16.513 Control Systems
Lecture Note #5

Last time:

- The base of a linear space: Basis
- Representations of a vector in terms of a basis
- Relationship among representations for different bases
- Generalization of the idea of length: Norms
- A sense of orientation: Inner Product
- The concept of perpendicularity: Orthogonality
- Gram-Schmidt Process to obtain orthonormal vectors
- Linear Operators and Representations

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Representation of vectors w.r.t. different basis

- What is a basis for R^n? What can be used as a basis?
  - A set of vectors $\{e_1, e_2, \ldots, e_n\}$ which can be used to represent every $x \in R^n$ uniquely as:

$$x = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n$$

- Every set of n LI vectors can be used as a basis.
- Given an old basis $(e_1 \quad e_2 \quad \ldots \quad e_n)$;
  - Let the new basis be:

$$(\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n) = (e_1 \quad e_2 \quad \ldots \quad e_n) Q$$

  - Equivalently,

$$(e_1 \quad e_2 \quad \ldots \quad e_n) = (\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n) Q^{-1}$$

  - For x such that

$$x = (e_1 \quad e_2 \quad \ldots \quad e_n) \beta$$

Different representations

  - We have

$$x = (\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n) Q^{-1} \beta$$

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- How to obtain an orthonormalized basis?
  - Gram-Schmidt process
- Linear operators and matrix representations
  - A linear operator is completely determined by how the basis are mapped
  - A matrix defines a linear operator
  - Any linear operator can be defined by a matrix
  - Matrix rep. under different basis

Consider the map

$x \to y$: $[e_1\ e_2\ ..\ e_n]\ \alpha \ \to [e_1\ e_2\ \ldots e_n]\ A\alpha$

Let the new basis be $[\hat{e}_1\ \hat{e}_2 \ldots \hat{e}_n] = [e_1\ e_2\ ..\ e_n]\ Q$

➢ The new rep. for the operator is $\bar{A} = Q^{-1}AQ$

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Today: More discussions on linear algebra (§3.3-3.5, 3.8)

- Linear algebraic equations, solutions
- Parameterization of all solutions
- Similarity transformation: companion form,
- Eigenvalues and eigenvectors, diagonal form
- Generalized eigenvectors, Jordan form

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Systems of Linear Algebraic Equations

- A system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + .. + a_{1n}x_n = y_1$$
$$a_{21}x_1 + a_{22}x_2 + .. + a_{2n}x_n = y_2$$
$$\vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = y_m$$

n variables, $x_1, x_2, \ldots, x_n$, to satisfy m equations

where $a_{ij}, y_i \in$ R or C are given, x_i's are to be solved.

- In matrix form: $Ax = y$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$m \times n \qquad\qquad n \times 1 \qquad\qquad m \times 1$$

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Let the ith column of A be a_i, i.e., $A=[a_1\ a_2\ \ldots\ a_n]$, then

$$Ax = [a_1 \quad a_2 \quad \cdots \quad a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

A linear combination of $\{a_1, a_2, \ldots, a_n\}$

The equation Ax=y has a solution if y is a linear combination of the columns of A.

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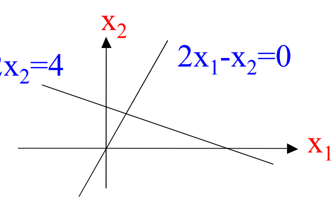
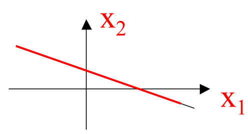
Example. $2x_1 - x_2 = 0$ and $x_1 + 2x_2 = 4$. Find x_1 and x_2.

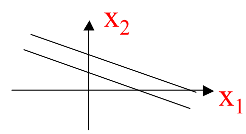
Geometric interpretation:

- Each equation represents a straight line in the plane.
- The solution is the intersection
- For this case, there is a unique intersection: $x=(0.8,1.6)^T$



- In general, there are 3 possibilities:
  – Unique sol.; inf. number of sol.; and no sol.





- For a system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + .. + a_{1n}x_n = y_1$$
$$a_{21}x_1 + a_{22}x_2 + .. + a_{2n}x_n = y_2$$
$$\vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = y_m$$

- When n=3, each equation represents a plane; The solution of Ax=y is the intersection of several planes
- For general n, each equation represents a hyperplane; The solution is the intersection of hyperplanes.

- How to analyze $Ax = y$ systematically?
  - We will examine from the viewpoint of a linear operator $A: R^n \rightarrow R^m$ (or $C^n \rightarrow C^m$)
  - Range of a linear operator A is

All possible linear combination of the columns of A

$\mathcal{R}(A) \equiv \{Ax: x \in R^n\} = \{y \in R^m: \exists x \in R^n \text{ s.t. } y = Ax\}$

  - Is $\mathcal{R}(A)$ a space? If so, it is a subspace of what?

