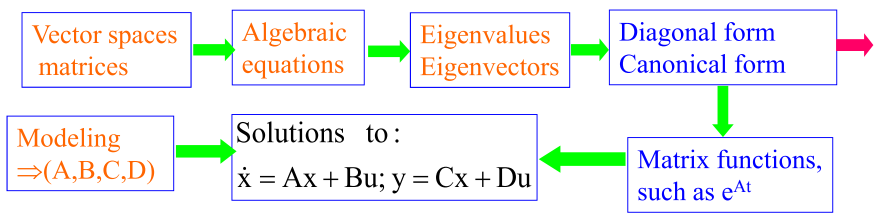
16.513 Control Systems (Lecture note #6)

- Last Time: Linear algebra review
  - Linear algebraic equations, solutions
  - Parameterization of all solutions
  - Similarity transformation: companion form
  - Eigenvalues and eigenvectors, diagonal form

A big picture: one branch of the course



There are more branches, mainly derived from linear algebra. 1

Review: A system of equations: $Ax = y$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$m \times n$ $\qquad$ $n \times 1$ $\qquad$ $m \times 1$

Let the ith column of A be a_i, i.e., $A=[a_1\ a_2\ \dots\ a_n]$, then

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

The existence of solution depends on the relationship between $\rho(A)$ and $\rho([A\ y])$

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Summary:

- If $\rho(A) \neq \rho([A : y])$ (i.e., $y \notin \mathcal{R}(A)$), then the equations are inconsistent, and there is no solution
- If $\rho(A) = \rho([A : y])$, then $\exists$ at least one solution
  - If $\rho(A) = \rho([A : y]) < n$ (i.e., $\nu(A) > 0$), then there are infinite number of solutions
  - If $\rho(A) = \rho([A : y]) = n$ (i.e., $\nu(A) = 0$), then there is a unique solution
- For an $n \times n$ matrix, $Ax = y$ has a unique solution $\forall\, y \in R^m$ iff A^{-1} exists, or $|A| \neq 0$

The ranks of matrices play an important role.

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Eigenvalues, eigenvectors and diagonal form

A scalar λ is called an eigenvalue of $A \in C^{n\times n}$ if $\exists$ a nonzero $x \in C^n$, such that $Ax = \lambda x$ and x is the eigenvector associated with λ.

Case 1: All eigenvalues are distinct

Theorem: the eigenvectors $\{v_1, v_2, \ldots, v_n\}$ are LI.
Let $Q=[v_1\ v_2\ \ldots\ v_n]$, then

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & .. & \lambda_n \end{bmatrix}$$

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On the other hand, if there exist a nonsingular Q and a diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & .. & \lambda_n \end{bmatrix}$$

such that $Q^{-1} AQ = \Lambda$, then λ_i's are the eigenvalues of A and the columns of Q are the eigenvectors:

$$Q^{-1} AQ = \Lambda \Rightarrow AQ=Q\Lambda \Rightarrow Av_i=\lambda_i v_i$$

- However, there are situations where there exists no such Q to make $Q^{-1}AQ$ a diagonal matrix.
- This case will be covered today.

Today: We are going to study

- Generalized eigenvectors, Jordan form
- Polynomial functions of a square matrix
- More general functions such as e^{At}, $(sI-A)^{-1}$

Tools for solving a state-space equation

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

Given x(0) and u(t), for $t \geq 0$, what is x(t) and y(t)?

- Next time, we will be able to do this.

Case 2: Eigenvalues with Multiplicity > 1

- What may happen when the multiplicity of an eigenvalue is greater than 1?
  - The matrix may not be diagonalizable

Example.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 1 & 0 & \lambda \end{vmatrix} = \lambda(\lambda - 1)^2$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = \lambda_3 = 1$$

$$(\lambda_1 I - A)v_1 = 0, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} v_1 = 0, \quad v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

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$$(\lambda_2 I - A)v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} v_2 = 0, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- What is v_3? Recall $\lambda_2 = \lambda_3 = 1$.
- We expect to have $\{v_1, v_2, v_3\}$ LI $\Rightarrow$ $\{v_2, v_3\}$ LI
- However, from $\rho(\lambda_2 I\text{-}A)=2 \Rightarrow \nu(\lambda_2 I\text{-}A)=3\text{-}2=1$.
- What does this mean?
- The null space of $\lambda_2 I\text{-}A$ has dimension 1;
  There doesn't exist LI $\{v_2, v_3\}$ s.t.
  $(\lambda_2 I\text{-}A)v_2 = (\lambda_2 I\text{-}A)v_3 = 0$
- If we take $v_3 = kv_2$, $\{v_1, v_2, v_3\}$ are not LI, and cannot be used as a basis

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– Have to think something different for v_2 and v_3
– We still choose v_2 as the solution to $(A-\lambda_2 I)v_2=0$
– For v_3, suppose that it satisfies

$$(A-\lambda_2 I)^2 v_3 = 0, \quad (A-\lambda_2 I)v_3 \neq 0$$

~ Different from the previous v_2

– Then $(A-\lambda_2 I)(A-\lambda_2 I)v_3=0 \Rightarrow (A-\lambda_2 I)v_3=kv_2$ for some k
– And we can just choose $v_2 = (A-\lambda_2 I)v_3$
– Then $\{v_1, v_2, v_3\}$ are LI (we just accept this).
– If we take $Q=[v_1\ v_2\ v_3]$, then $\bar{A}=Q^{-1}AQ$ can't be diagonal. But what does it look like?

