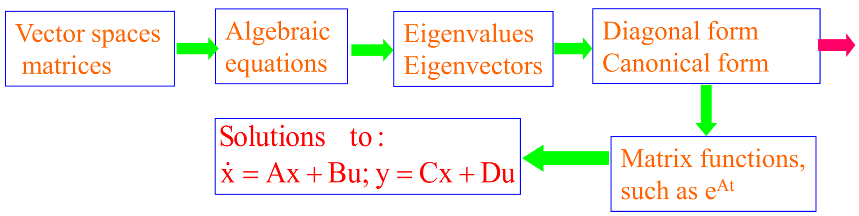
16.513 Control Systems (Lecture note #7)

- Last Time:
  - Generalized eigenvectors, Jordan form
  - Polynomial functions of a square matrix, e^{At}

A big picture: one branch of the course



The linear algebra tools will also be useful for other objectives.

Review: diagonal form and Jordan form

- All eigenvalues of A are distinct $\Rightarrow$ diagonalizable
- There are repeated eigenvalues, e.g., λ_i with multiplicity k.
  - If $\nu(A-\lambda_i I) = n - \rho(A-\lambda_i I) = k$, there exist k LI solutions to $(A-\lambda_i I)v=0$ and they are all eigenvectors. If this is the case for all repeated eigenvalues $\Rightarrow$ diagonalizable
  - If $\nu(A-\lambda_i I) = n - \rho(A-\lambda_i I) < k$, there exist generalized eigenvectors, $\Rightarrow$ not diagonalizable, there exist Jordan blocks

Definition. A vector v is a generalized eigenvector of grade k associated with λ if

$(A-\lambda I)^k v = 0, \quad \text{but } (A-\lambda I)^{k-1} v \neq 0$

Denote $v_k \equiv v$,

$v_{k-1} \equiv (A-\lambda I)v = (A-\lambda I)v_k, \qquad Av_k = v_{k-1} + \lambda v_k$

$v_{k-2} \equiv (A-\lambda I)^2 v = (A-\lambda I)v_{k-1}, \qquad Av_{k-1} = v_{k-2} + \lambda v_{k-1}$

$v_1 \equiv (A-\lambda I)^{k-1} v = (A-\lambda I)v_2, \qquad Av_2 = v_1 + \lambda v_2$

$(A-\lambda I)v_1 = (A-\lambda I)^k v = 0, \qquad Av_1 = \lambda v_1$

– What is the new representation w.r.t. $\{v_1, v_2, ., v_k\}$? i.e.,

$A[v_1\ v_2\ \ldots\ v_k] = [v_1\ v_2\ \ldots\ v_k]\bar{A}$

$$\bar{A} = \begin{bmatrix} \lambda & 1 & .. & 0 \\ 0 & \lambda & .. & 0 \\ : & : & & 1 \\ 0 & 0 & .. & \lambda \end{bmatrix}$$

A Jordan block

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Polynomial functions of a square matrix

- Let $f(A)=\Sigma_{i=1}^{k} \alpha_i A^i$ be a polynomial function of A. If $A=Q\bar{A}Q^{-1}$, then $f(A) = Qf(\bar{A})Q^{-1}$.
- Let $\Delta(\lambda)$ be the characteristic polynomial of A.

Cayley-Hamilton Theorem: $\Delta(A) = 0$

⬇

➢ Any polynomial can be expressed as a polynomial of degree n-1

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Theorem. Given $A \in C^{n \times n}$ and a polynomial $f(\lambda)$. Let the distinct eigenvalues of A be λ_i, i=1,2,...,m, each with multiplicity n_i, $(n_1+n_2+\ldots+n_m= n)$. Let

$$g(\lambda) = \beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1}$$

Then f(A)=g(A) iff

$$f^{(l)}(\lambda_i) = g^{(l)}(\lambda_i),\ l = 0, 1, .., n_i - 1,\ i = 1, .., m$$

$$\text{where } f^{(l)}(\lambda_i) = \left.\frac{d^{(l)}f(\lambda)}{d\lambda^l}\right|_{\lambda=\lambda_i}, \quad f^{(0)}(\lambda_i) = f(\lambda_i)$$

Under the above condition, the coefficients β_i's can be determined

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General functions of a square matrix:

Definition: Given $A \in C^{n \times n}$. Let the distinct eigenvalues of A be λ_i, i=1,2,...,m, each with multiplicity n_i, $(n_1+n_2+\ldots+n_m= n)$. Let $f(\lambda)$ be a general function with $\{f^{(l)}(\lambda_i)\}$ well defined. Suppose that $g(\lambda)$ is a polynomial satisfying

$$f^{(l)}(\lambda_i) = g^{(l)}(\lambda_i),\ l = 0, 1, .., n_i - 1,\ i = 1, .., m$$

Then $f(A) \equiv g(A)$.