Theorem. $\mathcal{R}(A)$ is a subspace of R^m

Proof: ▪ Clearly it is a subset of R^m.

▪ Need to show that if $y_1, y_2 \in \mathcal{R}(A)$, then $\alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{R}(A)$ for all $\alpha_1, \alpha_2 \in R$

– $\exists\ x_1, x_2 \in R^n$ s.t. $y_1 = Ax_1$, $y_2 = Ax_2$

$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 Ax_1 + \alpha_2 Ax_2 = A(\alpha_1 x_1 + \alpha_2 x_2)$

$\Rightarrow \alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{R}(A)$, and $\mathcal{R}(A)$ is a subspace

▪ What is the dimension of $\mathcal{R}(A)$?

– Recall that the dim. is the maximum number of LI vectors in $\mathcal{R}(A)$.

▪ Let $a_1, a_2, \ldots a_n$ be the columns of A, i.e., $A = [a_1\ \ a_2\ \ldots\ a_n]$. Then

– $\mathcal{R}(A)$ is a subspace spanned by a_i's:

$\mathcal{R}(A) = \{Ax: x \in R^n\} = \{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n: \ x \in R^n\}$

– The dim.of $\mathcal{R}(A)$ is the maximum number of a_i's which are LI, $\leq \min\{m,n\}$

– It also equals the rank of A : denoted $\rho(A)$

– If $\rho(A) = m$, then $\mathcal{R}(A) = R^m$

Example:

$$A=\begin{bmatrix}0&1&1&2\\1&2&3&4\\2&0&2&0\end{bmatrix}$$
$a_1 \quad a_2 \quad a_3 \quad a_4$

- a_1, a_2 are independent;
- How about a_1, a_2, a_3?
- How about a_1, a_2, a_4?

- $a_3 = a_1 + a_2$; $a_4 = 2a_2$;
- If $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$, then
  $z = a_1x_1 + a_2x_2 + (a_1 + a_2)x_3 + a_22x_4$
  $= a_1(x_1 + x_3) + a_2(x_2 + x_3 + 2x_4) = a_1y_1 + a_2y_2$
- All $z \in \mathcal{R}(A)$ can also be expressed as linear combinations of a_1 and a_2
- $\mathcal{R}(A) = \mathcal{R}([a_1 \ \ a_2])$, dim.=2

In general, consider $C=[A\ B]$; $A=[a_1 \dots a_{n1}]$, $B=[b_1 \dots b_{n2}]$
Every b_i can be expressed as linear combination of a_j's
if and only if $\mathcal{R}(C)=\mathcal{R}(A)$; $\rho(C)=\rho(A)$

Basis for the range space

- Let $a_1, a_2, \dots a_n$ be the columns of A, i.e., $A=[a_1 \ \ a_2 \ \dots \ a_n]$. Then
  - $\mathcal{R}(A)$ is a subspace of R^m. It is spanned by a_i's:
    $\mathcal{R}(A)=\{Ax: x \in R^n\}=\{a_1x_1+a_2x_2+\dots+a_nx_n: \ x \in R^n\}$
- If rank(A)=n, then $\{a_1, a_2, \dots a_n\}$ are LI, and they form the basis for the range space.
- If $\text{rank}(A)=n_1<n$, $\{a_1, a_2, \dots a_n\}$ are LD. We can choose n_1 columns of A that are LI. They form the basis of the range.

Meaning of the rank:

rank(A) = the maximal size of all square sub-matrices having none-zero determinant
rank(A) = the maximal number of vectors in a LI set which is formed by the columns of A.
rank(A) = the dimension of the range space.

Example. Find the rank and basis for the range space for the following

$$A_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \qquad \text{rank}(A_1) = 2,\ \mathcal{R}(A_1) = ?$$

Since m=2, $\mathcal{R}(A_1) \subseteq R^2$,
Since rank(A_1)=2, the dimension of $\mathcal{R}(A_1)$ is 2.
$\Rightarrow \mathcal{R}(A_1) = R^2$.
$\Rightarrow$ Basis for the range space:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

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Example. Find the rank and basis for the range space, given

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}, \quad m = 3,\ n = 2$$

$\rho(A_2) = 2 = n$, full column rank, $\{a_1, a_2\}$ are LI

Basis for the range space is $\{a_1, a_2\}$.

