16.513 Control Systems, Lecture Note #4

Last Time:

- Modeling of Selected Systems (§2.5, §2.6):
  - Electrical circuits, Mechanical systems, simple financial systems
- Linear Algebra,
  - Linear spaces over a field, subspace
  - Linear dependence and Linear independence

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Today:

- The base of a linear space: Basis
  - Representations of a vector in terms of a basis
  - Relationship among representations for different bases
- Linear Operators and Representations
- Current Research

  on power management for nanogenerators

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Basis and Representations

- **Basis:** The basic elements from which everything can be constructed.
- A set of vectors $\{e_1, e_2, .., e_n\}$ of R^n is said to be a basis of R^n if every vector in R^n can be uniquely expressed as a linear combination of them
  - They span R^n
  - For any $x \in R^n$, there exist n real numbers $\beta_1, \beta_2, .., \beta_n$ s.t.

$$x = \beta_1 e_1 + \beta_2 e_2 + .. + \beta_n e_n = \sum_{i=1}^{n} \beta_i e_i$$

$$x = [e_1 \quad e_2 \quad \ldots \quad e_n] \begin{bmatrix} \beta_1 \\ \beta_2 \\ : \\ \beta_n \end{bmatrix} \sim \equiv \beta$$

$$x = [e_1 \quad e_2 \quad \ldots \quad e_n] \beta$$

  - β: Representation of x with respect to the basis
  - x is uniquely identified with β

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Now given a set of vectors, $e_1, e_2, \ldots, e_n$. What is the condition for the set to be a basis for R^n?

$$\text{Let } Y = \left\{ \sum_{i=1}^{n} \beta_i e_i : \beta_1, \beta_i, \cdots \beta_n \in R \right\} = \{\beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n : \beta_i \in R\}$$

$$= \left\{ [e_1 \quad e_2 \quad \ldots \quad e_n] \begin{bmatrix} \beta_1 \\ \beta_2 \\ : \\ \beta_n \end{bmatrix} : \quad \beta_1, \beta_i, \cdots \beta_n \in R \right\}$$

Y is a subset of R^n ($Y \subseteq R^n$), a subspace, to be precise .
If $\{e_1, e_2, .., e_n\}$ is a basis, we need to have $R^n \subseteq Y$. To ensure this,

- For every $x \in R^n$, there exist $\{\beta_1, \beta_2, .., \beta_n\}$ s.t.

$$x = \beta_1 e_1 + \beta_2 e_2 + .. + \beta_n e_n = \sum_{i=1}^{n} \beta_i e_i$$

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A particular basis -- The orthonormal basis:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \cdots, e_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

For all $x \in R^n$, we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

And the representation is unique. 5

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Q. What else qualifies to be a basis?

Theorem: In an n-dimensional vector space, any set of n LI vectors qualifies as a basis

Proof:

- Let $\{e_1, e_2, .., e_n\}$ be linearly independent
- For any $x \in R^n$, $\{x, e_1, e_2, .., e_n\}$ are linearly dependent
- $\exists$ $\{\alpha_0, \alpha_1, \alpha_2, .., \alpha_n\}$, not all zero, such that

  $$\alpha_0 x + \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0$$

- $\alpha_0 \neq 0$. Otherwise, $\{e_1, e_2, .., e_n\}$ are not linearly independent. Thus

  $$x = -\frac{1}{\alpha_0}(\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n) = \sum_{i=1}^{n} \beta_i e_i, \quad \beta_i = -\frac{\alpha_i}{\alpha_0}$$

- Any x can be expressed as a linear combination of them

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- Is the combination unique here?
  - Suppose that $\exists$ another linear combination

$$x = \sum_{i=1}^{n} \tilde{\beta}_i e_i = \sum_{i=1}^{n} \beta_i e_i$$

$$\sum_{i=1}^{n} \left(\tilde{\beta}_i - \beta_i\right) e_i = 0$$

- What can be said now?

