16.513 Control Systems

Last Time:

- Math. Descriptions of Systems
  - Classification of systems
  - Linear systems
  - Linear-time-invariant systems
  - State variable description
  - Linearization
- Modeling of electric circuits

Today:

- Modeling of Selected Systems
  - Continuous-time systems (§2.5)
    - Mechanical systems
    - Integrator/Differentiator realization
  - Discrete-Time systems (§2.6)
    - Derive state-space equations – difference equations
    - Two simple financial systems
- Linear Algebra, Chapter 3
  - Linear spaces over a field
  - Linear dependence

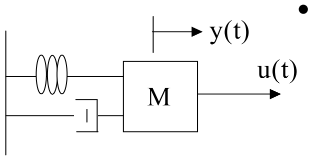
2.5 Modeling of Selected Systems

- Deriving state-space model for the following systems
  - Electrical Circuits
  - Operational Amplifiers
  - Mechanical Systems
  - Integrator/Differentiator Realization
- For any of the above system, we derive a state space description:

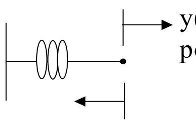
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

➢ Different engineering systems are unified into the same framework, to be addressed by system and control theory.

Mechanical Systems

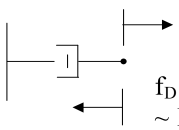


- Elements: Spring, dashpot, and mass
  - Spring:

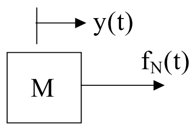


$f_S = Ky$, opposite direction
~ Hooke's law, K: Stiffness

  - Dashpot:



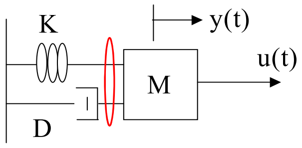
$f_D = Dy'$, opposite direction
~ D: Damping coefficient



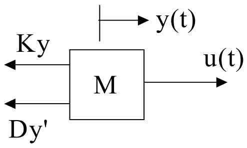
  - Mass: M, Newton's law of motion

$M\ddot{y} = f_N$ ~ Net force

  - LTI elements, LTI systems
  - Linear differential equations with constant coefficients



- How to describe the system?
- Free body diagram:



$$M\ddot{y} = u - Ky - D\dot{y} \qquad \ddot{y} + \frac{D}{M}\dot{y} + \frac{K}{M}y = \frac{1}{M}u$$

~ Input/Output description

- Number of state variables? Which ones?
  - 2 state variables: $x_1 \equiv y$, $x_2 \equiv x_1'$

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \ddot{y} = \frac{u - Ky - D\dot{y}}{M} = \frac{u - Kx_1 - Dx_2}{M}$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \qquad y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u$$

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- Steps to obtain state and output equations:
  - Step 1: Determine ALL junctions and label the displacement of each one
  - Step 2: Draw a free body diagram for each rigid body to obtain the net force on it
  - Step 3: Apply Newton's law of motion to each rigid body
  - Step 4: Select the displacement and velocity as state variables, and write the state and output equations in matrix form
- For rotational systems: $\tau = J\alpha$
  - τ: Torque = Tangential force·arm
  - J: Moment of inertia = $\int r^2 dm$
  - α: Angular acceleration
  - There are also angular spring/damper

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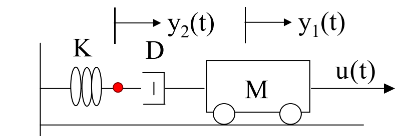
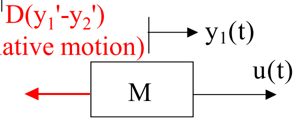
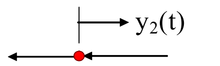


- How to describe the system?
- How many junctions are there?