Generally, g is a polynomial of degree n-1.

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Today:

- Some properties of e^{At};
- Solution to a continuous-time system

  $$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- Solution to the discrete-time system

  $$x[k+1] = A[k] + Bu[k]; \quad y[k] = Cx[k] + Du[k]$$

- Equivalent state equations

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Some properties for e^{At}

From the definition,

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!} + \cdots$$

The following can be verified

$$e^0 = I;$$

$$e^{A(t_1 + t_2)} = e^{At_1} e^{At_2};$$

$$e^{-At} = \left(e^{At}\right)^{-1};$$

Caution: e^{A+B} usually does not equal to $e^A e^B$.
We only have $e^{A+B} = e^A e^B$ when AB=BA

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More properties:

$$\frac{d\left(e^{At}\right)}{dt} = ?$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!} + \cdots$$

Let $\bar{k} = k - 1$

$$\frac{d\left(e^{At}\right)}{dt} = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A\left(\sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!}\right) = A\left(\sum_{\bar{k}=0}^{\infty} \frac{A^{\bar{k}} t^{\bar{k}}}{\bar{k}!}\right) = Ae^{At} = e^{At}A$$

$$\frac{d\left(e^{\lambda t}\right)}{dt} = \lambda e^{\lambda t} \quad \Leftrightarrow \quad \boxed{\frac{d\left(e^{At}\right)}{dt} = Ae^{At} = e^{At}A}$$

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More properties:

$$\int_0^t e^{A\tau} d\tau = ? \qquad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

~ Assuming that A^{-1} exists

$k+1 \to k$

$$\int_0^t e^{A\tau} d\tau = \sum_{k=0}^{\infty} \frac{A^k t^{k+1}}{(k+1)!} = A^{-1}\left(\sum_{k=0}^{\infty} \frac{A^{k+1} t^{k+1}}{(k+1)!}\right) = A^{-1}\left(\sum_{k=1}^{\infty} \frac{A^k t^k}{k!}\right)$$

$$= A^{-1}\left(\sum_{k=1}^{\infty} \frac{A^k t^k}{k!} + I - I\right) = A^{-1}\left(e^{At} - I\right) = \left(e^{At} - I\right)A^{-1}$$

$$\boxed{\int_0^t e^{A\tau} B d\tau = \left(\int_0^t e^{A\tau} d\tau\right) B = (e^{At} - I)A^{-1}B}$$

- This will be used to compute the output response under constant inputs.

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Example. Laplace Transform of e^{At}

$$L\{e^{At}\} = \sum_{k=0}^{\infty} L\left\{\frac{t^k}{k!}\right\}A^k = \sum_{k=0}^{\infty}\left(\frac{A^k}{s^{k+1}}\right) = \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{A}{s}\right)^k$$

$$\sum_{k=0}^{\infty}\lambda^k = \frac{1}{1-\lambda} \qquad \Rightarrow \sum_{k=0}^{\infty}\left(\frac{A}{s}\right)^k = \left(I - \frac{A}{s}\right)^{-1} = s(sI - A)^{-1}$$

~ Assuming $|\lambda| < 1$ ~ Assuming s is sufficiently large

$$L\{e^{At}\} = (sI - A)^{-1}, \quad \text{or } e^{At} = L^{-1}\{(sI - A)^{-1}\}$$

$$L\{e^{\lambda t}\} = \frac{1}{s-\lambda} \Longleftrightarrow \boxed{L\{e^{At}\} = (sI - A)^{-1}}$$

- How to compute $(sI - A)^{-1}$?

Example. $f(\lambda) = (s - \lambda)^{-1}$. Compute $f(A) = (sI - A)^{-1}$,

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}.$$

- $\Delta(\lambda) = (\lambda - \lambda_1)^3$, with λ_1 of multiplicity 3
- $f^{(0)}(\lambda_1) = (s - \lambda_1)^{-1}$, $f^{(1)}(\lambda_1) = (s - \lambda_1)^{-2}$, $f^{(2)}(\lambda_1) = 2(s - \lambda_1)^{-3}$
- $g(\lambda) = \beta_0 + \beta_1(\lambda - \lambda_1) + \beta_2(\lambda - \lambda_1)^2$
  $g^{(0)}(\lambda_1) = \beta_0 = (s - \lambda_1)^{-1}$, $g^{(1)}(\lambda_1) = \beta_1 = (s - \lambda_1)^{-2}$
  $g^{(2)}(\lambda_1) = 2\beta_2 = 2(s - \lambda_1)^{-3}$
- $g(\lambda) = (s - \lambda_1)^{-1} + (s - \lambda_1)^{-2}(\lambda - \lambda_1) + (s - \lambda_1)^{-3}(\lambda - \lambda_1)^2$
- $g(A) = (s - \lambda_1)^{-1}I + (s - \lambda_1)^{-2}(A - \lambda_1) + (s - \lambda_1)^{-3}(A - \lambda_1)^2$