$\mathcal{R}(A_2) = \{a_1 x_1 + a_2 x_2 :\ x_1, x_2 \in R\}, \ \Leftarrow$ spanned by a_1 and a_2

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Example: $A_3 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} = [a_1 \quad a_2 \quad a_3], \quad m = n = 3$

$\rho(A_3) = 2,$

$\{a_1, a_2\}$ are LI; $\{a_1, a_3\}$ are LI; $\{a_2, a_3\}$ are LI.

- Any pair of the columns can be used as a basis for the range space.

$$\mathcal{R}(A_3) = \{a_1x_1 + a_2x_2 : \ x_i \in R\} = \{a_2x_1 + a_3x_2 : \ x_i \in R\}$$
$$= \{a_1x_1 + a_3x_2 : \ x_i \in R\}$$

What about $A_4 = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & 4 \\ 1 & 2 & 3 \end{bmatrix}$?

$\quad a_1 \quad a_2 \quad a_3$

- More facts about the rank of a matrix ,

  Rank (A) = Number of LI columns

  = Number of LI rows $\leq$ min (n, m)

  – A is full rank if $\rho(A)$ = min (n, m)

  – $\rho(A)=\rho(A')=\rho(A^*)$

  – A square matrix (n×n) has full rank ($\rho(A) = n$) iff $|A| \neq 0$, or equivalently A^{-1} exists

- Question: Under what condition does y = Ax have a solution for a specific y? for every y in R^m?

Theorem.

$y = Ax$ has a solution iff $y \in \mathcal{R}(A)$, or $\rho(A) = \rho([A : y])$

$y = Ax$ has a solution $\forall\ y \in R^m$ iff $\mathcal{R}(A) = R^m$ $(\rho(A) = m)$

Question: Under what condition will the solution be not unique?

- We need to use null space to describe this.

Def. The null space of A, N(A), is defined as

$N(A) \equiv \{x|\ x \in R^n \text{ s.t. } Ax = 0\}$

– How can we see that N(A) is a subspace?

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Theorem. N(A) is a subspace of R^n

Proof: Need to show that *if* $x_1, x_2 \in N(A)$, *then*

$\alpha_1 x_1 + \alpha_2 x_2 \in N(A)$ *for all* $\alpha_1, \alpha_2 \in R$

$Ax_1 = 0, Ax_2 = 0 \Rightarrow A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 Ax_1 + \alpha_2 Ax_2 = 0$

– Note that $N(A) \subseteq R^n$ (domain) and

$\mathcal{R}(A) \subseteq R^m$ (Codomain)

– The dimension of N(A) is called the nullity, notation $\nu(A)$

Theorem. $\rho(A) + \nu(A) = n$

Proof. Will not be covered

Corollary. The number of linearly independent solutions of $Ax = 0$ is $\nu(A)$ $(= n - \rho(A))$

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$\rho(A) + \nu(A) = n$

Example. Find the nullity and null space for the following

$$A_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = [a_1 \ \ a_2 \ \ a_3], \quad N(A_1) = \{x | x \in R^3, s.t. A_1 x = 0\}$$

$$\text{rank}(A_1) = 2, \quad \nu(A_1) = n - \text{rank}(A_1) = 3 - 2 = 1$$

Note that a_3 can be expressed as a linear combination of a_1 and a_2

$$a_3 = 2a_1 + 3a_2 \Rightarrow 2a_1 + 3a_2 - a_3 = 0$$

$$[a_1 \ \ a_2 \ \ a_3]\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 0 \Rightarrow A\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 0$$

$h = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is a basis for the null space. $\Rightarrow N(A_1) = \{k\,h : \ k \in R\}$

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$$\text{Example:} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} = [a_1 \ \ a_2],$$

a_1 and a_2 are LI, rank(A)=2, n=2, $\nu(A) = \nu - \text{rank}(A) = 0$

Dimension of the null space is 0

The only x such that Ax=0 is x=0.

$$\text{Example:} \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} = [a_1 \ \ a_2 \ \ a_3], \quad n = m = 3$$

rank(A)=2, ν(A)=3-2=1

$\{a_1, a_2\}$ are LI, $a_3 = a_1/3 + a_2/2 \Rightarrow [a_1 \ \ a_2 \ \ a_3]\begin{bmatrix} 1/3 \\ 1/2 \\ -1 \end{bmatrix} = 0$

Let $h = \begin{bmatrix} 1/3 \\ 1/2 \\ -1 \end{bmatrix} \Rightarrow A\,h = 0 \Rightarrow N(A) = \{k\,h : \ k \in R\}$

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Example:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix} = [a_1 \quad a_2 \quad a_3], \quad n = m = 3,$$

rank(A)=2, ν(A)=3-2=1

$\{a_1, a_2\}$ are LI, $a_3 = 0\, a_1 + 0\, a_2$ ➡ $[a_1 \quad a_2 \quad a_3] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$

Let $h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ➡ $N(A_1) = \{k\, h : \quad k \in R\}$

Practice: find the null space and range space for

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$
$$a_1 \quad a_2 \quad a_3 \quad a_4$$

- What is the implication of $\nu(A) > 0$ for equ. $Ax = y$?
  - Suppose that x_s is a solution to $Ax = y$ ($Ax_s = y$), and $x_0\ (\neq 0) \in N(A)$. What can be said about $x_s + \alpha x_0$?