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– We need to find $\bar{A}$ such that $AQ=Q\bar{A}$. Observe that

$$Av_1 = \lambda_1 v_1 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix}$$

$$Av_2 = \lambda_2 v_2 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \end{bmatrix}$$

From $(A-\lambda_2 I)v_2 = (A-\lambda_2 I)^2 v_3 = 0$

We have $(A-\lambda_2 I)v_3 = v_2$

$$Av_3 = v_2 + \lambda_2 v_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \lambda_2 \end{bmatrix}$$

$$A[v_1\ v_2\ v_3] = [v_1\ v_2\ v_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Q

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Not diagonal, but close

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- For this particular example, how to get v_3 such that

$$(A-\lambda_2 I)^2 v_3 = 0,\ (A-\lambda_2 I)v_3 \neq 0 \qquad (v_3 \neq v_2)$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(A-\lambda_2 I)^2 v_3 = 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} v_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} v_3 \Rightarrow v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$$

$$v_2 = (A-\lambda_2 I)v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$Q = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \qquad Q^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\overline{A} = Q^{-1}AQ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} \sim \text{as expected}$$

- This example just show the complexity that may arise when we have repeated eigenvalues.
- To handle such a situation systematically, we need to define the generalized eigenvectors.

Definition. A vector v is a generalized eigenvector of grade k associated with λ if

$(A-\lambda I)^k v = 0,\quad$ but $(A-\lambda I)^{k-1} v \neq 0$

Denote $v_k \equiv v$,

$v_{k-1} \equiv (A-\lambda I)v = (A-\lambda I)v_k,\qquad Av_k = v_{k-1} + \lambda v_k$

$v_{k-2} \equiv (A-\lambda I)^2 v = (A-\lambda I)v_{k-1},\qquad Av_{k-1} = v_{k-2} + \lambda v_{k-1}$

$v_1 \equiv (A-\lambda I)^{k-1} v = (A-\lambda I)v_2,\qquad Av_2 = v_1 + \lambda v_2$

$(A-\lambda I)v_1 = (A-\lambda I)^k v = 0,\qquad Av_1 = \lambda v_1$

– What is the new representation w.r.t. $\{v_1, v_2, ., v_k\}$? i.e.,
$A[v_1\ v_2 \dots v_k] = [v_1\ v_2 \dots v_k]\bar{A}$

$$\bar{A} = \begin{bmatrix} \lambda & 1 & .. & 0 \\ 0 & \lambda & .. & 0 \\ : & : & & 1 \\ 0 & 0 & .. & \lambda \end{bmatrix}$$

A Jordan block

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Example: $A = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}\qquad \Delta(\lambda) = \begin{vmatrix} \lambda-6 & 4 \\ -9 & \lambda+6 \end{vmatrix} = \lambda^2,\ \lambda_1 = \lambda_2 = 0$

$A - \lambda_1 I = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix},\ \rho(A-\lambda_1 I) = 1, \nu(A-\lambda_1 I) = 2-1 = 1$

Only one LI v s.t. $Av = \lambda_1 v$ Have to use generalized eigenvectors.

First pick v_2 such that $(A-\lambda_1 I)^2 v_2 = 0$, but $(A-\lambda_1 I)v_2 \neq 0$

Need $(A-\lambda_1 I)^2 v_2 = 0$. $(A-\lambda_1 I)^2 = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}^2 = 0$

v_2 can be anything but $(A-\lambda_1 I)v_2 \neq 0$!

Pick $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $v_1 = (A-\lambda_1 I)v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$Q = [v_1\ \ v_2] = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix},\quad Q^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$ ✓

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An alternative approach: $A = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$ $\Delta(\lambda) = \begin{vmatrix} \lambda-6 & 4 \\ -9 & \lambda+6 \end{vmatrix} = \lambda^2,\ \lambda_1 = \lambda_2 = 0$

$$A - \lambda_1 I = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix},\ \rho(A - \lambda_1 I) = 1, \nu(A - \lambda_1 I) = 2 - 1 = 1$$

Need to find v_1, v_2 such that $(A-\lambda_1 I)v_1=0$ and $(A-\lambda_1 I)v_2=v_1$

You can also fine v_1 first, then solve $(A-\lambda_1)v_2=v_1$ to get v_2.

$$(A-\lambda_1 I)v_1 = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix} v_1 = 0,\quad v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

From $(A - \lambda_1 I)v_2 = v_1 \quad \Rightarrow \quad \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix} v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, v_2 not unique

$$v_2 = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 1/3 \\ 3 & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 2 & 0 \\ 3 & 1/2 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{for any of the above Q}$$

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Example: Find Jordan form for $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

It is easy to see that the eigenvalues of A_1 are

$$\lambda_1 = \lambda_2 = 1, \qquad \lambda_3 = 0$$

Does the matrix have generalized eigenvector for λ_1, λ_2?