$$A-\lambda_1 I=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\quad (A-\lambda_1 I)^2=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$g(A)=(s-\lambda_1)^{-1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+(s-\lambda_1)^{-2}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}+(s-\lambda_1)^{-3}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$(sI-A)^{-1}=g(A)=\begin{bmatrix}\frac{1}{s-\lambda_1}&\frac{1}{(s-\lambda_1)^2}&\frac{1}{(s-\lambda_1)^3}\\0&\frac{1}{s-\lambda_1}&\frac{1}{(s-\lambda_1)^2}\\0&0&\frac{1}{s-\lambda_1}\end{bmatrix}=L(e^{At})=L\left(\begin{bmatrix}e^{\lambda_1 t}&te^{\lambda_1 t}&t^2e^{\lambda_1 t}/2\\0&e^{\lambda_1 t}&te^{\lambda_1 t}\\0&0&e^{\lambda_1 t}\end{bmatrix}\right)$$

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Today:

- We will compute e^{At};
- Some of its properties;
- ➢ Solution to a continuous-time system

$$\dot{x}=Ax+Bu;\quad y=Cx+Du$$

- Solution to the discrete-time system

$$x[k+1]=A[k]+Bu[k];\quad y[k]=Cx[k]+Du[k]$$

- Equivalent state equations

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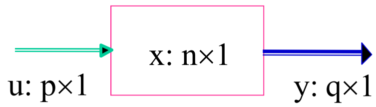
State-Space Solutions and Realizations

Solutions of Dynamic Equations



- Consider a linear system:

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

  – A: n×n real matrix; B: n×p real matrix
  – C: q×n real matrix; D: q×p real matrix
  – Given $x(t_0) = x_0$ and $u(\cdot) \Rightarrow$ A unique solution $x(\cdot)$, $y(\cdot)$
  – What is the solution?

- Recall that earlier we derived the solution for the input/output description based on superposition:

$$y(t) = \int_{t_0}^{t} G(t-\tau)u(\tau)d\tau, \quad G(t-\tau) = \begin{bmatrix} g_{11}(t-\tau) & g_{12}(t-\tau) & g_{1p}(t-\tau) \\ g_{21}(t-\tau) & g_{22}(t-\tau) & g_{2p}(t-\tau) \\ g_{q1}(t-\tau) & g_{q2}(t-\tau) & g_{qp}(t-\tau) \end{bmatrix}$$

Questions:

– Given system matrices, A,B,C,D, what is G(t)?
– What is the response due to initial state?

- Another approach is by using Laplace transform:

$$\hat{y}(s) = C(sI-A)^{-1}x(0) + [C(sI-A)^{-1}B + D]\hat{u}(s)$$

– A downside: the Laplace transform of u(t) may be not available, you may need to approximate it.

State-Space Solutions

The system: $\dot{x} = Ax + Bu; \quad y = Cx + Du$

Given x(0) and u(t) for $t \geq 0$. The solution for x and y is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau;$$
$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

$G(t-\tau)=Ce^{A(t-\tau)}B$

- Clearly two parts: zero-input resp. + zero-state resp.
- Linearity also obvious.
- We know how to compute e^{At}. The integration can be done numerically through discretization.

$$\int_0^{k\Delta} e^{A(t-\tau)}Bu(\tau)d\tau \approx \sum_{i=0}^{k-1} e^{A(k-i)\Delta}Bu(i\Delta)\,\Delta$$

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We first consider the state x:

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad (*)$$

Recall that $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ ← The key part

$$\frac{d}{dt}e^{-At}x = e^{-At}\dot{x} + \frac{d}{dt}(e^{-At})x = e^{-At}\dot{x} - e^{-At}Ax \qquad (**)$$

Plug (*) into (**) $\frac{d}{dt}e^{-At}x = e^{-At}Ax + e^{-At}Bu - e^{-At}Ax = e^{-At}Bu$

$$\Rightarrow de^{-At}x(t) = e^{-At}Bu(t)\,dt$$

Integrate from 0 to t; $\left[e^{-A\tau}x(\tau)\right]_0^t = \int_0^t e^{-A\tau}Bu(\tau)d\tau$

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

Premultiplying e^{At} to both sides, noting $e^{At}e^{-At} = I$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

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We verify that the solution

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

satisfies $\dot{x}(t)=Ax(t)+Bu(t);$

$$\begin{aligned}\dot{x}(t)&=\frac{d}{dt}\left[e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right]\\&=Ae^{At}x(0)+\int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau+e^{A(t-\tau)}Bu(\tau)|_{\tau=t}\\&=A\left(e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right)+Bu(t)\\&=Ax(t)+Bu(t)\ \checkmark\end{aligned}$$

Also, it is clear that the initial condition is satisfied.