D(y1'-y2')
(relative motion)



$$M\ddot{y}_1 = u - D(\dot{y}_1 - \dot{y}_2)$$
$$0 = -Ky_2 - D(\dot{y}_2 - \dot{y}_1)$$



Ky_2 D(y2'-y1')

- Number of state variables? How to select the state variables?

$$x_1 \equiv y_1; \quad x_2 \equiv \dot{y}_1; \quad x_3 \equiv y_2$$
$$\dot{x}_1 \equiv \dot{y}_1 = x_2$$
$$\dot{x}_2 \equiv \ddot{y}_1 = \frac{1}{M}(u - Dx_2 + D\dot{x}_3) = -\frac{K}{M}x_3 + \frac{1}{M}u$$
$$\dot{x}_3 = x_2 - \frac{K}{D}x_3$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{K}{M} \\ 0 & 1 & -\frac{K}{D} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

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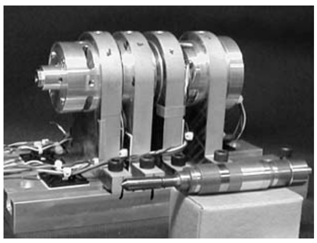
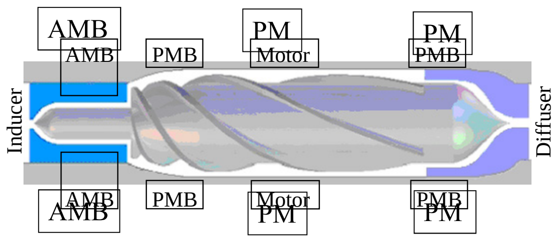
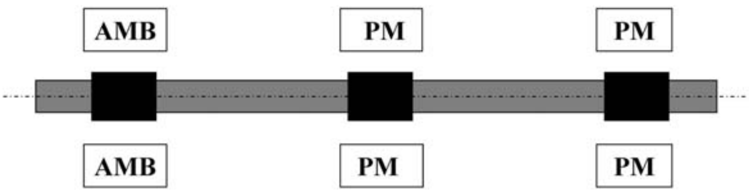
Example: an axial artificial heart pump

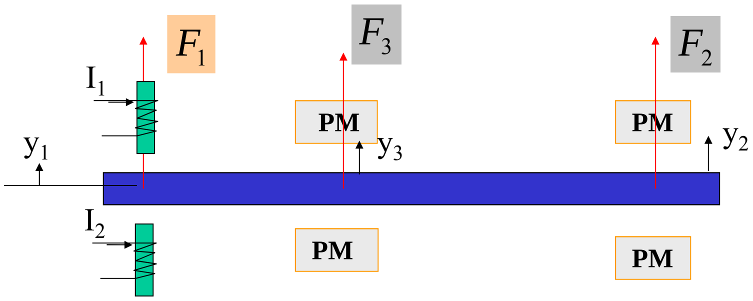




Illustration

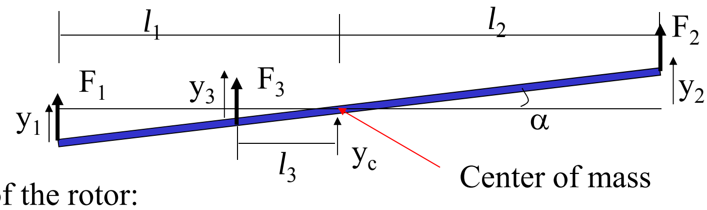


The forces acting on the rotor:



F_1: the active force that can be generated as desired
($F = k\, I_1^2 / (c+y_1)^2 - kI_2^2/(c-y_1)^2$
F_2, F_3: passive forces, $F_2 = -k_2y_2$, $F_3 = -k_3y_3$, similar to springs

Modeling



The motion of the rotor:

$$M\ddot{y}_c = F_1 + F_2 + F_3, \quad y_c = (l_1 y_2 + l_2 y_1)/l$$
$$J\ddot{\alpha} = -l_1 F_1 + l_2 F_2 - l_3 F_3, \quad \alpha = (y_2 - y_1)/l$$

$$l = l_1 + l_2$$

F_2 and F_3 depend on y_1 and y_2. Equation can be expressed in terms of y_1 and y_2

$$\ddot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1F_1$$
$$\ddot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2F_1$$

Let $x_1 = y_1, \quad x_2 = \dot{y}_1, \quad x_3 = y_2, \quad x_4 = \dot{y}_2$

$$\ddot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1F_1$$
$$\ddot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2F_1$$