Theorem. $x_s + \alpha x_0$ is also a solution to $Ax = y$

Proof:

$Ax_s = y, \; Ax_0 = 0$

$A(x_s + \alpha x_0) = Ax_s + \alpha Ax_0 = y + 0 = y$

➢ If $\nu(A) > 0$, then $Ax = y$ has infinite number of solutions if it has one.

Parameterization of all solutions

Theorem: Given m×n matrix A and a m×1 vector y.

- Let x_p be a solution to Ax = y.
- Let $\nu(A)=k$.
- Suppose k>0 and the null space is spanned by $\{n_1,n_2,\ldots n_k\}$

➢ The set of all solutions is given by
$\{x = x_p+\alpha_1 n_1+\alpha_2 n_2+\ldots+\alpha_k n_k:\ \alpha_i \in R\}$

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Summary:

- If $\rho(A) \neq \rho([A : y])$ (i.e., $y \notin \mathcal{R}(A)$), then the equations are inconsistent, and there is no solution
- If $\rho(A) = \rho([A : y])$, then ∃ at least one solution
  - If $\rho(A) = \rho([A : y]) < n$ (i.e., $\nu(A) > 0$), then there are infinite number of solutions
  - If $\rho(A) = \rho([A : y]) = n$ (i.e., $\nu(A) = 0$), then there is a unique solution
- For an n×n matrix, Ax = y has a unique solution $\forall\ y \in R^m$ iff A^{-1} exists, or $|A| \neq 0$

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Example:

$$\begin{matrix} & 0 & -4 & 0 & 0 \end{matrix}$$

$$Ax = y;\quad A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \end{bmatrix},\quad y = \begin{bmatrix} -4 \\ -8 \\ 0 \end{bmatrix}$$

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix}$$

- n=4; $\rho(A)=2$; $\Rightarrow$ $\nu(A)=4-2=2$. Two LI solutions for Ax=0.
- Observe that $y=-4a_2=0a_1-4a_2+0a_3+0a_4=[a_1\ a_2\ a_3\ a_4][0\ -4\ 0\ 0]'$
- ➢ A particular solution: $x_p=[0\ \ -4\ \ 0\ 0]'$
- Note that $a_3=a_1+a_2 \Leftrightarrow a_1+a_2-a_3+0a_4=0 \Leftrightarrow A[\ 1\ 1\ -1\ 0]'=0$
  $a_4=2a_2 \Leftrightarrow 0a_1+2a_2+0a_3-a_4=0 \Leftrightarrow A[0\ 2\ 0\ -1]'=0$
- ➢ Two solutions for Ax=0:

$$n_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},\quad n_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

- All solutions:

$$x = x_p + k_1 n_1 + k_2 n_2 = \begin{bmatrix} k_1 \\ -4 + k_1 + 2k_2 \\ -k_1 \\ -k_2 \end{bmatrix}, k_1, k_2 \in R$$

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Exercise:

$$Ax = y;\quad A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & -1 & 1 \end{bmatrix},\quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix}$$

Find null space,
Solution,
All solutions

Observe that a_1 and a_2 are LI, $a_3=a_2-a_1$, $a_4=a_1+a_2$,
$\rho(A)=2$, $\nu(A)=4-2=2$, $\rho[A\ y]=3 > \rho(A)$. $\Rightarrow$ No solution
From $a_3=a_2-a_1$, $\Rightarrow a_1-a_2+a_3+0a_4=0$, $[a_1\ a_2\ a_3\ a_4][1\ -1\ 1\ 0]'=0$
From $a_4=a_1+a_2$, $\Rightarrow a_1+a_2+0a_3-a_4=0$, $[a_1\ a_2\ a_2\ a_3][1\ 1\ 0\ -1]'=0$

➢ Two solutions for Ax=0

$$n_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix},\quad n_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

What if y=[1 3 2]'? Then $y=a_1+a_4=[a_1\ a_2\ a_3\ a_4][1\ 0\ 0\ 1]'$,
Hence a particular solution is $x_p=[1\ 0\ 0\ 1]'$

All solutions:

$$x = x_p + k_1 n_1 + k_2 n_2 = \begin{bmatrix} 1 + k_1 + k_2 \\ -k_1 + k_2 \\ k_1 \\ 1 - k_2 \end{bmatrix},\quad k_1, k_2 \in R$$

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Solution for xA=y:

- A: m×n; y: 1×n row vector;
  x: 1×m unknown row vector.
- Notice $xA=y \Leftrightarrow A^T x^T = y^T$
  A^T: n×m; y^T: n×1 column vector;
  x^T: m×1 unknown column vector.
- Transformed into the former systems of equations.