Let us check the nullity of $A_1 - \lambda_1 I$

$$A_1 - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ rank(A_1 - \lambda_1 I) = 2,$$

$$N(A_1 - \lambda_1 I) = 3 - 2 = 1$$

Cannot find two LI eigenvectors for λ_1, λ_2

Must have generalized eigenvectors for λ_1, λ_2

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Need to find v_1, v_2, such that

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A_1 - \lambda_1 I)v_1 = 0, \qquad (A_1 - \lambda_1 I)v_2 = v_1$$

$$\lambda_1 = \lambda_2 = 1, \qquad \lambda_3 = 0$$

Approach 1: Find v_2 first, then let $v_1 = (A_1 - \lambda_1 I)v_2$

v_2 should satisfy $(A_1 - \lambda_1 I)^2 v_2 = 0,\ (A_1 - \lambda_1 I)v_2 \neq 0$

$$v_2 \in N\big((A_1 - \lambda_1 I)^2\big), \qquad v_2 \notin N(A_1 - \lambda_1 I)$$

$A_1 - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, Basis for the null space: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$(A_1 - \lambda_1 I)^2 = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, Basis for the null space: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Pick $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, Then $v_1 = (A_1 - \lambda_1 I)v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For v_3, $(A_1 - \lambda_3 I)v_3 = 0$, $\quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$Q = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

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$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 1, \qquad \lambda_3 = 0$$

$$Q = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

What is $\bar{A} = Q^{-1}AQ$?

Based on the property of generalized eigenvalue, must have

$$\bar{A}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Did I get everything right? Check if $A_1 Q = Q\bar{A}_1$?

$$A_1 Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q\bar{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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Need to find v_1, v_2, such that

$(A_1 - \lambda_1 I)v_1 = 0, (A_1 - \lambda_1 I)v_2 = v_1$

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \lambda_1 = \lambda_2 = 1, \lambda_3 = 0$$

Approach 2: Find v_1 first, then solve $(A_1 - \lambda_1 I)v_2 = v_1$ for v_2.

v_1 should satisfy $(A_1 - \lambda_1 I)v_1 = 0$

$$A_1 - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ Basis for the null space: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. v_2 needs to satisfy $(A_1 - \lambda_1 I)v_2 = v_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{or } v_2 = \begin{bmatrix} x \\ 1 \\ 1 \end{bmatrix} \text{ for any } x$$

$$Q = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & x & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_1 Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x+1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q\bar{A}_1 = \begin{bmatrix} 1 & x & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

✓

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Theorem. The generalized eigenvectors associated with a particular eigenvalue are LI

Theorem. The generalized eigenvectors associated with different eigenvalues are LI

- The eigenvectors and generalized eigenvectors span C^n
- A good basis ~ $\bar{A}$ is the Jordan Canonical Form

$$\begin{bmatrix} L_{11}(\lambda_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{12}(\lambda_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{1k}(\lambda_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{21}(\lambda_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{22}(\lambda_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{mp}(\lambda_m) \end{bmatrix}$$

$$L_{ik} = \begin{bmatrix} \lambda & 1 & .. & 0 \\ 0 & \lambda & .. & 0 \\ : & : & & 1 \\ 0 & 0 & .. & \lambda \end{bmatrix} \qquad \Delta(\lambda) = |\lambda I - A| = \prod_i (\lambda - \lambda_i)^{n_i}$$

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For the same eigenvalue λ_1, it may have more than one Jordan blocks such as

$$\left[\begin{array}{c|cc|ccc} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 \end{array}\right]$$

- Another case:
  - A matrix with repeated eigenvalues could still be diagonalizable

Example. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. $\Delta(\lambda) = \begin{bmatrix} \lambda-1 & 0 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda-2 \end{bmatrix} = (\lambda-1)^2(\lambda-2)$

$\lambda = 1, 1, 2.$

$$\lambda = 1: \quad (\lambda_i I - A)\, v_i = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} v_i = 0 \quad \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

~ 2 LI eigenvectors! ☺

$$\lambda = 2: \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0 \qquad \Rightarrow v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \overline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- A is diagonalizable even with repeated eigenvalues.

In summary, we have the following cases:

- All eigenvalues of A are distinct $\Rightarrow$ diagonalizable
- There are repeated eigenvalues, e.g., λ_i with multiplicity k.
  - If $\nu(A-\lambda_i I) = n - \rho(A-\lambda_i I) = k$, there exist k LI solutions to $(A-\lambda_i I)v=0$ and they are all eigenvectors.
    If this is the case for all repeated eigenvalues
    $\Rightarrow$ diagonalizable
  - If $\nu(A-\lambda_i I) = n - \rho(A-\lambda_i I) < k$, there exist generalized eigenvectors,
    $\Rightarrow$ not diagonalizable, there exist Jordan blocks

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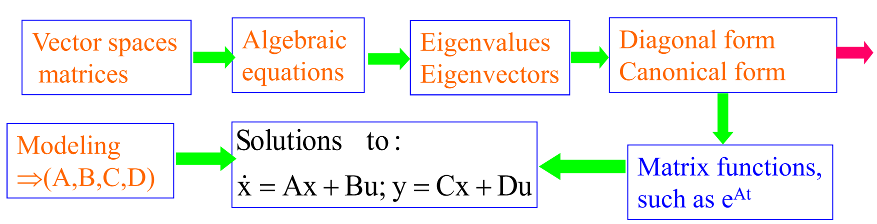
Today: Linear algebra (continued)