Let $x_1 = y_1, \quad x_2 = \dot{y}_1, \quad x_3 = y_2, \quad x_4 = \dot{y}_2$

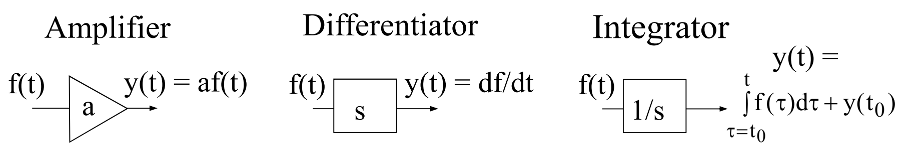
$$\dot{x}_1 = \dot{y}_1 = x_2$$
$$\dot{x}_2 = \ddot{y}_1 = a_{11}x_1 + a_{12}x_3 + b_1F_1$$
$$\dot{x}_3 = \dot{y}_2 = x_4$$
$$\dot{x}_4 = \ddot{y}_2 = a_{21}x_1 + a_{22}x_3 + b_2F_1$$

In matrix form?

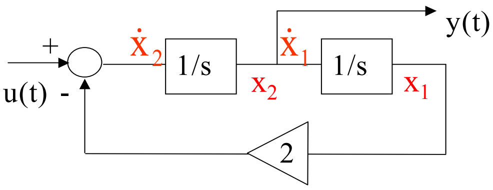
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{11} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 1 \\ a_{21} & 0 & a_{22} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix} F_1$$
$$= Ax + Bu$$

Integrator/Differentiator Realization

- **Elements:** Amplifiers, differentiators, and integrators



- Are they LTI elements? Yes
- Which one has memory? What are their dimensions?
  - Integrator has memory. Dimensions: 0, 0, and 1, respectively
- They can be connected in various ways to form LTI systems
  - Number of state variables = number of integrators
  - Linear differential equations with constant coefficients



- What are the state variables?
- Select output of integrators as SVs
- What are the state and output equations?

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u - 2x_1 \qquad y = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{0}_{D} u$$

- Linear differential equations with constant coefficients

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- Steps to obtain state and output equations:

  Step 1: Select outputs of integrators as state variables

  Step 2: Express inputs of integrators in terms of state variables and input based on the interconnection of the block diagram

  Step 3: Put in matrix form

  Step 4: Do the same thing for y in terms of state variables and input, and put in matrix form

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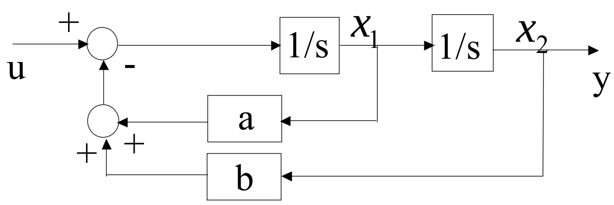
Exercise: derive state equations for the following sys.



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- Modeling of Selected Systems
  - Continuous-time systems (§2.5)
    - Electrical circuits
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  - Discrete-Time systems (§2.6):
    - Derive state-space equations – difference equations
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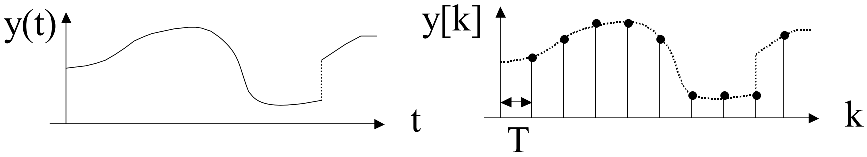
2.6 Discrete-Time Systems

- Thus far, we have considered continuous-time systems and signals



- In many cases signals are defined only at discrete instants of time
  - T: Sampling period
  - No derivative and no differential equations
  - The corresponding signal or system is described by a set of difference equations

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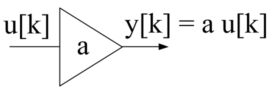
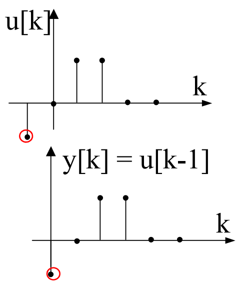
Elements: Amplifiers, delay elements, sources (inputs)

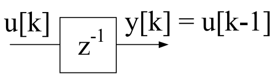
Amplifiers:





~ LTI and memoryless

- Delay Element:



~ LTI with memory (1 initial condition)