Today: Linear algebra (continued)

- Linear algebraic equations, solutions
- Parameterization of all solutions
- ➢ Similarity transformation: companion form
- Eigenvalues and eigenvectors, diagonal form
- Generalized eigenvectors, Jordan form

Similarity transformation: Companion form

Review: Let A be a n×n matrix and x the representation of a vector w.r.t the basis $\{e_1, e_2, \ldots, e_n\}$, where

$$e_i = [0 \cdots 1 \; 0 \cdots 0]^T \quad (i^{th} \text{ element}) \Longrightarrow [e_1 \; e_2 \ldots e_n] = I$$

- The linear operator L w.r.t the basis is: $x \to Ax$
- Let the new basis be $[\hat{e}_1, \hat{e}_2 \; \ldots \hat{e}_n] = [e_1 \; e_2 \; \ldots e_n]Q = Q$

➢ Then the operator w.r.t the new basis is: $z \to \bar{A} z$

$$\boxed{\bar{A} = Q^{-1}AQ} \Longleftarrow \text{Similarity transformation}$$

Question: How to choose Q so that $\bar{A}$ has a desired form? Which forms are desired?

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The companion form

$$\bar{A}_1 = \begin{bmatrix} 0 & 0 & -a \\ 1 & 0 & -b \\ 0 & 1 & -c \end{bmatrix} \qquad \bar{A}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -\beta_1 \\ 1 & 0 & \cdots & 0 & 0 & -\beta_2 \\ 0 & 1 & \cdots & 0 & 0 & -\beta_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -\beta_{n-1} \\ 0 & 0 & \cdots & 0 & 1 & -\beta_n \end{bmatrix}$$

What are $\det\bar{A}_1, \det\bar{A}_2$?

$$\det\bar{A}_1 = -a, \; \det\bar{A}_2 = (-1)^n \beta_1,$$

$$\det(\lambda I - \bar{A}_1) = \begin{vmatrix} \lambda & 0 & a \\ -1 & \lambda & b \\ 0 & -1 & \lambda + c \end{vmatrix} = \lambda^3 + c\lambda^2 + b\lambda + a$$

$$\det(\lambda I - \bar{A}_2) = \lambda^n + \beta_n \lambda^{n-1} + \beta_{n-1}\lambda^{n-2} + \cdots + \beta_1$$

- Clean structures, easy for analysis.
- How to get them?

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Transformation to companion form

Example: $A = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$ Let $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

We have: $Ab = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, A^2b = \begin{bmatrix} -4 \\ 2 \\ -3 \end{bmatrix}, A^3b = \begin{bmatrix} -5 \\ 10 \\ -13 \end{bmatrix},$

It can be verified that $A^3b = 17b - 15Ab + 5A^2b$

Also, b, Ab, A^2b are linearly independent. We can choose

$$[\hat{e}_1 \;\; \hat{e}_2 \;\; \hat{e}_3] = [b \;\; Ab \;\; A^2b] =: Q$$

The new rep. for the linear operator is $\bar{A} = Q^{-1}AQ \;\; \Rightarrow \;\; AQ = Q\bar{A}$

Observe that:

$$Ab = [b \;\; Ab \;\; A^2b]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A^2b = [b \;\; Ab \;\; A^2b]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, A^3b = [b \;\; Ab \;\; A^2b]\begin{bmatrix} 17 \\ -15 \\ 5 \end{bmatrix},$$

$$\underbrace{[Ab \; A^2b \; A^3b]}_{AQ} = \underbrace{[b \; Ab \; A^2b]}_{Q}\underbrace{\begin{bmatrix} 0 & 0 & 17 \\ 1 & 0 & -15 \\ 0 & 1 & 5 \end{bmatrix}}_{\bar{A}} \;\; \Rightarrow \;\; \bar{A} = \begin{bmatrix} 0 & 0 & 17 \\ 1 & 0 & -15 \\ 0 & 1 & 5 \end{bmatrix}$$

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In general: consider n×n matrix A. Choose b such that