$\tilde{\beta}_i - \beta_i = 0$ or $\tilde{\beta}_i = \beta_i$ for all i $\sim$ since $\{e_1, e_2, .., e_n\}$ are LI

  - The linear combination is unique
  - $\{e_1, e_2, .., e_n\}$ is a basis, and the proof is completed

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Theorem: In an n-dimensional vector space, any set of n LI vectors qualifies as a basis

Another explanation: Let $\{e_1, e_2, .., e_n\}$ be linearly independent. Form a square matrix $A=[e_1\ \ e_2\ \ ..\ \ e_n]$. Then A is nonsingular, i.e., $\det A \neq 0$

Consider the equation

$$A\beta = x \quad \Leftrightarrow \quad \beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n = x$$

There exists a unique solution $\beta=A^{-1}x$.

➢ For any $x \in R^n$, there exists a unique β such that

$$\beta_1 e_1 + \beta_2 e_2 + \cdots \beta_n e_n = x$$

➢ $\{e_1, e_2, .., e_n\}$ is a basis.

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Change of Basis

- Any set of n LI vectors qualify as a basis

$$(e_1 \quad e_2 \quad \ldots \quad e_n); \qquad (\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n)$$

 – For a particular x, the representation is unique for each basis $x = \sum_{i=1}^{n} \beta_i e_i; \quad x = \sum_{i=1}^{n} \bar{\beta}_i \bar{e}_i;$

Example: Consider R^2 ,

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \bar{e}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \bar{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

 Given $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, what are β and $\bar{\beta}$?

$$x = e_1 + 2e_2 = (e_1 \quad e_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \Rightarrow \beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

– Given β, how to find $\bar{\beta}$? 9

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$$x = \sum_{i=1}^{n} \beta_i e_i; \quad x = \sum_{i=1}^{n} \bar{\beta}_i \bar{e}_i;$$

- Problem: Given β, find $\bar{\beta}$; or the otherwise

➢ We need to express one basis in terms of the other

$$e_j = \sum_{i=1}^{n} p_{ij} \bar{e}_i = (\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n) \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix}$$

Or the other way,

$$\bar{e}_j = \sum_{i=1}^{n} q_{ij} e_i = (e_1 \ e_2 \ \ldots \ e_n) \begin{pmatrix} q_{1j} \\ q_{2j} \\ \vdots \\ q_{nj} \end{pmatrix}$$

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$$e_j = \sum_{i=1}^{n} p_{ij}\bar{e}_i = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix} = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)p_j$$

p_j

Recall $[Ep_1 \quad Ep_2 \quad \dots \quad Ep_n] = E[p_1 \quad p_2 \quad \cdots \quad p_n]$

$$(e_1 \quad e_2 \quad \dots \quad e_n) = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)(p_1 \quad p_2 \quad \dots \quad p_n)$$
$$= (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\, P \qquad \equiv P \sim n{\times}n$$

Given $x = \sum_{i=1}^{n} \beta_i e_i$

$$x = (e_1 \quad e_2 \quad \dots \quad e_n)\beta = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)P\beta$$
$$= (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\bar{\beta}$$

A new representation: $\bar{\beta} = P\beta$

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– Conversely, we can express $\bar{e}_j$ in terms of $(e_1 \quad e_2 \quad \dots \quad e_n)$;

$$(\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n) = (e_1 \quad e_2 \quad \dots \quad e_n)(q_1 \quad q_2 \quad \dots \quad q_n)$$
$$= (e_1 \quad e_2 \quad \dots \quad e_n)Q$$

$$x = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\bar{\beta} = (e_1 \quad e_2 \quad \dots \quad e_n)Q\bar{\beta}$$

$$\Rightarrow \quad \beta = Q\bar{\beta}$$

– What is the relationship between P and Q?

$$\bar{\beta} = P\beta = PQ\bar{\beta} \Rightarrow PQ = I, \quad \text{or } P = Q^{-1}$$

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In summary: Suppose we have an old basis: $(e_1 \quad e_2 \quad \dots \quad e_n)$;

And a new basis $(\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)$

Suppose $x = \sum_{i=1}^{n} \beta_i e_i = [e_1 \quad e_2 \quad \cdots \quad e_n]\beta$;

How to find $\bar{\beta}$ such that $x = \sum_{i=1}^{n} \bar{\beta}_i \bar{e}_i$?