- Generalized eigenvectors, Jordan form
- ➢ Polynomial functions of a square matrix
- Exponential function of a square matrix



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Functions of a Square Matrix

Polynomials of a Square Matrix

Example. $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$

What is A^1? A^2? A^3? A^0? $A^1 = A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$,

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 12 & 13 \end{bmatrix}$$

$$A^3 = A \cdot A \cdot A = A \cdot A^2 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -3 & -4 \\ 12 & 13 \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ -39 & -40 \end{bmatrix}$$

$A^0 = I$

- In general, suppose $A: C^n \to C^n$
  - $A^1 = A$, $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$
  - $A^k = A \cdot A \cdots A$, k terms, $k \geq 1$
  - $A^0 = I$
- Let $f(\lambda)$ be a polynomial, e.g.,

  $f(\lambda) = 5\lambda^3 + 4\lambda^2 + 7\lambda - 2$

  What is f(A)?
- $f(A) = 5A^3 + 4A^2 + 7A - 2A^0$

$$= 5\begin{bmatrix} 12 & 13 \\ -39 & -40 \end{bmatrix} + 4\begin{bmatrix} -3 & -4 \\ 12 & 13 \end{bmatrix} + 7\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} - 2I$$

$$= \begin{bmatrix} 46 & 56 \\ -168 & -178 \end{bmatrix}$$

– Is there an easier way to compute f(A)?

– Would the process be easier for a diagonal or block diagonal matrix? How to proceed?

$$A = Q\bar{A}Q^{-1}, \quad A^2 = \left(Q\bar{A}Q^{-1}\right)\left(Q\bar{A}Q^{-1}\right) = Q\bar{A}^2Q^{-1}$$

$$A^3 = \left(Q\bar{A}Q^{-1}\right)^2\left(Q\bar{A}Q^{-1}\right) = \left(Q\bar{A}^2Q^{-1}\right)\left(Q\bar{A}Q^{-1}\right) = Q\bar{A}^3Q^{-1}$$

$$A^k = Q\bar{A}^kQ^{-1}$$

$$f(A) = 5A^3 + 4A^2 + 7A - 2A^0$$

$$= 5Q\bar{A}^3Q^{-1} + 4Q\bar{A}^2Q^{-1} + 7Q\bar{A}Q^{-1} - 2I$$

$$= Q\left(5\bar{A}^3 + 4\bar{A}^2 + 7\bar{A} - 2I\right)Q^{-1} = Q\,f\left(\bar{A}\right)Q^{-1}$$

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Example (Continued) $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$

$$\lambda_1 = -1, \lambda_2 = -3, \quad \bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$Q = (v_1 \quad v_1) = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}, \qquad Q^{-1} = \frac{-1}{2}\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$f(A) = Q\left(5\bar{A}^3 + 4\bar{A}^2 + 7\bar{A} - 2I\right)Q^{-1}$$

$$= Q\left\{5\begin{bmatrix} -1 & 0 \\ 0 & -27 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} + 7\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} - 2I\right\}Q^{-1}$$

$$= \frac{-1}{2}\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -10 & 0 \\ 0 & -122 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 46 & 56 \\ -168 & -178 \end{bmatrix}$$

~ as expected

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- In general,

$$f(A)=\sum_i \alpha_i A^i=\sum_i \alpha_i\left(Q\bar{A}Q^{-1}\right)^i=\sum_i \alpha_i Q\bar{A}^i Q^{-1}$$

$$=Q\left(\sum_i \alpha_i \bar{A}^i\right)Q^{-1}=\boxed{Qf(\bar{A})Q^{-1}}$$

– Advantages to use diagonal or Jordan canonical form?

$$\text{If } \bar{A}=\begin{bmatrix}\bar{A}_1 & 0\\ 0 & \bar{A}_2\end{bmatrix},\quad \bar{A}^2=\begin{bmatrix}\bar{A}_1^2 & 0\\ 0 & \bar{A}_2^2\end{bmatrix},\quad \bar{A}^k=\begin{bmatrix}\bar{A}_1^k & 0\\ 0 & \bar{A}_2^k\end{bmatrix}$$

$$f(\bar{A})=\sum_i \alpha_i \bar{A}^i=\sum_i \alpha_i \begin{bmatrix}\bar{A}_1^i & 0\\ 0 & \bar{A}_2^i\end{bmatrix}=\begin{bmatrix}f(\bar{A}_1) & 0\\ 0 & f(\bar{A}_2)\end{bmatrix}$$

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Cayley Hamilton Theorem

$$\Delta(\lambda)=|\lambda I-A|=\prod_{i=1}^{m}(\lambda-\lambda_i)^{n_i}=\lambda^n+\alpha_1\lambda^{n-1}+\cdots+\alpha_n$$

$$\Downarrow$$

$$\Delta(A)=\prod_{i=1}^{m}(A-\lambda_i I)^{n_i}=A^n+\alpha_1 A^{n-1}+\cdots+\alpha_n I$$

– There is something special about $\Delta(A)$.