$b, \; Ab, \; A^2b, \cdots A^{n-1}b$ are linearly independent

Then $A^nb = -\beta_1 b - \beta_2 Ab - \cdots - \beta_n A^{n-1}b$

If we choose $Q = [b \; Ab \; A^2b \cdots A^{n-1}b]$,

$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -\beta_1 \\ 1 & 0 & \cdots & 0 & 0 & -\beta_2 \\ 0 & 1 & \cdots & 0 & 0 & -\beta_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -\beta_{n-1} \\ 0 & 0 & \cdots & 0 & 1 & -\beta_n \end{bmatrix}$$

- A and $\bar{A}$ are said similar to each other
- The transformation $A \rightarrow Q^{-1}AQ$ similar transformation

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The dual case: consider n×n matrix A. Choose c such that

$$Q := \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} \text{ is nonsingular, i.e., } \{c', A'c', \ldots A'^{n-1} c'\} \quad \text{LI}$$

Then $cA^n = -\beta_1 c - \beta_2 cA - \cdots - \beta_n cA^{n-1}$

$$\bar{A} = QAQ^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\beta_1 & -\beta_2 & -\beta_3 & \cdots & -\beta_{n-1} & -\beta_n \end{bmatrix}$$

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- We next discuss how to transform A into a diagonal matrix. i.e., find a matrix Q such that

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

❖ Why this form?

- Stability of the system $\dot{x} = Ax$ is reflected by these diagonal elements λ_i's
- $|A| = \lambda_1 \lambda_2 \ldots \lambda_n$
- $|sI - A| = (s - \lambda_1)(s - \lambda_2) \ldots (s - \lambda_n)$

➢ These λ_i's are called eigenvalues of A

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Today: Linear algebra (continued)

- Linear algebraic equations, solutions
- Parameterization of all solutions
- Similarity transformation: companion form,
- ➢ Eigenvalues and eigenvectors, diagonal form
- Generalized eigenvectors, Jordan form

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Eigenvalues and Eigenvectors

Definition. Let A be a linear operator from C^n to C^n.

- A scalar λ is called an eigenvalue of A if $\exists$ a nonzero $x \in C^n$, such that $Ax = \lambda x \Leftrightarrow (\lambda I - A)x = 0$.
- $(\lambda I - A)x = 0$ has a non-zero sol. iff $\Delta(\lambda) = |\lambda I - A| = 0$
  ~ Characteristic polynomial of A with degree n
- λ must be a root of $\Delta(\lambda)$.
- A has n eigenvalues, not necessarily distinct, and some of them could be complex ~ So we consider C^n instead of R^n.
- x is the eigenvector associated with λ. What can be said?
- $(\lambda I - A)x = 0 \Leftrightarrow x \in N(\lambda I - A)$
- The set of eigenvalues of A, or, the set of the roots of $\Delta(\lambda)$, is called the spectrum and denoted eig(A)

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Example

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix},\ \text{find } \lambda_1, \lambda_2, x_1, \text{ and } x_2$$

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 3 & \lambda+4 \end{vmatrix} = \lambda^2 + 4\lambda + 3$$

$$= (\lambda+1)(\lambda+3) \quad \Rightarrow \lambda_1 = -1, \lambda_2 = -3$$

$$(\lambda_1 I - A)x_1 = 0 \qquad \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} x_1 = 0, \qquad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(\lambda_2 I - A)x_2 = 0, \qquad \begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} x_2 = 0, \qquad x_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

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- We will see later that
  - Eigenvalues are associated with system stability
  - Eigenvectors form a convenient set of basis
  - Will examine two cases of eigenvalues and eigenvectors
    - Case 1: All eigenvalues are distinct
    - Case 2: Eigenvalues with multiplicity > 1

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Case 1: All Eigenvalues are Distinct

- Consider first the case where all the eigenvalues of A are distinct, i.e., $\Delta(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$,

  $\lambda_i \neq \lambda_j$ for $i \neq j$.

  Let v_i be the associated eigenvector for λ_i

  – What can we say about $\{v_1, v_2, .., v_n\}$?

Theorem. $\{v_1, v_2, .., v_n\}$ are linearly independent

How to proof this theorem?

