16.513 Control Systems: Lecture note #2

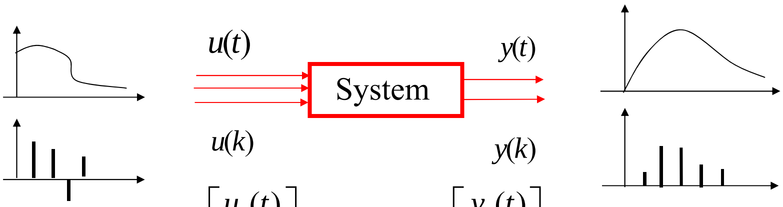
- Last Time:
  - Introduction
    - Motivation
    - Course Overview
    - Course project
  - Matrix Operations -- Fundamental to Linear Algebra
    - Determinant
    - Matrix Multiplication
    - Eigenvalue
    - Rank

Today:

- Math. Descriptions of Systems
  - Classification of systems
  - Linear systems
  - Linear-time-invariant systems
  - State variable description
  - Linearization
- Modeling of electric circuits

2. Mathematical Descriptions of Systems

(Review)



$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_q(t) \end{bmatrix}$$

- Classification of systems
- Linear systems
- Linear time invariant (LTI) systems

2.1 Classification of Systems

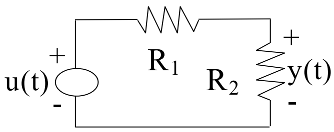
- Basic assumption: When an input signal is applied to the system, a unique output is obtained

Q. How do we classify systems?

- Number of inputs/outputs; with/without memory; causality; dimensionality; linearity; time invariance

- **The number of inputs and outputs**
  - When p = q = 1, it is called a single-input single-output (SISO) system
  - When p > 1 and q > 1, it is called a multi-input multi-output (MIMO) system
  - MISO, SIMO defined similarly

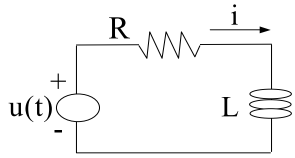
- **Memoryless vs. with Memory**
  - If $y(t)$ depends on $u(t)$ only, the system is said to be memoryless, otherwise, it has memory
  - An example of a memoryless system?



A purely resistive circuit

$$y(t) = \frac{R_2}{R_1 + R_2} u(t) \quad \sim \text{ Memoryless}$$

  - An example of a system with memory?



$$Ri + L\frac{di}{dt} = u \quad or \quad \frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} u$$

$$i(t) = e^{-\frac{R}{L}(t-t_0)} i(t_0) + \frac{1}{L}\int_{t_0}^{t} e^{-\frac{R}{L}(t-\tau)} u(\tau) d\tau$$

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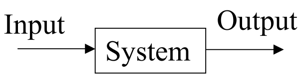
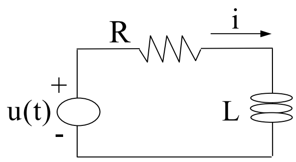
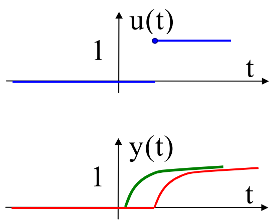
  - $i(t)$ depends on $i(t_0)$ and $u(\tau)$ for $t_0 \le \tau \le t$, not just $u(t)$
  - A system with memory
- **Causality:** No output before an input is applied



  - A system is causal or non-anticipatory if $y(t_0)$ depends only on $u(t)$ for $t \le t_0$ and is independent of $u(t)$ for $t > t_0$
  - Is the circuit discussed last time causal?





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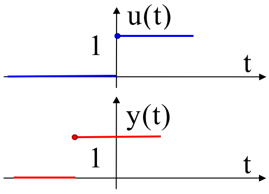
– An example of a non-causal system?

– $y(t) = u(t + 2)$



– Can you truly build a physical system like this?

– All physical systems are causal!

- **The Concept of State**
  - The state of a system at t_0 is the information at t_0 that, together with $u_{[t_0,\infty)}$, uniquely determines the behavior of the system for $t \geq t_0$
  - The number of state variables = the number of ICs needed to solve the problem
  - For an RLC circuit, the number of state variables = the number of C + the number of L (except for degenerated cases)
  - A natural way to choose state variables as what we have done earlier: $\{v_c\}$ and $\{i_L\}$
  - Is this the unique way to choose state variables?