– First consider a diagonalizable A.

$$\Delta(A)=Q\Delta(\bar{A})Q^{-1}=Q\begin{bmatrix}\Delta(\lambda_1) & 0 & 0 & 0\\ 0 & \Delta(\lambda_2) & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \Delta(\lambda_n)\end{bmatrix}Q^{-1}=0$$

This is true even if A has Jordan blocks.

Cayley-Hamilton Theorem: $\Delta(A)=0$

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We only need to consider a Jordan block. For example,

$$\bar{A}_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}, \qquad (\bar{A}_i - \lambda_i I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

$$(\bar{A}_i - \lambda_i I)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0 \qquad (\bar{A}_i - \lambda_i I)^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

For a $n_i \times n_i$ Jordan block $\bar{A}_i$, $(\bar{A}_i - \lambda_i I)^{n_i} = 0$

Note $\Delta(\bar{A}_i) = \prod_j (\bar{A}_i - \lambda_j I)^{n_j}$ contains the term $(\bar{A}_i - \lambda_i I)^{n_i}$

$$\Rightarrow \quad \Delta(A) = Q\Delta(\bar{A})Q^{-1} = Q\begin{bmatrix} \Delta(\bar{A}_1) & 0 & 0 & 0 \\ 0 & \Delta(\bar{A}_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Delta(\bar{A}_n) \end{bmatrix} Q^{-1} = 0$$

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In summary:

Let $\Delta(\lambda) = |\lambda I - A| = \prod_i (\lambda - \lambda_i)^{n_i}$, $\Delta(A) = \prod_i (A - \lambda_i I)^{n_i}$,

Have $\Delta(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \ldots + \alpha_n$,

$\Delta(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_n I$,

Conclusion: $\Delta(A)=0$

Implication:

$$A^n = -\alpha_1 A^{n-1} - \alpha_2 A^{n-2} - \ldots - \alpha_n I,$$

- A^n can be expressed as linear combination of $I, A, A^2, \ldots A^{n-1}$.
- Inductively, A^k can be expressed as linear combination of these terms for all integer k
- Furthermore, all polynomials of A can be expressed so.

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– Any polynomial of a square matrix can be expressed as a polynomial of the same matrix of degree n-1

– If there is a polynomial $\psi(\lambda)$ of degree m < n such that $\psi(A) = 0$, then any polynomial can be expressed as a polynomial of degree m-1

– The minimal polynomial $\psi(\lambda)$ of A is the monic polynomial (with highest power coefficient = 1) of least degree such that $\psi(A) = 0$

Example: Motivation for a general problem.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, f(\lambda) = \lambda^{85}. \text{ Find } f(A) = A^{85}$$

– How to solve this problem?

$$\Delta(\lambda) = \begin{bmatrix} \lambda - 2 & -3 \\ 0 & \lambda - 1 \end{bmatrix} = (\lambda - 2)(\lambda - 1)$$

– We should be able to represent f(A) as

$$A^{85} = \beta_0 I + \beta_1 A = g(A)$$

~ Much easier to compute

– What is β_0? β_1? How to obtain them?

– A general problem: Find g(A) that is equivalent to f(A) but simpler to evaluate

–Under what conditions would f(A) = g(A)?

Theorem. Given $A \in C^{n\times n}$ and a polynomial $f(\lambda)$. Let the distinct eigenvalues of A be λ_i, i=1,2,...,m, each with multiplicity n_i, $(n_1+n_2+\ldots+n_m= n)$. Let

$$g(\lambda) = \beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1}$$

Then f(A)=g(A) iff

$f^{(l)}(\lambda_i) = g^{(l)}(\lambda_i)$, $l = 0, 1, .., n_i - 1$, $i = 1, .., m$

where $f^{(l)}(\lambda_i) = \left.\frac{d^{(l)}f(\lambda)}{d\lambda^l}\right|_{\lambda=\lambda_i}$, $f^{(0)}(\lambda_i) = f(\lambda_i)$

Under the above condition, the coefficients β_i's can be determined

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Definition. $\{f^{(l)}(\lambda_i), l = 0, 1, .., n_i - 1, i = 1, .., m\}$ are called the values of f on the spectrum of A

– Any two polynomials having the same values on the spectrum of A define the same matrix function

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Example (continued)

$$f(\lambda)=\lambda^{85}.\ \text{Find } f(A)=A^{85} \text{ for } A=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}$$

$$\Delta(\lambda)=\begin{bmatrix}\lambda-2 & -3\\ 0 & \lambda-1\end{bmatrix}=(\lambda-2)(\lambda-1),\quad \lambda_1=2, \lambda_2=1$$

$$f^{(0)}(\lambda_1)=\lambda_1^{85}=2^{85},\qquad f^{(0)}(\lambda_2)=\lambda_2^{85}=1^{85}=1$$

– What is a good $g(\lambda) = \beta_0 + \beta_1\lambda$?