- Express the old basis in terms of the new basis

$$e_j = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix} = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\ p_j \quad \text{for each } j$$

Form $P = [p_1 \ p_2 \ \dots \ p_n]$. Then $\boxed{\bar{\beta} = P\beta}$

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Alternative approach:

An old base: $(e_1 \quad e_2 \quad \dots \quad e_n)$; A new base $(\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)$

Suppose $x = \sum_{i=1}^{n} \beta_i e_i = [e_1 \quad e_2 \quad \cdots \quad e_n]\beta$;

- Express the new basis in terms of the old basis

$$\bar{e}_j = (e_1 \quad e_2 \quad \dots \quad e_n)\begin{pmatrix} q_{1j} \\ q_{2j} \\ \vdots \\ q_{nj} \end{pmatrix} = (e_1 \quad e_2 \quad \dots \quad e_n)\ q_j \quad \text{for each } j$$

Form $Q = [q_1 \ q_2 \ \dots \ q_n]$. Then $\boxed{\bar{\beta} = Q^{-1}\beta}$

$\bar{\beta}$ is the representaion in terms of the new basis: $x = \sum_{i=1}^{n} \bar{\beta}_i \bar{e}_i$.

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Example: Consider R^2 , $e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ $\bar{e}_1 = \begin{bmatrix}1\\-1\end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

$$\bar{e}_1 = e_1 - e_2 = \begin{bmatrix}1\\-1\end{bmatrix}, \quad \bar{e}_2 = e_1 + e_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

$$(\bar{e}_1 \ \ \bar{e}_2) = (e_1 \ \ e_2)\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} \Rightarrow Q = \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}, \quad P = \frac{1}{2}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

- Given $x = \begin{bmatrix}1\\2\end{bmatrix}$, what are β and $\bar{\beta}$?

$$x = e_1 + 2e_2 = (e_1 \ \ e_2)\begin{bmatrix}1\\2\end{bmatrix}, \Rightarrow \beta = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\bar{\beta} = P\beta = \frac{1}{2}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-0.5\\1.5\end{bmatrix}$$

- Verify: $(\bar{e}_1 \ \ \bar{e}_2)\ \bar{\beta} = \begin{bmatrix}1\\-1\end{bmatrix}\times(-0.5) + \begin{bmatrix}1\\1\end{bmatrix}\times 1.5 = \begin{bmatrix}1\\2\end{bmatrix} = x$ √☺

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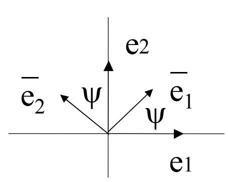
Example. Consider R^2 with $e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}$



$$\bar{e}_1 = (\cos\psi)e_1 + (\sin\psi)e_2 = (e_1 \ \ e_2)\begin{pmatrix}\cos\psi\\\sin\psi\end{pmatrix} \sim q_1$$

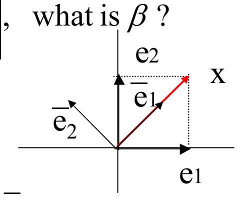
$$\bar{e}_2 = (-\sin\psi)e_1 + (\cos\psi)e_2 = (e_1 \ \ e_2)\begin{pmatrix}-\sin\psi\\\cos\psi\end{pmatrix} \sim q_2$$

$$Q = \begin{pmatrix}\cos\psi & -\sin\psi\\\sin\psi & \cos\psi\end{pmatrix}, \quad P = Q^{-1} = \begin{pmatrix}\cos\psi & \sin\psi\\-\sin\psi & \cos\psi\end{pmatrix}$$

$\bar{\beta} = P\beta$ and $\beta = Q\bar{\beta}$

– For $\psi = 45°$ and $x = \begin{bmatrix}1\\1\end{bmatrix} = [e_1 \ \ e_2]\begin{bmatrix}1\\1\end{bmatrix}$, what is $\bar{\beta}$?

$$\bar{\beta} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\0\end{pmatrix}$$



$$x = \sqrt{2}\,\bar{e}_1$$

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In summary:

- Given a basis $(e_1 \quad e_2 \quad \dots \quad e_n)$;
- Let the new basis be $(\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n) = (e_1 \quad e_2 \quad \dots \quad e_n)Q$

  Then $(e_1 \quad e_2 \quad \dots \quad e_n) = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\, Q^{-1}$
- For x such that $x = (e_1 \quad e_2 \quad \dots \quad e_n)\beta$

  We have $x = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)Q^{-1}\beta \;\Rightarrow\; \bar{\beta} = Q^{-1}\beta$
- For x such that $x = (\bar{e}_1 \quad \bar{e}_2 \quad \dots \quad \bar{e}_n)\bar{\beta}$