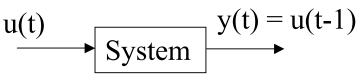
– Any invertible transformation of the above can serve as a state, e.g.,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} 2v(t)+i(t) \\ i(t) \end{bmatrix}$$

– Although the number of state variables = 2, there are infinite numbers of representations

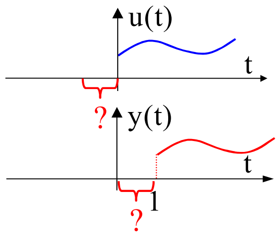
- **Order of dimension of a system:** The number of state variables
  – If the dimension is a finite number ⇒ Finite dimensional (or lumped) system
  – Otherwise, an infinite dimensional (or distributed) system

- An example of an infinite dimensional system



A delay line

– Given u(t) for t ≥ 0, what information is needed to know y(t) for t ≥ 0?



– We need an infinite amount of information ⇒ An infinite dimensional system

2.2 Linear Systems

Linearity

- Double the efforts double the outcome?
  - Suppose we have the following (state,input)-output pairs:

$$\left.\begin{array}{c} x_1(t_0) \\ u_1(t), t \geq t_0 \end{array}\right\} \rightarrow y_1(t), t \geq t_0$$

$$\left.\begin{array}{c} x_2(t_0) \\ u_2(t), t \geq t_0 \end{array}\right\} \rightarrow y_2(t), t \geq t_0$$

  - What would be the output of

$$\left.\begin{array}{c} x_1(t_0)+x_2(t_0) \\ u_1(t)+u_2(t), t \geq t_0 \end{array}\right\} \rightarrow \quad y_1(t)+y_2(t), t \geq t_0$$

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  - If this is true ~ Additivity
  - How about

$$\left.\begin{array}{c} \alpha\, x_1(t_0) \\ \alpha\, u_1(t), t \geq t_0 \end{array}\right\} \rightarrow \quad \alpha\, y_1(t), t \geq t_0$$

  - If this is true ~ Homogeneity
  - Combined together to have:

$$\left.\begin{array}{c} \alpha_1 x_1(t_0)+\alpha_2 x_2(t_0) \\ \alpha_1 u_1(t)+\alpha_2 u_2(t), t \geq t_0 \end{array}\right\} \rightarrow \ \alpha_1 y_1(t)+\alpha_2 y_2(t)$$

  - If this is true ~ Superposition or linearity property
  - A system with such a property: a Linear System

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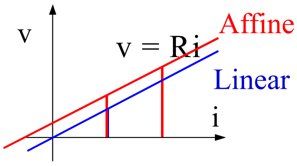
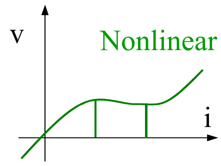
- Are R, L, and C linear elements?

$$v_R = Ri_R, \quad v_L = L\frac{di_L}{dt}, \quad i_C = C\frac{dv_C}{dt}$$

  - Yes (differentiation is a linear operation)





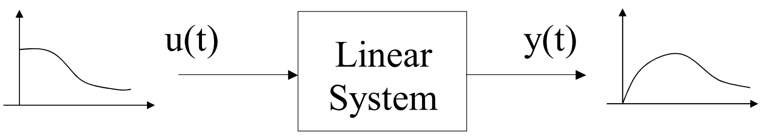
- Also, KVL and KCL are linear constraints. When put together, we have a linear system

- The additivity property implies that

$$y(t) \text{ due to } \begin{cases} x_1(t_0) \\ u_1(t), t \ge t_0 \end{cases} = y(t) \text{ due to } \begin{cases} x_1(t_0) \\ u_1(t) \equiv 0 \end{cases} + y(t) \text{ due to } \begin{cases} x_1(t_0) = 0 \\ u_1(t), t \ge t_0 \end{cases}$$

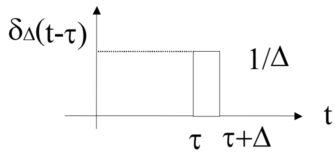
  - Response = zero-input response + zero-state response

Response of a Linear System



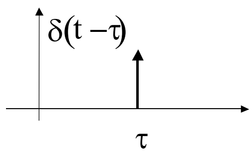
- How can we determine the output y(t)?
- Can be derived from u(t) + the unit impulse response based on linearity

- Let $\delta_\Delta(t-\tau)$ be a square **pulse** at time τ with width Δ and height $1/\Delta$



Area = 1

- As $\Delta \to 0$, we obtain a shifted unit impulse



- Let the unit impulse response be $g(t,\tau)$. Based on linearity,

$$y(t) = \int_{-\infty}^{\infty} g(t,\tau)u(\tau)d\tau$$

- If the system is causal,

$$g(t,\tau) = 0 \text{ for } t < \tau \qquad y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau$$

