10\\ 3 & 4 & 11 & 12\\ 0 & 0 & 5 & 6\\ 0 & 0 & 7 & 8\end{bmatrix}$

10:43 54

An example:

$$\det\begin{bmatrix}1&2&3\\1&3&5\\1&4&8\end{bmatrix}$$ r2-r1 → r2

$$=\det\begin{bmatrix}1&2&3\\0&1&2\\1&4&8\end{bmatrix}$$ r3-r1 → r3

$$=\det\begin{bmatrix}1&2&3\\0&1&2\\0&2&5\end{bmatrix}$$ r3-2r2 → r3

$$=\det\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}=1$$

The row/column that is changed should not be scaled by any number except 1;

Why elementary operations preserve the determinant?

- Because elementary operation is equivalent to multiplying the matrix with another one whose determinant is 1.

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$$ Add column 3 scaled by x to column 1 $$\begin{bmatrix}a+cx&b&c\\d+fx&e&f\\g+ix&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\x&0&1\end{bmatrix}$$

Since $\det\begin{bmatrix}1&0&0\\0&1&0\\x&0&1\end{bmatrix}=1,$

$$\det\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\det\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\x&0&1\end{bmatrix}=\det\begin{bmatrix}a+cx&b&c\\d+fx&e&f\\g+ix&h&i\end{bmatrix}$$

What about $\begin{bmatrix}1&0&0\\0&1&0\\x&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=?$

- If two rows (columns) are switched, the determinant changes the sign;
- If a whole row (column) is scaled by a number k, the determinant is scaled by a number k.
- What if the whole matrix is scaled by a number k?

Exercise:

$$\det\begin{bmatrix} x & -x & 0 \\ 0 & x & -1 \\ ax & bx & x+c \end{bmatrix} =$$

$$\det\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix} =$$

A square matrix is said to be nonsingular if its determinant is nonzero.

Eigenvalue of a square matrix: s is an eigenvalue of A if

$$\det[sI - A] = 0 \qquad I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

If A is n×n, det[sI-A] is a polynomial of order n. An eigenvalue is a root of the polynomial. A has n eigenvalues.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},$$

$$\det[sI - A] = \det\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} = s^2 + 3s + 2 = (s+1)(s+2)$$

Roots are $s_1 = -1$ and $s_2 = -2$.

Exercise: $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$

The inverse of a square matrix: If det A≠0, A has a unique inverse X such that AX=XA=I, denoted as $X=A^{-1}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solving a system of equations:

$$\begin{matrix} ax+by=e \\ cx+dy=f \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$$

Exercise: What is the inverse of

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}?$$

$$\boxed{AX = b \Rightarrow X = A^{-1}b}$$

The inverse of a block partitioned matrix:

$$A = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$$

Assume that A_1 and A_2 are square and nonsingular, then

$$A^{-1} = \begin{bmatrix} A_1^{-1} & X \\ 0 & A_2^{-1} \end{bmatrix}$$

for certain X. What is X?

Sub-matrix of a matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 9 & 11 & 12 \end{bmatrix}$$ a 2 by 3 matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \\ 9 & 11 & 12 \end{bmatrix}$$ a 3 by 3 square matrix

There are ? 3 by 3 sub-matrices
? 2 by 2 sub-matrices
12 1 by 1 matrices

Rank: The rank of M is the highest dimension of a square sub-matrix whose determinant is nonzero. denoted as $\rho(M)$, or, rank(M).

For example, a 3×4 matrix $M = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$

3×3 sub-matrices $\rightarrow$?
2×2 submatrices $\rightarrow$?
1×1 sub-matrices $\rightarrow$ 12

Suppose that

- the number of 3×3 sub-matrices with nonzero det is N(3)
- the number of 2×2 sub-matrices with nonzero det is N(2)
- the number of 1x1 sub-matrices with nonzero det is N(1)

If $N(3)\neq 0$, then $\rho(M)=3$
If $N(3)=0$ and $N(2)\neq 0$, $\rho(M)=2$
If $N(3)=N(2)=0$ and $N(1)\neq 0$, $\rho(M)=1$
$\rho(M)=0$ only if $M=0$

You need to work from the highest order submatrices. The procedure stops whenever you find one nonzero det.

Example 1: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

3×3 submatrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix};\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix};\ A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix};\ A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix};$$

$\det = 0$ $\det = 0$ $\det = 12$ $\det = 8$

$\boxed{\rho(A)=3}$

Example 2: $A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

All the 3×3 submatrices have 0 det. And there is at least one 2×2 nonsingular submatrix $\rightarrow$ $\rho(A)=2$

It can be very tedious to check all sub-matrices.

- A systematic way to find the rank is to use elementary operation to transform the matrix into a special form,.e.g., block diagonal.

Elementary operations that preserve the rank:

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

All operations that keep the determinant or scale the determinant by a nonzero number

1) Add one row scaled by a number to another row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{r_1\times 4+r_3\rightarrow r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix}$$

Add row 1 scaled by 4 to row 3

2) Add one column scaled by a number to another column

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{(-1)c_4+c_1\rightarrow c_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

3) Exchange two columns or two rows

$$M = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Add row 2 scaled by -1 to row 3: $r_2*(-1)+r_3 \rightarrow r_3$

$$\begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Add row 1 scaled by -1 to row 2: $r_1*(-1)+r_2 \rightarrow r_2$

$$\begin{bmatrix} 1 & 4 & 7 & 10 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Add row 2 scaled by -1 to row 3

$$\begin{bmatrix} 1 & 4 & 7 & 10 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank < 3, but $\det\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} = 1 - 4 = -3 \neq 0$

Rank = 2

- Suppose M is n×m. rank(M) ≤ min{m,n}
- If M is multiplied with a nonsingular matrix (square and has non-zero determinant), the rank is preserved. This is why elementary operations preserve the rank.

Since there are usually many sub-matrices, a systematic procedure to compute the rank is to use elementary operation to make it into upper or lower triangular form.

Another simple operation that preserves the rank is to reorder the columns or rows (permutation).

$[A_1 \quad A_2 \quad A_3]X = A_3,\ X = ?$

$[A_1 \quad A_2 \quad A_3]Y = A_2,\ Y = ?$

$$\rho\left(\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}\right) = \rho\left(\begin{bmatrix} A_3 \\ A_1 \\ A_2 \end{bmatrix}\right); \quad \rho\left(\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}\right) = \rho\left(\begin{bmatrix} A_2 & A_3 & A_1 \end{bmatrix}\right)$$

This operation is equivalent to multiply the matrix with a matrix whose determinant is 1 or -1.

Example: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix}$

- **Next Time :**
  - Math. Descriptions of Systems
    - Classification of systems
    - Linear systems
    - Linear-time-invariant systems
    - State variable description
    - Linearization

Homework Set #1:

1. Compute the eigenvalues for

$$A_1 = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 6 \\ 2 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

2. Compute the ranks for

$$B_1 = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & 0 \\ -2 & -6 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 2 & 4 & 8 \\ -1 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 5 \\ 3 & -1 & 2 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

3. Compute the determinant for

$$C_1 = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 0 & 1 \\ 2 & -1 & 2 & 0 \\ 4 & 0 & 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 3 & 4 & 0 \\ 2 & 0 & 3 & 1 \\ -1 & -2 & 2 & 1 \end{bmatrix}$$

(Use elementary operation to simplify the matrices for Pb.2 and 3.
Please show the steps.)

You may use computer to check your answer.