  We have $x = (e_1 \quad e_2 \quad \dots \quad e_n)Q\bar{\beta} \;\Rightarrow\; \beta = Q\bar{\beta}$

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Inverse for block diagonal matrices:

For a block diagonal matrix

$$S = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \Rightarrow \quad S^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

$$S = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \Rightarrow \quad S^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{bmatrix}$$

Exercise: Compute the inverse for

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

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Exercise: The old basis: $e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$

The new basis is $\bar{e}_1=e_1+e_2$; $\bar{e}_2=e_2+e_3$; $\bar{e}_3=e_3$

1) For $x=e_1+e_2+e_3$, find a,b,c such that $x=a\bar{e}_1+b\bar{e}_2+c\bar{e}_3$,

- What is the representation of the new basis in terms of the old one?

$$[\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3]=[e_1\ \ e_2\ \ e_3]\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}=[e_1\ \ e_2\ \ e_3]Q,\quad Q=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$$

- The representation of the old basis in terms of the new basis:

$$P=Q^{-1}=\begin{bmatrix}1&0&0\\-1&1&0\\1&-1&1\end{bmatrix},\quad [e_1\ \ e_2\ \ e_3]=[\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3]\begin{bmatrix}1&0&0\\-1&1&0\\1&-1&1\end{bmatrix}$$

$$x=[e_1\ \ e_2\ \ e_3]\begin{bmatrix}1\\1\\1\end{bmatrix}=[\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3]P\begin{bmatrix}1\\1\\1\end{bmatrix}=[\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3]\begin{bmatrix}1\\0\\1\end{bmatrix}$$

→ a=1

→ b=0

→ c=1

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Exercise: The old basis: $e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$

The new basis is $\bar{e}_1=e_1+e_2$; $\bar{e}_2=e_2+e_3$; $\bar{e}_3=e_3$

2) Given $x=\bar{e}_1-\bar{e}_2+\bar{e}_3$, what are a,b,c such that $x=ae_1+be_2+ce_3$?

Need the representation of the new basis in terms of the old one:

$$[\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3]=[e_1\ \ e_2\ \ e_3]\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}=[e_1\ \ e_2\ \ e_3]Q,\quad Q=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$$

$$x=[\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3]\begin{bmatrix}1\\-1\\1\end{bmatrix}=[e_1\ \ e_2\ \ e_3]Q\begin{bmatrix}1\\-1\\1\end{bmatrix}=[e_1\ \ e_2\ \ e_3]\begin{bmatrix}1\\0\\0\end{bmatrix}$$

→ a=1

→ b=0

→ c=0

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- So far, we have addressed a few issues on linear vector spaces:
  - Relation between vectors: LD/LI,
  - Representation: basis

➢ Next we will discuss operations between vector spaces
□ linear operations and matrices

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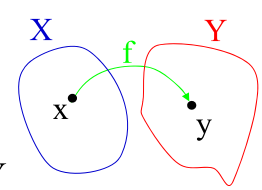
Linear Operations

- Functions
  - A function f is a mapping from domain X to codomain Y that assigns each $x \in X$ one and only one element of Y



  - Range: $\{y \in Y \mid \exists\, x \in X, \text{ s.t. } f(x) = y\} \subseteq Y$
- What is a "linear function"?
  - A function L that maps from X to Y is said to be a linear operator (linear function, linear mapping, or linear transformation) iff

$$L(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 L(x_1) + \alpha_2 L(x_2)$$
$$\forall\, \alpha_1, \alpha_2 \in R \text{ (or } C\text{), and } \forall\; x_1, x_2 \in X$$

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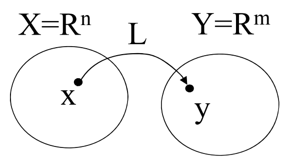
Linear Operations Associated with Matrices

- Consider $X=R^n$ and $Y=R^m$
- Given an $m \times n$ real matrix S;
- For $x \in R^n$, define