- A system is said to be relaxed at t_0 if the initial state at t_0 is 0
  - In this case, $y(t)$ for $t \geq t_0$ is caused exclusively by $u(t)$ for $t \geq t_0$

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau$$

- How about a system with p inputs and q outputs?
  - Have to analyze the relationship for input/output pairs

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

$$G(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & & g_{1p}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & & g_{2p}(t,\tau) \\ & & & \\ g_{q1}(t,\tau) & g_{q2}(t,\tau) & & g_{qp}(t,\tau) \end{bmatrix}$$

$g_{ij}(t,\tau)$: The impulse response between the j^{th} input and i^{th} output

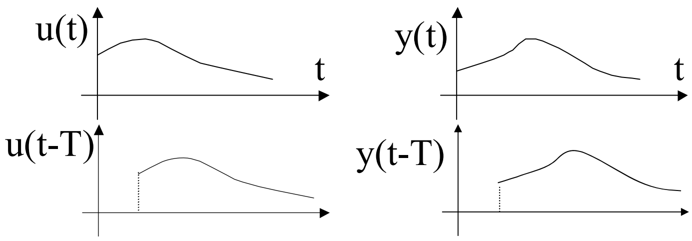
State-Space Description

- A linear system can be described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

2.3 Linear Time-Invariant (LTI) Systems

- Time Invariance: The characteristics of a system do not change over time
  - What are some of the LTI examples? Time-varying examples?
  - What happens for an LTI system if u(t) is delayed by T?



If the same IC is also shifted by T

– This property can be stated as:

$$\left.\begin{matrix} x(0)=x_0 \\ u(t),\ t\ge 0 \end{matrix}\right\} \rightarrow y(t),\ t\ge t_0$$

$$\Downarrow$$

$$\left.\begin{matrix} x(T)=x_0 \\ u(t-T),\ t\ge T \end{matrix}\right\} \rightarrow y(t-T),\ t\ge T$$

Practice: Suppose u(t) → y(t)=1-exp(-t), y(t)=0 for t<0.
What is the response to u(t+1)+u(t-1)?

- What happens to the unit impulse response when the system is LTI?

$$g(t,\tau)=g(t+T,\tau+T) \quad \text{for any } T$$

$$g(t,\tau)=g(t-\tau,\tau-\tau)=g(t-\tau,0)=g(t-\tau)$$

  – Only the difference between t and τ matters
  – What happens to y(t)?

$$\begin{aligned} y(t) &= \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau \\ &= \int_{t_0}^{t} g(t-\tau)u(\tau)d\tau \\ &= \int_{t_0}^{t} g(\tau)u(t-\tau)d\tau \\ &= g(t)*u(t) \quad \sim \text{Convolution integral} \end{aligned}$$

$$\Rightarrow \hat{y}(s)=\hat{g}(s)\hat{u}(s)$$

Proof of $\hat{y}(s) = \hat{g}(s)\,\hat{u}(s)$

$$\hat{y}(s) \equiv \int_0^\infty y(t)e^{-st}dt$$

$$= \int_{t=0}^{\infty}\left(\int_{\tau=0}^{\infty} g(t-\tau)u(\tau)d\tau\right)e^{-st}dt$$

$$= \int_{t=0}^{\infty}\left(\int_{\tau=0}^{\infty} g(t-\tau)u(\tau)d\tau\right)e^{-s(t-\tau)}e^{-s\tau}dt$$

$$= \int_{\tau=0}^{\infty}\left(\int_{t=0}^{\infty} g(t-\tau)e^{-s(t-\tau)}dt\right)u(\tau)e^{-s\tau}d\tau,$$

(Let $\upsilon = t-\tau$)

$$\hat{y}(s) = \int_{\tau=0}^{\infty}\left(\int_{\upsilon=-\tau}^{\infty} g(\upsilon)e^{-s\upsilon}d\upsilon\right)u(\tau)e^{-s\tau}d\tau,$$

(Note $g(\upsilon)=0$ for $\upsilon<0$)

$$= \int_{\tau=0}^{\infty}\left(\int_{\upsilon=0}^{\infty} g(\upsilon)e^{-s\upsilon}d\upsilon\right)u(\tau)e^{-s\tau}d\tau$$

$$= \left(\int_{\upsilon=0}^{\infty} g(\upsilon)e^{-s\upsilon}d\upsilon\right)\left(\int_{\tau=0}^{\infty} u(\tau)e^{-s\tau}d\tau\right)$$

$$\hat{y}(s) = \hat{g}(s)\cdot\hat{u}(s)$$

Transfer-Function Matrix

- For SISO system, $\hat{y}(s) = \hat{g}(s) \cdot \hat{u}(s)$

- $\hat{g}(s)$ ~ Transfer function, the Laplace transform of the unit impulse response