$g^{(0)}(\lambda_1) = \beta_0 + \beta_1\lambda_1 = \beta_0 + 2\beta_1$

$g^{(0)}(\lambda_2) = \beta_0 + \beta_1\lambda_2 = \beta_0 + \beta_1$

f and g having the same values on the spectrum of A requires

$g^{(0)}(\lambda_1) = f^{(0)}(\lambda_1) \Rightarrow \beta_0 + 2\beta_1 = 2^{85}$

$g^{(0)}(\lambda_2) = f^{(0)}(\lambda_2) \Rightarrow \beta_0 + \beta_1 = 1$

$\beta_1 = 2^{85} - 1,\ \beta_0 = 2 - 2^{85} \Rightarrow g(\lambda) = (2 - 2^{85}) + (2^{85} - 1)\lambda$

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$g(A) = (2 - 2^{85})\cdot I + (2^{85} - 1)A$

$$=\begin{bmatrix}2-2^{85} & 0\\ 0 & 2-2^{85}\end{bmatrix}+\begin{bmatrix}2(2^{85}-1) & 3(2^{85}-1)\\ 0 & (2^{85}-1)\end{bmatrix}$$

$$=\begin{bmatrix}2^{85} & 3(2^{85}-1)\\ 0 & 1\end{bmatrix}=\begin{bmatrix}3.86856\times10^{25} & 1.16057\times10^{26}\\ 0 & 1\end{bmatrix}$$

– One way to compute f(A):

- Form $\Delta(\lambda)$, and find $\{\lambda_i\}$ and $\{f^{(l)}(\lambda_i)\}$
- Construct an (n - 1)th order polynomial

  $g(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + .. + \beta_{n-1}\lambda^{n-1}$

  s.t. f and g have the same values on the spectrum of A:

  $g^{(l)}(\lambda_i) = f^{(l)}(\lambda_i)$, for all l, i
- f(A) = g(A)

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Example: Compute A^{100} for $A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$

In other words: Given $f(\lambda)=\lambda^{100}$, find $f(A)$.
First, $\Delta(\lambda)=(\lambda+1)^2$
A has one distinct eigenvalue $\lambda_1= -1$ with multiplicity 2.
Let $g(\lambda)=\beta_0+\beta_1\lambda$, on the spectrum of A, have

$$f(-1)=g(-1) \Rightarrow (-1)^{100}=\beta_0-\beta_1;$$
$$f'(-1)=g'(-1) \Rightarrow 100(-1)^{99}=\beta_1$$

Note:
$f'(\lambda) = 100\lambda^{99}$
$g'(\beta) = \beta_1$

$$\beta_1=-100,\ \beta_0=1+\beta_1=-99 \Rightarrow g(\lambda)=-99-100\lambda$$

$$A^{100}=f(A)=g(A)=\beta_0 I+\beta_1 A$$

$$A^{100}=\beta_0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\beta_1\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}=\begin{bmatrix}-199 & -100\\ 100 & 101\end{bmatrix}$$

Example: Compute A^k, $k \geq 3$ for $A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$

A has eigenvalue $\lambda_1=1$ with multiplicity 3.
Consider $f(\lambda) = \lambda^k$, $g(\lambda)=\beta_0+\beta_1\lambda+\beta_2\lambda^2$

$$f(1)=g(1) \Rightarrow (1)^k=\beta_0+\beta_1+\beta_2;$$
$$f'(1)=g'(1) \Rightarrow k=\beta_1+2\beta_2;$$
$$f''(1)=g''(1) \Rightarrow k(k-1)=2\beta_2$$

$$\beta_2=\frac{k(k-1)}{2},\ \beta_1=k-2\times\frac{k(k-1)}{2}=2k-k^2;$$
$$\beta_0=0.5k^2-1.5k+1$$
$$\Rightarrow g(\lambda)=(0.5k^2-1.5k+1)+(2k-k^2)\lambda+\frac{k(k-1)}{2}\lambda^2$$

$$g(\lambda) = (0.5k^2 - 1.5k + 1) + (2k - k^2)\lambda + \frac{k(k-1)}{2}\lambda^2$$

$$g(A) = (0.5k^2 - 1.5k + 1)I + (2k - k^2)A + \frac{k(k-1)}{2}A^2$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g(A) = (0.5k^2 - 1.5k + 1)I + (2k - k^2)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \frac{k(k-1)}{2}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & k & k(k-1)/2 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^k = f(A) = g(A) = \begin{bmatrix} 1 & k & k(k-1)/2 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

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Example: Compute A^k for $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

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General Functions of a Square Matrix

- Polynomials of a square matrix are naturally defined. How about non-polynomial functions?
- Suppose $f(\lambda) = e^{\lambda}$, $\sin\lambda$, or $1/(s - \lambda)$. What is $f(A)$?
- Two definitions
  - By means of a polynomial $g(\lambda)$ having the same values on the spectrum of A
  - By Taylor expansion
- These two turn out to be equivalent.
- We will have a lot of discussions on $f(A)=e^{At}$.

  The solution of a LTI system relies on this function.

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Definition: Given $A \in C^{n\times n}$. Let the distinct eigenvalues of A be λ_i, i=1,2,...,m, each with multiplicity n_i, $(n_1+n_2+\ldots+n_m= n)$. Let $f(\lambda)$ be a general function with $\{f^{(l)}(\lambda_i)\}$ well defined. Suppose that $g(\lambda)$ is a polynomial satisfying

$$f^{(l)}(\lambda_i) = g^{(l)}(\lambda_i),\ l = 0, 1, .., n_i -1,\ i = 1, .., m$$

Then $f(A) \equiv g(A)$.