They can be interconnected to form an LTI system

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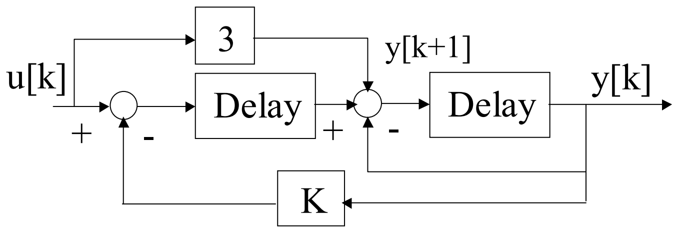
Example



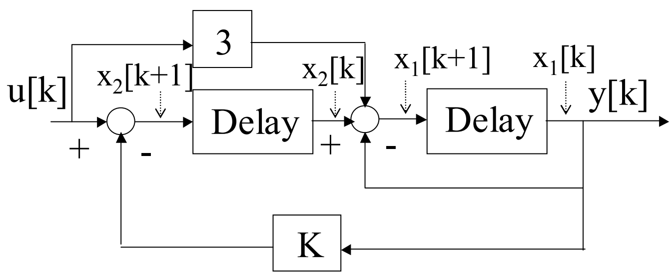
- How to describe the above mathematically?
  - I/O description:

    $$y[k+1] = -y[k] + 3u[k] + (u[k-1] - Ky[k-1]), \text{ or}$$

    $$y[k+1] = -y[k] - Ky[k-1] + 3u[k] + u[k-1]$$

  - A linear difference equation with constant coefficient

- State space description: Select output of delay elements as state variables

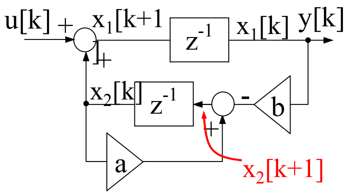


$$x_1[k+1] = -x_1[k] + x_2[k] + 3u[k]$$

$$x_2[k+1] = -K\, x_1[k] + u[k]$$

$$y[k] = x_1[k]$$

Exercise:



Derive state equations for the discrete-time system.

- Two state variables, $x_1[k]$, $x_2[k]$

  $x_1[k+1] = u[k] + x_2[k]$

  $x_2[k+1] = -bx_1[k] + ax_2[k]$

  $y[k] = x_1[k]$

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ -b & a \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k] \qquad y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[k]$$

Example 1: Balance in your bank account

- A bank offers interest r compounded every day at 12am
  - u[k]: The amount of deposit during day k (u[k] < 0 for withdrawal)
  - y[k]: The amount in the account at the beginning of day k
  - What is y[k+1]?

    $y[k+1] = (1 + r)\, y[k] + u[k]$

Example 2: Amortization

- How to describe paying back a car loan over four years with initial debt D, interest r, and monthly payment p?
  - Let x[k] be the amount you owe at the beginning of the kth month. Then

$$x[k+1] = (1 + r)\, x[k] - p$$

  - Initial and terminal conditions: x[0] = D and final condition x[48] = 0
    - How to find p?

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The system:

$$x[k+1] = \underbrace{(1 + r)}_{A}\, x[k] + \underbrace{(-1)}_{B}\, \underbrace{p}_{u}$$

Solution:

$$x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-m-1} B u[m]$$

$$= (1+r)^k x[0] + \sum_{m=0}^{k-1} (1+r)^{k-m-1} (-1) p$$

$$= (1+r)^k D - \left(\sum_{m=0}^{k-1} (1+r)^{k-m-1} \right) p = (1+r)^k D - \frac{(1+r)^k - 1}{r} p$$

Given D=20000; r=0.004; x[48]=0;

Your monthly payment

$$0 = (1+0.004)^{48}\, 20000 - \frac{(1+0.004)^{48} - 1}{0.004} p$$

p=458.7761

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Linear Algebra:
Tools for System Analysis and Design

- Our modeling efforts lead to a state-space description of LTI system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x[k+1] = Ax[k] + Bu[k]$$
$$y(t) = Cx(t) + Du(t) \qquad y[k] = Cx[k] + Du[k]$$

- Analysis problems: stability; transient performances; potential for improvement by feedback control, …