- For MIMO system,

$$\hat{y}(s) = \hat{G}(s) \cdot \hat{u}(s)$$

$$\hat{G}(s) = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) & & \hat{g}_{1p}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & & \hat{g}_{2p}(s) \\ & & & \\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & & \hat{g}_{qp}(s) \end{bmatrix}$$

~ Transfer-function matrix, or transfer matrix

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Today:

- Math. Descriptions of Systems
  - Classification of systems
  - Linear systems
  - Linear-time-invariant systems
  - State variable description
  - Linearization
- Modeling of electric circuits

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State Variable Description

- Start with a general lumped (finite-dimensional) system:

$$\dot{x}(t) = h(x(t), u(t), t)$$
$$y(t) = f(x(t), u(t), t)$$

- If the system is linear, the above reduces to:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

- If the system is linear and time-invariant, then:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- To find an LTI system's response to a particular input u(t), we can use Laplace transform:

$$s\hat{x}(s) - x_0 = A\hat{x}(s) + B\hat{u}(s)$$
$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

- Solve the above linear algebraic equations:

$$\hat{x}(s) = (sI - A)^{-1}B\hat{u}(s) + (sI - A)^{-1}x_0$$
$$\hat{y}(s) = \boxed{\left[C(sI - A)^{-1}B + D\right]}\hat{u}(s) + C(sI - A)^{-1}x_0$$

Transfer function matrix $\hat{G}(s)$

- x_0 is the information needed to determine x(t) and y(t) for t>0, in addition to the input u(t).

⇨ x is the state

2.4 Linearization

- There are many results on linear systems while nonlinear systems are generally difficult to analyze
  – What to do with a nonlinear system described by

$$\dot{x}(t) = h(x(t), u(t), t)$$
$$y(t) = f(x(t), u(t), t)$$

- Linearization. How? Under what conditions?
  – Using Taylor series expansion based on a nominal trajectory, ignoring second order terms and higher
  – Effects are not bad if first order Taylor series expansion is a reasonable approximation over the duration under consideration

- Suppose that with $x_o(t)$ and $u_o(t)$, we have

$$\dot{x}_o(t) = h(x_o(t), u_o(t), t)$$

  – Suppose that the input is perturbed to $u_o(t) + \bar{u}(t)$
  – Assume the solution is $x_o(t) + \bar{x}(t)$, with $\bar{x}(t)$ satisfying

$$\dot{x}_o(t) + \dot{\bar{x}}(t) = h(x_o(t) + \bar{x}(t), u_o(t) + \bar{u}(t), t)$$
$$= h(x_o(t), u_o(t), t) + \left.\frac{\partial h}{\partial x}\right|_o \bar{x} + \left.\frac{\partial h}{\partial u}\right|_o \bar{u} + \ldots$$

$$\frac{\partial h}{\partial x} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdot\cdot & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdot\cdot & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \cdot\cdot & \vdots \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdot\cdot & \frac{\partial h_n}{\partial x_n} \end{pmatrix}, \quad \frac{\partial h}{\partial u} = \begin{pmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} & \cdot\cdot & \frac{\partial h_1}{\partial u_p} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} & \cdot\cdot & \frac{\partial h_2}{\partial u_p} \\ \vdots & \vdots & \cdot\cdot & \vdots \\ \frac{\partial h_n}{\partial u_1} & \frac{\partial h_n}{\partial u_2} & \cdot\cdot & \frac{\partial h_n}{\partial u_p} \end{pmatrix}$$

~ Jacobians

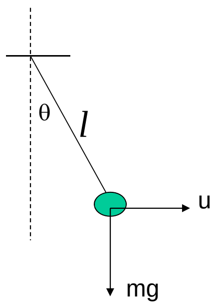
- Then the perturbed system can be described by

$$\dot{\bar{x}}(t) = \left.\frac{\partial h}{\partial x}\right|_o \bar{x} + \left.\frac{\partial h}{\partial u}\right|_o \bar{u} \qquad \sim \text{A linear system}$$

  – The above is valid if the first order Taylor series expansion works out well within the time duration under consideration. It may lead to wrong prediction.
- What to do with the output y(t) = f(x(t), u(t), t)?
  – The output equation can be similarly linearized, but most often there is no need for linearization unless with output feedback

➢ There is another approach to deal with nonlinear time-varying systems: Conservative but reliable