Generally, g is a polynomial of degree n-1.

Example: $f(\lambda) = e^{\lambda t}$. Find $f(A) = e^{At}$ with $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

$$\Delta(\lambda) = \begin{bmatrix} \lambda-1 & 2 \\ -1 & \lambda-4 \end{bmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

$$\lambda_1 = 2,\ \lambda_2 = 3$$

$$f^{(0)}(\lambda_1) = e^{2t},\ f^{(0)}(\lambda_2) = e^{3t}$$

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– Now let $g(\lambda) = \beta_0 + \beta_1\lambda$

$g^{(0)}(\lambda_1) = \beta_0 + 2\beta_1 = e^{2t} \quad (=f^{(0)}(\lambda_1))$

$g^{(0)}(\lambda_2) = \beta_0 + 3\beta_1 = e^{3t} \quad (=f^{(0)}(\lambda_2))$

$\beta_1 = e^{3t} - e^{2t},\ \beta_0 = e^{2t} - 2\beta_1 = 3e^{2t} - 2e^{3t}$

– $g(\lambda) = (3e^{2t} - 2e^{3t}) + (- e^{2t} + e^{3t})\lambda$

$f(A) = g(A) = (3e^{2t} - 2e^{3t})I + (- e^{2t} + e^{3t})A$

$$= \begin{bmatrix} (3e^{2t} - 2e^{3t}) + (-e^{2t} + e^{3t}) & -2(-e^{2t} + e^{3t}) \\ (-e^{2t} + e^{3t}) & (3e^{2t} - 2e^{3t}) + 4(-e^{2t} + e^{3t}) \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} - e^{3t} & 2e^{2t} - 2e^{3t} \\ -e^{2t} + e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix} = e^{At}$$

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- Steps to calculate $f(A)$ given $f(\lambda)$ and A:
  - Form $\Delta(\lambda)$, and find $\{\lambda_i\}$ and $f^{(l)}(\lambda_i)$
  - Construct an $(n - 1)^{th}$ order polynomial $g(\lambda)$ such that $g^{(l)}(\lambda_i) = f^{(l)}(\lambda_i)$ for all i and l
  - $f(A) = g(A)$

Definition 2. Let $f(\lambda) \equiv \Sigma_{i=1}^{\infty} \alpha_i\lambda^i$ with the radius of convergence ρ. Then

$$f(A) \equiv \Sigma_{i=1}^{\infty} \alpha_i A^i$$

if $|\lambda_j| < \rho$ for all j.

- It can be shown that Definitions 1 and 2 are equivalent

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Example. Find e^{At} for a diagonal A and for A in Jordan canonical form

$$f(\lambda) = e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} \qquad f(A) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

$$A = \begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ : & : & : & : \\ 0 & 0 & .. & \lambda_n \end{bmatrix}, \qquad A^k = \begin{bmatrix} \lambda_1^k & 0 & .. & 0 \\ 0 & \lambda_2^k & .. & 0 \\ : & : & : & : \\ 0 & 0 & .. & \lambda_n^k \end{bmatrix}$$

$$f(A) = \begin{bmatrix} \sum \frac{\lambda_1^k t^k}{k!} & .. & 0 \\ : & : & : \\ 0 & .. & \sum \frac{\lambda_n^k t^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & .. & 0 \\ : & : & : \\ 0 & .. & e^{\lambda_n t} \end{bmatrix}$$

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- Now suppose that A is a Jordan block. Find e^{At}

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

$f(\lambda) = e^{\lambda t}$,

$f^{(0)}(\lambda) = e^{\lambda t}$, $f^{(1)}(\lambda) = te^{\lambda t}$,

$f^{(2)}(\lambda) = t^2 e^{\lambda t}$, $f^{(3)}(\lambda) = t^3 e^{\lambda t}$.

Derivative with respect to λ, not t.

  - $\Delta(\lambda) = (\lambda - \lambda_1)^4$, with λ_1 of multiplicity 4

    $f^{(0)}(\lambda_1) = e^{\lambda_1 t}$, $\quad f^{(1)}(\lambda_1) = te^{\lambda_1 t}$

    $f^{(2)}(\lambda_1) = t^2 e^{\lambda_1 t}$, $\quad f^{(3)}(\lambda_1) = t^3 e^{\lambda_1 t}$

  - $g(\lambda) = \beta_0 + \beta_1(\lambda - \lambda_1) + \beta_2(\lambda - \lambda_1)^2 + \beta_3(\lambda - \lambda_1)^3$

    $g^{(0)}(\lambda_1) = \beta_0 = e^{\lambda_1 t}$ $(= f^{(0)}(\lambda_1))$,

    $g^{(1)}(\lambda_1) = \beta_1 = te^{\lambda_1 t}$ $(=f^{(1)}(\lambda_1))$,

    $g^{(2)}(\lambda_1) = 2\beta_2 = t^2 e^{\lambda_1 t}$ $(=f^{(2)}(\lambda_1))$,

    $g^{(3)}(\lambda_1) = 6\beta_3 = t^3 e^{\lambda_1 t}$ $(=f^{(3)}(\lambda_1))$.