- Consider an LTI continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- For a practical system, usually there is a natural way to choose the state variables, e.g.,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} \longrightarrow i_L \\ \longrightarrow v_c \end{matrix}$$

- However, the natural state selection may not be the best for analysis. There may exist other selection to make the structure of A,B,C,D simple for analysis
- If T is a nonsingular matrix, then $z = Tx$ is also the state and satisfies

$$\begin{aligned} \dot{z}(t) &= TAT^{-1}z(t) + TBu(t) \\ y(t) &= CT^{-1}z(t) + Du(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{z}(t) &= \tilde{A}z(t) + \tilde{B}u(t) \\ y(t) &= \tilde{C}z(t) + \tilde{D}u(t) \end{aligned}$$

- Two descriptions

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \dot{z}(t) &= \tilde{A}z(t) + \tilde{B}u(t) \\ y(t) &= \tilde{C}z(t) + \tilde{D}u(t) \end{aligned}$$

are equivalent when I/O relation is concerned.

- For a particular analysis problem, a special form of $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ may be the most convenient, e.g.,

$$\tilde{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

- We need to use tools from Linear Algebra to get a desirable description.

- The operation $x \rightarrow z = Tx$ is called a linear transformation.
  - It plays the essential role in obtaining a desired state-space description

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}u(t)$$
$$y(t) = \tilde{C}z(t) + \tilde{D}u(t)$$

- Linear algebra will be needed for the transformation and analysis of the system
  - Linear spaces over a vector field
  - Relationship among a set of vectors: LD and LI
  - Representations of a vector in terms of a basis
  - The concept of perpendicularity: Orthogonality
  - Linear Operators and Representations

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3.1 Linear Vector Spaces and Linear Operators

Notation:

R^n: n-dimensional real linear vector space
C^n: n-dimensional complex linear vector space
$R^{n\times m}$: the set of n×m real matrices (also a vector space)
$C^{n\times m}$: the set of n×m complex matrices (a vector space)

- A matrix $T \in R^{n\times m}$ represents a linear operation from R^m to R^n: $x \in R^m \rightarrow Tx \in R^n$.
- All the matrices A,B,C,D in the state space equation are real

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Linear Vector Spaces R^n and C^n

R: The set of real numbers; C: The set of complex numbers
If x is a real number, we say $x \in R$;
If x is a complex number, we say $x \in C$
R^n: n-dimensional real vector space
C^n: n-dimensional complex vector space

$$R^n = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : \quad x_1, x_2, \cdots, x_n \in R \right\}, \quad C^n = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : \quad x_1, x_2, \cdots, x_n \in C \right\}$$

If $x,y \in R^n$, $a,b \in R$, then $ax + by \in R^n$ ⇨ R^n is a linear space.

If $x,y \in C^n$, $a,b \in C$, then $ax + by \in C^n$ ⇨ C^n is a linear space.

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Subspace

- Consider $Y \subset R^n$. Y is a **subspace** of R^n iff Y itself is a linear space
  - Y is a subspace iff $\alpha_1 y_1 + \alpha_2 y_2 \in Y$ for all $y_1, y_2 \in Y$ and $\alpha_1, \alpha_2 \in R$ (linearity condition)
  - Subspace of C^n can be defined similarly

Example: Consider R^2. The set of (x_1,x_2) satisfying $x_1 - 2x_2 + 1 = 0$ can be written as

$$Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \; x_1 - 2x_2 + 1 = 0, \; x_1, x_2 \in R \right\}$$

Is the linearity condition satisfied?

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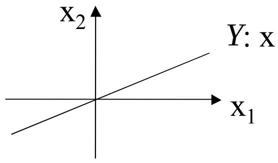
- Then how about the set of (x_1,x_2) satisfying $x_1 - 2x_2 = 0$?



Y: $x_1-2x_2=0$

$$Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \ x_1 - 2x_2 = 0,\ x_1, x_2 \in \mathrm{R} \right\}$$

  – Yes. In fact, any straight line passing through 0 form a subspace
- What would be a subspace for R^3 ?
  – Any plane or straight line passing through 0

  $\{(x_1,x_2,x_3):\ ax_1+bx_2+cx_3=0\}$ for constants a,b,c denote a plane. How to represent a line in the space?
- The set of solutions to a system of homogeneous equation is a subspace: $\{x \in R^n: Ax=0\}$.
- How about $\{x \in R^n: Ax=c\}$?