Example: A model for a pendulum



$x_1 = \theta$(the angle) , $x_2 = \dot{\theta}$ (angular velocity) ,

The state is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

The model is derived from Newton's law,

$$\dot{x}_1 = h_1(x) = x_2$$

$$\dot{x}_2 = h_2(x) = -\frac{g}{l}\sin x_1 + \frac{1}{ml}\cos x_1 \quad u$$

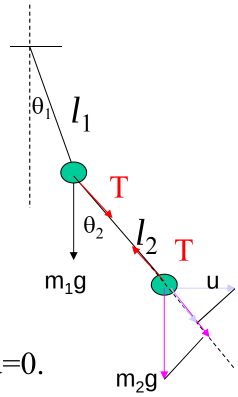
torque = force×arm

Linearize the system at x_1=0, x_2=0, u=0,

$$\left.\frac{\partial h_1}{\partial x_1}\right|_0 = 0, \quad \left.\frac{\partial h_1}{\partial x_2}\right|_0 = 1, \quad \left.\frac{\partial h_1}{\partial u}\right|_0 = 0$$

$$\left.\frac{\partial h_2}{\partial x_1}\right|_0 = \left(-\frac{g}{l}\cos x_1 - \frac{1}{ml}\sin x_1\right)\Big|_0 = -\frac{g}{l}, \quad \left.\frac{\partial h_2}{\partial x_2}\right|_0 = 0, \quad \left.\frac{\partial h_2}{\partial u}\right|_0 = \frac{1}{ml}\cos x_1 \Big|_0 = \frac{1}{ml}$$

$$\frac{\partial h}{\partial x}=\begin{bmatrix}\frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2}\\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{g}{l} & 0\end{bmatrix},\quad \frac{\partial h}{\partial u}=\begin{bmatrix}\frac{\partial h_1}{\partial u}\\ \frac{\partial h_2}{\partial u}\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}$$



Linearized system:

$$\dot{x}=\begin{bmatrix}0 & 1\\ -\frac{g}{l} & 0\end{bmatrix}x+\begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u$$

Exercise: Linearize the following system at x =0, u=0.

$$\dot{x}_1=x_2;$$
$$\dot{x}_2=-\frac{g}{l_1}\sin x_1+\frac{m_2 g}{m_1 l_1}\cos x_3\sin(x_3-x_1)+\frac{1}{m_1 l_1}\sin x_3\sin(x_3-x_1)u$$
$$\dot{x}_3=x_4;$$
$$\dot{x}_4=-\frac{g}{l_2}\sin x_3+\frac{1}{m_2 l_2}\cos x_3\, u$$

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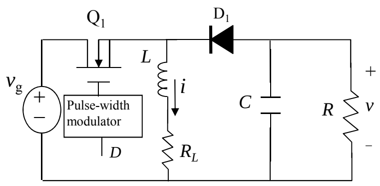
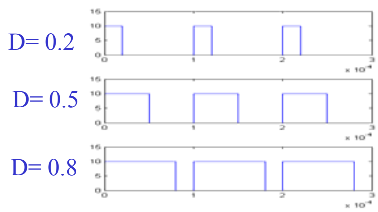
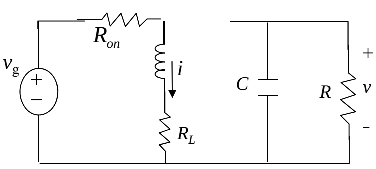
Modeling the buck-boost converter





When MOSFET is on



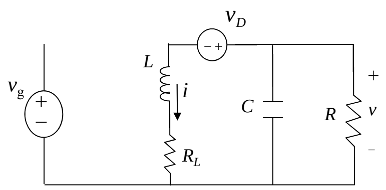
$$\begin{bmatrix}\frac{di}{dt}\\ \frac{dv}{dt}\end{bmatrix}=\begin{bmatrix}-\frac{R_{on}+R_L}{L} & 0\\ 0 & -\frac{1}{RC}\end{bmatrix}\begin{bmatrix}i\\ v\end{bmatrix}+\begin{bmatrix}\frac{1}{L} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_g\\ v_D\end{bmatrix}$$

A_1 B_1

$$y=\begin{bmatrix}0 & 1\end{bmatrix}v$$

C

When MOSFET is off



$$\begin{bmatrix}\frac{di}{dt}\\ \frac{dv}{dt}\end{bmatrix}=\begin{bmatrix}-\frac{R_L}{L} & \frac{1}{L}\\ -\frac{1}{C} & -\frac{1}{RC}\end{bmatrix}\begin{bmatrix}i\\ v\end{bmatrix}+\begin{bmatrix}0 & -\frac{1}{L}\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_g\\ v_D\end{bmatrix}$$