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$\beta_0 = e^{\lambda_1 t}$, $\beta_1 = te^{\lambda_1 t}$, $\beta_2 = t^2 e^{\lambda_1 t}/2$, $\beta_3 = t^3 e^{\lambda_1 t}/6$

– $g(\lambda) = e^{\lambda_1 t} + te^{\lambda_1 t}(\lambda - \lambda_1) + t^2 e^{\lambda_1 t}(\lambda - \lambda_1)^2/2 + t^3 e^{\lambda_1 t}(\lambda - \lambda_1)^3/6$

– $f(A) = g(A) = e^{\lambda_1 t} I + te^{\lambda_1 t}(A - \lambda_1 I) + t^2 e^{\lambda_1 t}(A - \lambda_1 I)^2/2 + t^3 e^{\lambda_1 t}(A - \lambda_1 I)^3/6$

$$A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (A - \lambda_1 I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A - \lambda_1 I)^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & t^3 e^{\lambda_1 t}/3! \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! \\ 0 & 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

Components: $t^k e^{\lambda_1 t}$, $0 \le k \le n-1$

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- For lower order submatrices of

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t}/2! & t^3 e^{\lambda t}/3! \\ 0 & e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t}/2! \\ 0 & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & 0 & e^{\lambda t} \end{bmatrix}$$

- For higher order matrices, you can extend from the pattern

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- For matrices in Jordan canonical form

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix} \quad \Rightarrow \quad e^{\bar{A}t} = \begin{bmatrix} e^{\bar{A}_1 t} & 0 \\ 0 & e^{\bar{A}_2 t} \end{bmatrix}$$

- For a general matrix A:

$$A = Q\bar{A}Q^{-1}$$

$$f(A) = f(Q\bar{A}Q^{-1}) = Qf(\bar{A})Q^{-1}$$

$$e^{At} = Qe^{\bar{A}t}Q^{-1}$$

➢ The similar transformation makes things easier.

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Example: Compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\Delta(\lambda) = \lambda(\lambda - 2)(\lambda + 1), \quad \lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 2$$

Approach 1: through the diagonal form.

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, Q^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$e^{At} = Qe^{\bar{A}t}Q^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 1 & e^{2t} \\ -e^{-t} & 0 & 2e^{2t} \\ e^{-t} & -1 & e^{2t} \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2e^{-t} + 3 + e^{2t} & -2e^{-t} + 2e^{2t} & 2e^{-t} - 3 + e^{2t} \\ -2e^{-t} + 2e^{2t} & 2e^{-t} + 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 3 + e^{2t} & -2e^{-t} + 2e^{2t} & 2e^{-t} + 3 + e^{2t} \end{bmatrix}$$

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$$\Delta(\lambda) = \lambda(\lambda-2)(\lambda+1),\ \lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 2$$

Approach 2: through the values of $f(\lambda) = e^{\lambda t}$ at the spectrum of A.

Let $g(\lambda)=a\lambda^2+b\lambda+c$,

$$g(\lambda_1) = a(-1)^2 + b(-1) + c = e^{\lambda_1 t} \Rightarrow a - b + c = e^{-t}$$
$$g(\lambda_2) = a(0)^2 + b(0) + c = e^{\lambda_2 t} \Rightarrow c = 1$$
$$g(\lambda_3) = a(2)^2 + a(2) + c = e^{\lambda_3 t} \Rightarrow 4a + 2b + c = e^{2t}$$

$$a = (2e^{-t} - 3 + e^{2t})/6$$
$$b = (-4e^{-t} + 3 + e^{2t})/6$$
$$c = 1$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{At} = \frac{(2e^{-t} - 3 + e^{2t})}{6}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{(-4e^{-t} + 3 + e^{2t})}{6}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2e^{-t} + 3 + e^{2t} & -2e^{-t} + 2e^{2t} & 2e^{-t} - 3 + e^{2t} \\ -2e^{-t} + 2e^{2t} & 2e^{-t} + 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 3 + e^{2t} & -2e^{-t} + 2e^{2t} & 2e^{-t} + 3 + e^{2t} \end{bmatrix}$$

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Next Time:

- Properties of e^{At};
- Solution to a continuous-time system

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- Solution to the discrete-time system

$$x[k+1] = A[k] + Bu[k]; \quad y[k] = Cx[k] + Du[k]$$

- Equivalent state equations
- Dealing with complex eigenvalues

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Homework set #6

1. Find Jordan-form representations $\bar{A}$ and transformation matrix Q for the following matrices:

$$A_1 = \begin{bmatrix} -1 & 5 & 1 \\ 0 & -3 & 0 \\ -1 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -3 & 0 \\ -1 & -1 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

2. Consider the matrices in Problem 1. Find $\alpha_i, \beta_i, \gamma_i, i = 1,2,3$ such that $A_i^k = \alpha_i + \beta_i A_i + \gamma_i A_i^2$

3. Let f(A) be a polynomial. Suppose that v is an eigenvector of A with corresponding eigenvalue λ, show that v is also an eigenvector of f(A) with corresponding eigenvalue $f(\lambda)$.

Note: Show the detailed procedure.

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4. Compute e^{At} for the following matrices:

$$A_1 = \begin{bmatrix} 4 & 0 \\ -3 & -2 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}; \quad A_3 = \begin{bmatrix} -2 & 0 & -2 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

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