  $y=L(x)=Sx$

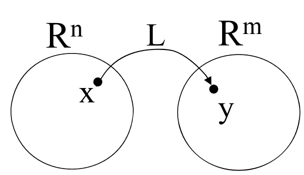
  – You can show that L is a linear map



- Thus we can use a matrix to define a linear map.
- What if we define $y=f(x)=Sx+c$ for a nozero $c \in R^m$?
  – Of course not a linear map
- More to say? Is every linear map from R^m to R^n associated with a matrix?
- Yes ! ☺

Matrix Representation of Linear Operators

- Still consider $X=R^n$ to $Y=R^m$.
- A linear operator is uniquely determined by how the basis are mapped



Theorem. Suppose that

– $\{e_1, e_2, .., e_n\}$ is a basis of R^n

– Then $L: X \rightarrow Y$ is uniquely determined by n pairs of mapping

$e_i \rightarrow y_i$, $i = 1, 2, .., n$

- Let the basis of X be $\{e_1, e_2, .., e_n\}$
- Let the basis of Y be $\{w_1, w_2, .., w_m\}$
- Suppose that

$$e_i \Rightarrow L(e_i) = y_i = [w_1 \ \ w_2 \ \ \cdots \ \ w_m] s_i, \quad s_i = \begin{bmatrix} s_{1i} \\ s_{2i} \\ \vdots \\ s_{mi} \end{bmatrix}$$

- Then by linearity, for any $x \in X$, with representation α, i.e., $x = \Sigma_{i=1}^{n} \alpha_i e_i$, we have

$$x \Rightarrow y = L(x) = \sum_{i=1}^{n} \alpha_i y_i = \sum_{i=1}^{n} [w_1 \ \ w_2 \ \ \cdots \ \ w_m] \alpha_i s_i$$

$$= [w_1 \ \ w_2 \ \ \cdots \ \ w_m] \sum_{i=1}^{n} \alpha_i s_i = [w_1 \ \ w_2 \ \ \cdots \ \ w_m] \underbrace{\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha}$$

- Representation of y: $\beta = S\alpha$

➢ The linear operator is determined by how the basis are mapped

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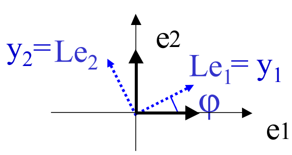
Example. Rotating counter-clock-wise in R^2 by φ



$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

How to proceed?

$e_1 \to y_1, e_2 \to y_2$

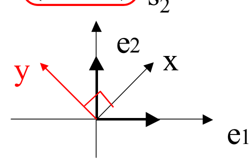
– What is y_1? What is y_2?

$$y_1 = (\cos\varphi) w_1 + (\sin\varphi) w_2 = (w_1 \ \ w_2) \underbrace{\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}}_{s_1} \qquad S = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$y_2 = (-\sin\varphi) w_1 + (\cos\varphi) w_2 = (w_1 \ \ w_2) \underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}}_{s_2}$$

– If $x = \alpha_1 e_1 + \alpha_2 e_2$, e.g., $\alpha = (1, 1)^T$, and $\varphi = 90°$, then $y = \beta_1 w_1 + \beta_2 w_2$, with

$$\beta = S\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



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Given 3 matrices:

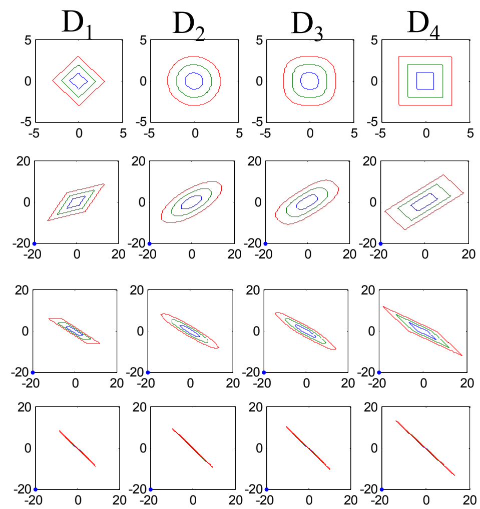
$$S_1 = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \times 1.5, \quad S_2 = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \times 2, \quad S_3 = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \times 1.5,$$

Subsets in R^2

$\{y=S_1x: \ x \in D_i\}$

$\{y=S_2x: \ x \in D_i\}$

$\{y=S_3x: \ x \in D_i\}$



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Change of Basis

– L: x → y ~ The mapping is independent of bases

– $\beta = A\alpha$ ~ The i^{th} column of A is the representation of Le_i ($= y_i$) w.r.t. $\{w_1, w_2, .., w_m\}$

- The representation A depends on the bases for X and Y

– Consider a special case where L: $R^n \to R^n$ (or C^n to C^n)

– With $\{e_1, e_2, .., e_n\}$ as a basis, under operator L
  $x \to y$: $[e_1\ e_2\ ..\ e_n]\ \alpha \ \to [e_1\ e_2\ \ldots e_n]\ A\alpha$

– For simplicity, we denote L: $\alpha \to \beta = A\alpha$

– Suppose the basis is changed to $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ for X,Y.