Proof. By contradiction

Suppose that they are linearly dependent, then assume without loss of generality that

$\Sigma_i \alpha_i v_i = 0$. At least one α_i nonzero. Assume $\alpha_1 \neq 0$

$(A - \lambda_2 I)(\Sigma_i \alpha_i v_i) = 0$

$= \Sigma_i \alpha_i (A - \lambda_2 I) v_i = \Sigma_i \alpha_i (A v_i - \lambda_2 v_i) = \Sigma_i \alpha_i (\lambda_i v_i - \lambda_2 v_i)$

$= \Sigma_i \alpha_i (\lambda_i - \lambda_2) v_i$

$= \Sigma_{i \neq 2} \alpha_i (\lambda_i - \lambda_2) v_i$ ~ The second term drops out

$(A - \lambda_3 I)[\Sigma_{i \neq 2} \alpha_i (\lambda_i - \lambda_2) v_i] = 0$

$= \Sigma_{i \neq 2} \alpha_i (\lambda_i - \lambda_2)(A - \lambda_3 I) v_i$

$= \Sigma_{i \neq 2} \alpha_i (\lambda_i - \lambda_2)(\lambda_i - \lambda_3) v_i$

$= \Sigma_{i \neq 2,3} \alpha_i (\lambda_i - \lambda_2)(\lambda_i - \lambda_3) v_i$

~ The third term drops out

Finally, $\alpha_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) .. (\lambda_1 - \lambda_n) v_1 = 0$

Since $\lambda_i \neq \lambda_j$ for $i \neq j$, the above implies $v_1 = 0$

~ Contradiction $\Rightarrow \{v_1, v_2, .., v_n\}$ are LI

- Let $Q=[v_1 \ \ v_2 \ \ \dots \ v_n]$, choose the new basis as

$$[\hat{e}_1, \hat{e}_2 \ \dots \hat{e}_n] = [e_1 \ e_2 \ \dots e_n]Q = Q$$

- What is the new rep. of A in terms of the new basis?
  What is $\bar{A}=Q^{-1}AQ$, or an $\bar{A}$ such that $AQ = Q\bar{A}$?

Notice that $Av_i=\lambda_i v_i$ for all i

$$AQ = A[v_1 \ \ v_2 \ \ \cdots \ \ v_n] = [Av_1 \ \ Av_2 \ \ \cdots \ \ Av_n] = [\lambda_1 v_1 \ \ \lambda_2 v_2 \ \ \cdots \ \ \lambda_n v_n]$$

$$AQ = [v_1 \ \ v_2 \ \ \cdots \ \ v_n]\begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & .. & \lambda_n \end{bmatrix} = Q\begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & .. & \lambda_n \end{bmatrix}$$

$\bar{A}$

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Example (Continued)

$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$, find $\bar{A}$

$\Rightarrow \lambda_1 = -1, \lambda_2 = -3, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

– First by inspection: $\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$

– Then by similar transformation:

$$Q = [v_1 \ \ v_2] = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}, \quad Q^{-1} = \frac{-1}{2}\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\bar{A} = Q^{-1}AQ = \frac{-1}{2}\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

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Another way to understand the Example

$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$, Find a diagonal matrix $\bar{A}$ and a nonsingular matrix Q such that $\bar{A} = Q^{-1}AQ$

$\lambda_1 = -3; \lambda_2 = -1, \quad v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Let $Q = [v_1 \;\; v_2] = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}$ Must have $\bar{A} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$

To verify, check $AQ \overset{?}{=} Q\bar{A}$

$$AQ = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 9 & 1 \end{bmatrix}$$

$$Q\bar{A} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 9 & 1 \end{bmatrix}$$

Indeed, $AQ = Q\bar{A}, \Rightarrow Q^{-1}AQ = \bar{A}$

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Similarity transformation for a LTI system:

$$\dot{x} = Ax + Bu \qquad (*)$$
$$y = Cx + Du$$

Let the new state be $z = Q^{-1}x$. Then $x = Qz$ and

$$\dot{z} = Q^{-1}\dot{x} = Q^{-1}(Ax + Bu) = \underbrace{Q^{-1}AQ}_{\bar{A}}z + \underbrace{Q^{-1}B}_{\bar{B}}u$$
$$y = Cx + Du = \underbrace{CQ}_{\bar{C}}z + \underbrace{D}_{\bar{D}}u$$

$$\dot{z} = \bar{A}z + \bar{B}u; \qquad (**)$$
$$y = \bar{C}z + \bar{D}u$$

If we pick $Q=[v_1 \;\; v_2 \; \dots \; v_n]$, then $\bar{A}$ has a diagonal form, making the analysis easy.