- Consider R^n,
  – Given any set of vectors $\{x_i\}_{i=1 \text{ to } n}$, $x_i \in R^n$.
  – Form the set of linear combinations

$$Y \equiv \left\{ \sum_{i=1}^{n} \alpha_i x_i \ : \ \alpha_i \in \mathrm{R} \right\}$$

  – Then Y is a linear space, and is a subspace of R^n.
  – It is the space spanned by $\{x_i\}_{i=1 \text{ to } n}$

Linear Independence

➢ Relationship among a set of vectors.

- A set of vectors $\{x_1, x_2, .., x_m\}$ in R^n is linearly dependent (LD) iff $\exists$ $\{\alpha_1, \alpha_2, .., \alpha_m\}$ in R, not all zero, s.t.

  $\alpha_1 x_1 + \alpha_2 x_2 + .. + \alpha_n x_m = 0$ (*)

  – If (*) holds and assume for example that $\alpha_1 \neq 0$, then

  $x_1 = -[\alpha_2 x_2 + .. + \alpha_n x_m]/\alpha_1$

  i.e., x_1 is a linear combination of $\{\alpha_i\}_{i=2 \text{ to } m}$

- If the only set of $\{\alpha_i\}_{i=1 \text{ to } m}$ s.t. the above holds is

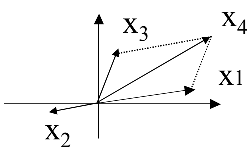
  $\alpha_1 = \alpha_2 = .. = \alpha_m = 0$

  then $\{x_i\}_{i=1 \text{ to } m}$ is said to be linearly independent (LI)

  – None of x_i can be expressed as a linear combination of the rest

– A linearly dependent set ~ Some redundancy in the set

Example. Consider the following vectors:



- For the following sets, are they linearly dependent (LD) or independent (LI)?
  - $\{x_1, x_2\}$
  - $\{x_1, x_3\}$
  - $\{x_1, x_3, x_4\}$
  - $\{x_1, x_2, x_3, x_4\}$

If you have a LD set, $\{x_1, x_2, \ldots x_m\}$, then $\{x_1, x_2, \ldots x_m, y\}$ is LD for any y.

- Given a set of vectors, $\{x_1, x_2, \ldots, x_m\} \subset R^n$, how to find out if they are LD or LI?

- A general way to detect LD or LI:
  - $\{x_1, x_2, .., x_m\}$ are LD iff $\exists$ $\{\alpha_1, \alpha_2, .., \alpha_m\}$, not all zero, s.t. $\alpha_1 x_1 + \alpha_2 x_2 + .. + \alpha_m x_m = 0$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m = \underbrace{\begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix}}_{= A \in R^{n\times m}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}}_{= \alpha \in R^m} = 0,$$

$\Rightarrow$ $A\alpha = 0$

  - $\{x_1, x_2, .., x_m\}$ are LD iff $A\alpha = 0$ has a nonzero solution

Need to understand the solution to a homogeneous equation.
There is always a solution $\alpha = 0$.
Question: under what condition is the solution unique? 37

- Detecting LD and LI through solutions to linear equations

Given $\{x_1, x_2, .., x_m\}$, form $A = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix}$

Consider the equation $A\alpha = 0$

If the equation has a unique solution, LI;
If the equation has nonunique solution, LD.

This is related to the rank of A.

If rank(A)=m, (A has full column rank), the solution is unique;
If rank(A)<m, the solution is not unique.

- If n=m and A is nonsingular, $\det(A) \neq 0$, rank(A)=m
  only $\alpha=0$ satisfies. $A\alpha=0$, hence LI
- If n=m and A is singular, $\det(A) = 0$, rank(A) < m
  $\exists\ \alpha \neq 0$ s.t. $A\alpha=0$, hence LD

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- Are the following vectors LD or LI?