– What would be the new rep. of L?

– Suppose the new rep. is $\bar{A}$. How are A and $\bar{A}$ related?

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– Still consider the map

$x \rightarrow y$: $[e_1\ e_2\ ..\ e_n]\ \alpha \ \rightarrow [e_1\ e_2\ \ldots e_n]\ A\alpha$

 Let the new basis be

$[\hat{e}_1\ \hat{e}_2 \ldots \hat{e}_n] = [e_1\ e_2\ ..\ e_n]\ Q$

 Then equivalently,

$[\hat{e}_1\ \hat{e}_2 \ldots \hat{e}_n]\ P = [e_1\ e_2\ ..\ e_n]$, where $P=Q^{-1}$

 Under the new basis, we have

$x \rightarrow y$: $[\hat{e}_1\ \hat{e}_2\ ..\ \hat{e}_n]\ P\alpha \ \rightarrow [\hat{e}_1\ \hat{e}_2\ \ldots \hat{e}_n]\ PA\alpha$

 If we let $a=P\alpha$, $b=PA\alpha$, then $a \rightarrow b=PAP^{-1}a$

- The new rep. for the operator is $\bar{A}=PAP^{-1}= Q^{-1}AQ$

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$$A \Rightarrow \overline{A} = PAP^{-1}$$

$$\text{or } \boxed{A = P^{-1}\overline{A}P = Q\overline{A}Q^{-1}}$$

– This is called the similar transformation, and A and $\overline{A}$ are similar matrices

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Example. Consider $\dot{x} = Ax, A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

What is the dynamics for a new representation with

$$\bar{x} = Px = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} x \quad \Rightarrow \quad \dot{\bar{x}} = \bar{A}\bar{x}, \text{ with } \bar{A} \equiv PAP^{-1}$$

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\dot{\bar{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \bar{x}$$

□ The diagonal elements capture the main characteristic of the system

- A much simpler structure
- Later on we will discuss how to find such a P matrix

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Today:

- The base of a linear space: Basis
  - Representations of a vector in terms of a basis
  - Relationship among representations for different bases
- Linear Operators and Representations

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- **Next Time: 3.3 to 3.6**
  - Systems of linear algebraic equations
  - Similarity transformation: the Companion form
  - Eigenvalues and Eigenvectors
  - Diagonal form and Jordan form
  - Functions of a square matrix

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Homework Set #4

Problem 1: Given a basis for R^3, $\left\{\begin{bmatrix}-1\\2\\-3\end{bmatrix},\begin{bmatrix}0\\-1\\2\end{bmatrix},\begin{bmatrix}0\\0\\-1\end{bmatrix}\right\}$

Express $x=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ and $y=\begin{bmatrix}1\\0\\1\end{bmatrix}$ in terms of the basis.

Problem 2: Let the old basis be $e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}, e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$

The new basis is $\bar{e}_1=e_1$; $\bar{e}_2=-2e_1+e_2$; $\bar{e}_3=e_1-2e_2+e_3$

1) For $x=2\bar{e}_1-3\bar{e}_2+\bar{e}_3$,
   find a,b,c such that $x=ae_1+be_2+ce_3$

2) For $x=e_1-e_2+e_3$, find α,β,γ such that $x=\alpha\bar{e}_1+\beta\bar{e}_2+\gamma\bar{e}_3$,

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Problem 4: Consider $\dot{x} = Ax, \; A = \begin{bmatrix} 0 & 0 & -1 \\ 0.5 & 0 & 0.5 \\ 0 & -2 & 1 \end{bmatrix}$

1) What is the dynamics for a new representation with

$$\overline{x} = Px, \; P = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & -0.5 \end{bmatrix}$$

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