A_2 B_2

$$y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}i\\ v\end{bmatrix}$$

C

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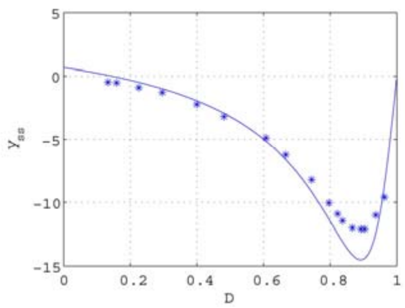
Let → Average over one switching period

$$\bar{i}(t)=\frac{1}{T}\int_t^{t+T} i(\tau)\,d\tau,\quad \bar{v}(t)=\frac{1}{T}\int_t^{t+T} v(\tau)\,d\tau,\quad \bar{x}=\begin{bmatrix}\bar{i}\\ \bar{v}\end{bmatrix},\quad v=\begin{bmatrix}v_g\\ v_D\end{bmatrix}$$

The averaged model is: $\dot{\bar{x}}=(DA_1+(1-D)A_2)\bar{x}+(DB_1+(1-D)B_2)v$
$\bar{y}=C\bar{x}$

Let the nominal working point be $D=D_0,\ \bar{x}=\bar{x}_0,\ \bar{y}=\bar{y}_0$

At steady state, $0=(D_0A_1+(1-D_0)A_2)\bar{x}_0+(D_0B_1+(1-D_0)B_2)v$ (1)
$\bar{y}_0=C\bar{x}_0$



Relationship between output voltage y_{ss} and duty cycle D
* : by experiment
__ : by equation (1)

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At nominal working condition:

$$0=(D_0A_1+(1-D_0)A_2)\bar{x}_0+(D_0B_1+(1-D_0)B_2)v \qquad (1)$$
$$\bar{y}_0=C\bar{x}_0$$

To achieve robust stability and tracking, so that the same output y_0 is produced when parameters have changed, we obtain a perturbation model around the nominal working point:

Define $x_p=\bar{x}-\bar{x}_0,\quad u=D-D_0,\quad y=\bar{y}-\bar{y}_0$

$$\dot{x}_p=\bar{A}x_p+\bar{A}_b x_p u+\bar{B}u,\quad y=Cx_p \qquad \bar{A}=D_0A_1+(1-D_0)A_2,\ \bar{A}_b=A_1-A_2,\ \bar{B}=(A_1-A_2)\bar{x}_0+(B_1-B_2)v$$

This is obtained by subtracting (1) from the averaged model:

$$\dot{\bar{x}}=(DA_1+(1-D)A_2)\bar{x}+(DB_1+(1-D)B_2)v$$
$$\bar{y}=C\bar{x}$$

If the perturbation is small, x_pu can be ignored as a second-order term
The approximate linear model is

$$\dot{x}_p=\bar{A}x_p+\bar{B}u,\ \ y=Cx_p$$

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Linear Differential Inclusion (LDI)

An LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

In many situations, A,B,C,D are not constant, but nonlinear time varying , and/or depend on a parameter α, such as,

$$\dot{x}(t) = A(x,\alpha,t)x(t) + B(x,\alpha,t)u(t)$$
$$y(t) = C(x,\alpha,t)x(t) + D(x,\alpha,t)u(t)$$

We can find a set Ω such that $\begin{bmatrix} A(x,\alpha,t) & B(x,\alpha,t) \\ C(x,\alpha,t) & D(x,\alpha,t) \end{bmatrix} \in \Omega$

The system satisfies $\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Omega \right\}$

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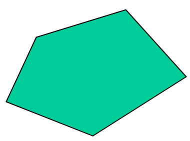
$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Omega \right\}$$

- This is a linear differential inclusion (LDI)
- An LDI uses a set of linear systems to describe a complicated nonlinear system.
- In many cases Ω is a polytope: the behavior of an LDI can be characterized by finite many linear systems, e,g.,

$$\dot{x}(t) = A_i x(t) + B_i u(t)$$
$$y(t) = C_i x(t) + D_i u(t), \quad i = 1, \cdots, N$$

- Like a polygon, its properties are determined by finite many vertices.