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, x_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{vmatrix} \begin{aligned} &= 3\times 6\times 1 + 2\times 5\times 3 + 2\times 4\times 4 \\ &\quad -3\times 3\times 4 - 2\times 2\times 6 - 5\times 4\times 1 \\ &= 18 + 30 + 32 - 36 - 24 - 20 \\ &= 0 \end{aligned}$$

⇒ LD

- How about $x_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ det(A)=?

$$\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 4 \\ 4 & 5 & 7 \end{vmatrix} = -10 \neq 0$$

⇒ LI

- All depends on the uniqueness of solution for $A\alpha=0$
- If m> n, A is a wide matrix, rank(A)<m, always has a nonzero solution, e.g.,

m
n [A]

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

  - If m < n and rank(A) = m, LI;
    If m < n and rank(A)<m, LD;

m
n [A]

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 1 & -1 \end{bmatrix},$$

rank(A_1)=2=m, LI

rank(A_2)=1<m, LD

- Examples: determine the LD/LI for the following group of vectors

$$\left\{\begin{bmatrix}\sin\theta \\ \cos\theta\end{bmatrix}, \begin{bmatrix}-\cos\theta \\ \sin\theta\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}\sin\theta \\ \cos\theta\end{bmatrix}, \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1 \\ 0\end{bmatrix}, \begin{bmatrix}1 \\ 1\end{bmatrix}, \begin{bmatrix}1 \\ 3\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}1 \\ 1 \\ 1\end{bmatrix}, \begin{bmatrix}0 \\ 1 \\ 3\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1 \\ 1 \\ 1\end{bmatrix}, \begin{bmatrix}0 \\ 1 \\ 3\end{bmatrix}, \begin{bmatrix}2 \\ 3 \\ 5\end{bmatrix}\right\}$$

Dimension

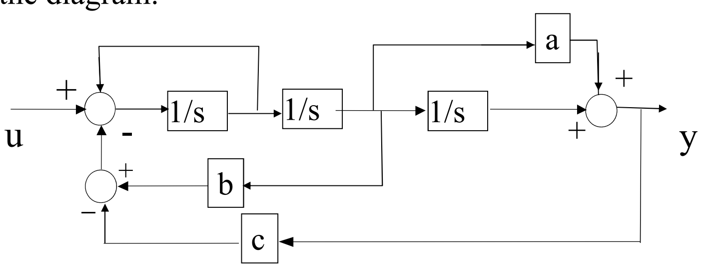
- For a linear vector space, the maximum number of LI vectors is called the dimension of the space, denoted as D
- Consider $\{x_1, x_2, \ldots, x_m\} \subset R^n$
  - If m>n, they are always dependent, $D \le n$
  - For m=n, there exist $x_1, x_2, \ldots x_n$ such that with $A=[x_1, x_2, \ldots x_n]$, $|A| \ne 0$, x_i's LI, $D \ge n$
  - Hence D = n

Today:

- Modeling of Selected Systems
  - Continuous-time systems (§2.5)
    - Electrical circuits, Mechanical systems
    - Integrator/Differentiator realization
    - Operational amplifiers
  - Discrete-Time systems (§2.6):
    - Derive state-space equations – difference equations
    - Two simple financial systems
- Linear Algebra, Chapter 3
  - Linear spaces over a field
  - Linear dependence
- Next time: More linear algebra.

Homework Set #3

1. Derive state-space description for the diagram:



2. Are the following sets subspace of R² ?

$$Y_1 = \left\{ a\begin{bmatrix}0\\1\end{bmatrix} + b\begin{bmatrix}0\\-2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} : a,b \in R \right\},$$

$$Y_2 = \left\{ a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} : a,b \in R \right\},$$

$$Y_2 = \left\{ a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} : a \geq 0, b \in R \right\}.$$

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3. Are the following groups of vectors LD or LI?

$$1) \left\{ \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}3\\1\\3\end{bmatrix}, \begin{bmatrix}2\\2\\4\end{bmatrix} \right\}, \quad 2) \left\{ \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}a\\b\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$$

$$3) \left\{ \begin{bmatrix}a\\1\end{bmatrix}, \begin{bmatrix}1\\-a\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}, \quad 4) \left\{ \begin{bmatrix}\cos\theta\\2\sin\theta\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}-\sin\theta\\\cos\theta\\1\end{bmatrix} \right\}$$

$$5) \left\{ \begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}, \quad 6) \left\{ \begin{bmatrix}1\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\\-1\end{bmatrix} \right\},$$

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