The similar transformation does not change the input-output relationship

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Example. $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

Find the matrix Q:

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 2$$

$$= (\lambda+1)(\lambda+2) \quad \Rightarrow \lambda_1 = -1, \lambda_2 = -2$$

$$(\lambda_1 I - A)v_1 = 0, \quad \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} v_1 = 0, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(\lambda_2 I - A)v_2 = 0, \quad \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} v_2 = 0, \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$Q = [v_1 \quad v_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \qquad Q^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

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– What is the system dynamics in terms of $z=Q^{-1}x$?

$$z \equiv Q^{-1}x, \qquad x = Qz$$

$$\dot{z} = Q^{-1}\dot{x} = Q^{-1}(Ax + Bu)$$

$$= Q^{-1}AQz + Q^{-1}Bu = \bar{A}z + \bar{B}u$$

$$\boxed{\bar{A} \equiv Q^{-1}AQ} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \boxed{\bar{B} \equiv Q^{-1}B} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$y = Cx = CQz = \bar{C}z, \quad \boxed{\bar{C} = CQ} = [1 \quad 0]\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = [1 \quad 1]$$

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} 5 \\ -3 \end{bmatrix} u, \quad y = [1 \quad 1] z$$

– Two decoupled modes and can be easily analyzed

– The system is stable since $Re(\lambda_i) < 0 \; \forall \; i$

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Theorem. All similar matrices have the same eigenvalues

– How to prove this?

$$|\lambda I - \overline{A}| = |\lambda I - Q^{-1}AQ|$$

$= |\lambda Q^{-1}Q - Q^{-1}AQ|$

$= |Q^{-1}(\lambda I - A)Q|$

$= |Q^{-1}|\cdot|(\lambda I - A)|\cdot|Q|$

$= |\lambda I - A|$

– The two matrices have the same characteristic polynomial, and therefore have the same set of eigenvalues

Today: Linear algebra (continued)

- Linear algebraic equations, solutions
- Parameterization of all solutions
- Similarity transformation: companion form,
- Eigenvalues and eigenvectors, diagonal form
- ➢ Generalized eigenvectors, Jordan form

Case 2: Eigenvalues with Multiplicity > 1

- What may happen when the multiplicity of an eigenvalue is greater than 1?
  - The matrix may not be diagonalizable

Example.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 1 & 0 & \lambda \end{vmatrix} = \lambda(\lambda-1)^2$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = \lambda_3 = 1$$

$$(\lambda_1 I - A)v_1 = 0, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} v_1 = 0, \quad v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

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$$(\lambda_2 I - A)v_2 = 0, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} v_2 = 0, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- What is v_3? Recall $\lambda_2 = \lambda_3$.
- $v_3 = v_2$
- $\{v_1, v_2, v_3\}$ are not LI, and cannot be used as a basis
- Q formed by them is not invertible, and there is no similar transformation to diagonalize A. ☹
- Have to think something different for v_2 and v_3
- Let us find v_3 such that

$$(A - \lambda_2 I)^2 v_3 = 0, \quad (A - \lambda_2 I)v_3 \neq 0$$ ~ Different from the previous v_2

- Then $\{v_1, v_2, v_3\}$ are LI.
- If we take $Q = [v_1\ v_2\ v_3]$, what is $\bar{A} = Q^{-1}AQ$?

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– We need to find $\bar{A}$ such that $AQ=Q\bar{A}$. Observe that

$$Av_1 = \lambda_1 v_1 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix}$$

$$Av_2 = \lambda_2 v_2 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \end{bmatrix}$$

From $(A-\lambda_2 I)v_2 = (A-\lambda_2 I)^2 v_3 = 0$

We have $(A-\lambda_2 I)v_3 = v_2$

$$Av_3 = v_2 + \lambda_2 v_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \lambda_2 \end{bmatrix}$$

$$A\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Q

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Not diagonal, but close

Today: Linear algebra (continued)

- Linear algebraic equations, solutions
- Parameterization of all solutions
- Similarity transformation: companion form,
- Eigenvalues and eigenvectors, diagonal form

Next Time: More on linear algebra §3.5,3.6,3.8
State space solutions §4.1,4.2

- Generalized eigenvectors, Jordan form
- Some useful results, matrix norms
- Functions of a square matrix

Homework Set #5:

1. Find

1) nullities,
2) bases for the range spaces and
3) bases for the null spaces

for the following matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 1 & -3 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

2. Find the general solutions for the following equations

$$a).\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} x = 1, \quad b).\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

$$c).\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad d).\begin{bmatrix} 1 & 3 & 0 & -1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note: these matrices are the same as those in Problem 1.

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3. Compute the eigenvalues, eigenvectors and diagonal forms for these matrices

$$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 3 & 1 \\ -2 & 0 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 6 & -2 & -3 \\ 4 & 0 & -3 \\ 8 & -2 & -5 \end{bmatrix}$$

4. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 4 \\ 1 & 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Form $Q=[A^2b \;\; Ab \;\; b]$. Compute $M=Q^{-1} A Q$.
Observe how M is related to the polynomial det(λI-A).

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