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Example: A model for a pendulum

$$\dot{x}_1 = h_1(x) = x_2$$
$$\dot{x}_2 = h_2(x) = -\frac{g}{l}\sin x_1 + \frac{1}{ml}\cos x_1 \quad u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\frac{\sin x_1}{x_1} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml}\cos x_1 \end{bmatrix} u = A(x_1)x + B(x_1)u$$

If the angle is restricted between 0 and $\pi/4$, we can write

$$\dot{x} \in \left\{ \left[A(x_1), \quad B(x_1) \right] \begin{bmatrix} x \\ u \end{bmatrix} : \ x_1 \in [0, \pi/4] \right\}$$

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Today:

– Math. Descriptions of Systems
  - Classification of systems
  - Linear systems
  - Linear-time-invariant systems
  - State variable description
  - Linearization

– Modeling of electric circuits

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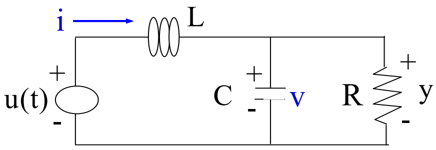
2.5 Modeling of Selected Systems

- We will briefly go over the following systems
  - Electrical Circuits
  - Operational Amplifiers
  - Mechanical Systems
  - Integrator/Differentiator Realization
- For any of the above system, we derive a state space description:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

➢ Different engineering systems are unified into the same framework, to be addressed by system and control theory.

Electrical Circuits



State variables?

– i of L and v of C

- How to describe the evolution of the state variables?

$$L\frac{di}{dt} = v_L = u - v \quad \Rightarrow \quad \frac{di}{dt} = -\frac{1}{L}v + \frac{1}{L}u$$

$$C\frac{dv}{dt} = i_C = i - \frac{v}{R} \quad \Rightarrow \quad \frac{dv}{dt} = \frac{1}{C}i - \frac{v}{RC}$$

State Equation: Two first-order differential equations in terms of state variables and input

In matrix form:

$$\underbrace{\begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix}}_{\dot{x}} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \underbrace{\begin{bmatrix} i \\ v \end{bmatrix}}_{x} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Output equation:

$$y = v = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + 0u$$

- Steps to obtain state and output equations:

  Step 1: Pick $\{i_L, v_C\}$ as state variables

  Step 2: $L\dfrac{di_L}{dt} = v_L = f_1(i_L, v_C, u)$ Linear functions

  $C\dfrac{dv_C}{dt} = i_C = f_2(i_L, v_C, u)$ By using KVL and KCL

  Step 3: $\dfrac{di_L}{dt} = (1/L)f_1(i_C, v_L, u)$

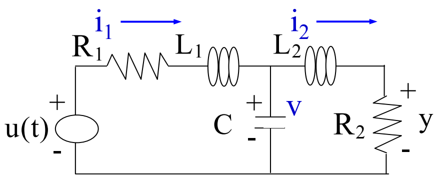
  $\dfrac{dv_C}{dt} = (1/C)f_2(i_C, v_L, u)$

  Step 4: Put the above in matrix form

  Step 5: Do the same thing for y in terms of state variables and input, and put in matrix form

Example

- State variables?
  - i_1, i_2, and v,



- State and output equations?

$$L_1\frac{di_1}{dt} = v_{L_1} = u - R_1 i_1 - v$$

$$L_2\frac{di_2}{dt} = v_{L_2} = v - R_2 i_2$$

$$C\frac{dv}{dt} = i_C = i_1 - i_2$$

$$\frac{di_1}{dt} = -\frac{R_1}{L_1}i_1 - \frac{1}{L_1}v + \frac{1}{L_1}u$$

$$\frac{di_2}{dt} = -\frac{R_2}{L_2}i_2 + \frac{1}{L_2}v$$

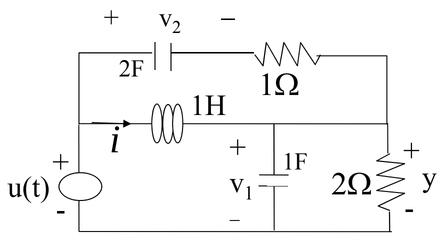
$$\frac{dv}{dt} = \frac{1}{C}i_1 - \frac{v}{C}i_2$$

$$\underbrace{\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dv}{dt} \end{bmatrix}}_{\dot{x}} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ v \end{bmatrix}}_{x} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = R_2 i_2 = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

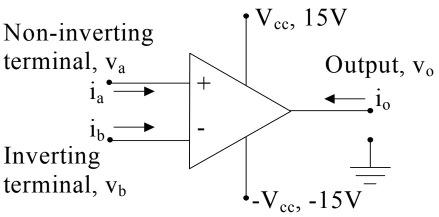
Practice: Derive the state space model for the following circuit:

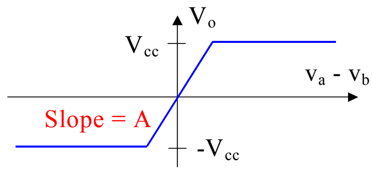


$$x = \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix}$$

Operational Amplifiers (Op Amps)

$v_0 = A\,(v_a - v_b)$, with $-V_{CC} \le v_0 \le V_{CC}$

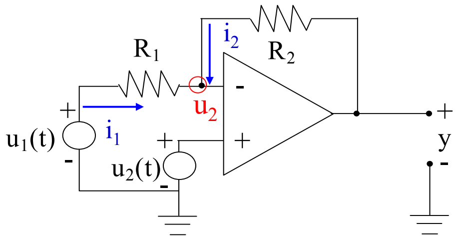




- Usually, $A > 10^4$

– Ideal Op Amp:
- $A \to \propto$ ~ Implying that $(v_a - v_b) \to 0$, or $v_a \to v_b$
- $i_a \to 0$ and $i_b \to 0$

– Problem: How to analyze a circuit with ideal Op Amps



$$i_1 + i_2 = 0$$

$$\frac{u_1 - u_2}{R_1} + \frac{y - u_2}{R_2} = 0$$

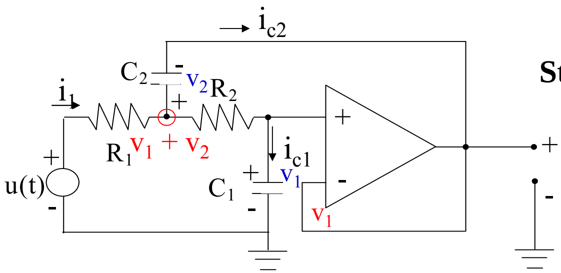
$$y = -\frac{R_2}{R_1} u_1 + \left(1 + \frac{R_2}{R_1}\right) u_2$$

Delineate the relationship between input and output

Input/Output description

Pure gain, no SVs

- Key ideas:
  - Make effective use of $i_a = i_b = 0$ and $v_a = v_b$
  - Do not apply the node equation to output terminals of op amps and ground nodes, since the output current and power supply current are generally unknown



State and output equations

What are the state variables?

State variables: v_1 and v_2

$$C_1 \frac{dv_1}{dt} = \frac{(v_1 + v_2) - v_1}{R_2} = \frac{v_2}{R_2} \qquad C_2 \frac{dv_2}{dt} = \frac{u - (v_1 + v_2)}{R_1} - \frac{v_2}{R_2}$$

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{R_2 C_1} \\ \frac{-1}{R_1 C_2} & \frac{-1}{C_2}\left[\frac{1}{R_1} + \frac{1}{R_1}\right] \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{R_1 C_2} \end{bmatrix} u$$

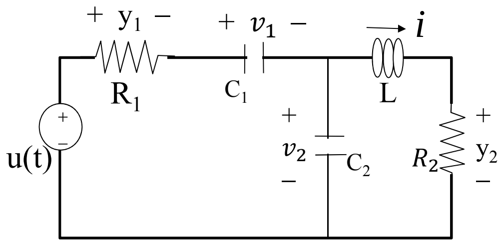
$$y = v_1 = [1 \quad 0] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + 0u$$

- **Today:**
  - Math. descriptions of systems
  - Modeling of electric circuit
- **Next Time:**
  - Modeling of Selected Systems
    - Continuous-time systems (§2.5)
      - mechanical systems, integrator/differentiator realization
    - Discrete-Time systems (§2.6)
      - difference equations, simple financial systems
  - Advanced Linear Algebra, Chapter 3

Problem Set #2:

1. Give examples for nonlinear systems and infinite dimensional systems respectively. What are the inputs, outputs and states?
2. Suppose we have a linear time-invariant system. Its response to u_1 is $y_1(t)= t+3$, for $t\geq 0$, and its response to u_2 is $y_2(t)= 2t$, for $t \geq 0$. For $t< 0$, $y_1(t) = y_2(t)=0$. Assume zero initial conditions. What is the response to $2u_1(t-1)-u_2(t+1)$? Plot the response for $t\in[-2,4]$ with Matlab.
3. An LTI system is described by
$$\ddot{y} + 4\dot{y} + 3y = u,$$
What is y(t) for u=0 and y(0)=1, y'(0)=-1?
What is y(t) for a unit step u (u=1(t)) and y(0)=y'(0)=0?
What is y(t) for u=1(t) and y(0)=2, y'(0)=-2?
What is the state of the system?

4. Derive state-space description for the circuit:



Input: u(t)

Output: $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

State: $x = \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix}$

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