Finally,
$$\begin{aligned}y(t)&=Cx(t)+Du(t)\\&=Ce^{At}x(0)+\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau+Du(t)\end{aligned}$$

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Different ways to compute e^{At}:

- From Definition 1:
  - Form $\Delta(\lambda)$, and find $\{\lambda_i\}$ and $(e^{\lambda t})^{(l)}|_{\lambda=\lambda i}$
  - Construct an $(n-1)^{th}$ order polynomial such that $g^{(l)}(\lambda_i)=(e^{\lambda t})^{(l)}|_{\lambda=\lambda i}$ for all i and l
  - $e^{At}=g(A)$
- From Definition 2: $e^{At}=\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}$, suitable for computer
- Use Jordan form $A=Q\bar{A}Q^{-1}$, $e^{At}=Qe^{\bar{A}t}Q^{-1}$
- Use the inverse Laplace transform of $(sI-A)^{-1}$.

$$e^{At}=\mathcal{L}^{-1}(sI-A)^{-1}$$

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Example: An LTI system:

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}x(t)+\begin{bmatrix}0\\1\end{bmatrix}u(t);\quad y=[1\ \ 0]x$$

Given x(0)=0; u(t)=1, for $t \geq 0$. Compute y(t), $t \geq 0$.

Step 1: Compute e^{At}. Eigenvalues of A are $\lambda_1=-1$; $\lambda_2=-2$.

Let $g(\lambda) = a\lambda+b$; $f(\lambda)=e^{\lambda t}$.

From $g(-1)=-a+b=e^{-t}$; $g(-2)=-2a+b=e^{-2t}$. $\Rightarrow a=e^{-t}-e^{-2t}$; $b=2e^{-t}-e^{-2t}$;

$$e^{At}=aA+bI=(e^{-t}-e^{-2t})\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}+(2e^{-t}-e^{-2t})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

Step 2: From $y(t)=Ce^{At}x(0)+\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$

$$y(t)=\int_0^t[1\ \ 0]e^{A(t-\tau)}\begin{bmatrix}0\\1\end{bmatrix}u(\tau)d\tau=\int_0^t(e^{-(t-\tau)}-e^{-2(t-\tau)})\,d\tau$$

$$=e^{-t}[e^{\tau}]_0^t-\frac{1}{2}e^{-2t}[e^{2\tau}]_0^t=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}$$

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Some properties about the zero-input response

$$x(t)=e^{At}x_0$$

Consider a Jordan block $e^{At}=\begin{bmatrix}e^{\lambda t} & te^{\lambda t} & t^2e^{\lambda t}/2! & t^3e^{\lambda t}/3!\\ 0 & e^{\lambda t} & te^{\lambda t} & t^2e^{\lambda t}/2!\\ 0 & 0 & e^{\lambda t} & te^{\lambda t}\\ 0 & 0 & 0 & e^{\lambda t}\end{bmatrix}$

For a general A, the terms of e^{At} are linear combinations of

$$e^{\lambda_i t}, te^{\lambda_i t}, t^2e^{\lambda_i t}, \cdots, t^{n_i-1}e^{\lambda_i t}, \qquad i=1,2,\cdots,m$$

- $Re(\lambda_i)<0$, for all .i, then as $t\to\infty$, all terms converges to 0, $e^{At}\to 0$, x(t) always converges to 0. $\to$ Stable system.
- $Re(\lambda_i)>0$, for some .i, then as $t\to\infty$, some terms diverge. There exist x_0 such that x(t) grows unbounded. Unstable
- $Re(\lambda_i)\leq 0$ for all .i, all eigenvalues with 0 real parts are simple, e^{At} is bounded for all t but not converge to 0. critical case
- $Re(\lambda_i)\leq 0$ for all .i, some eigenvalues with 0 real parts are repeated, e^{At} unbounded; x(t) unbounded for some x_0. unstable

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Today:

- We will compute e^{At};
- Some of its properties;
- Solution to a continuous-time system

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- ➢ Solution to the discrete-time system

$$x[k+1] = A[k] + Bu[k]; \quad y[k] = Cx[k] + Du[k]$$

- Equivalent state equations

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Discretization

$$x(t) \rightarrow x(T), x(2T), \ldots ,x(kT), \ldots$$

A continuous-time system $\dot{x} = Ax + Bu; \quad y = Cx + Du$

We use discretization for

- Digital simulation with computer;
- Implementation through a digital controller

Approach 1: Suppose we know x(kT). If T is small enough,

$$x(kT+T) - x(kT) \approx \dot{x}(kT)T = (Ax(kT) + Bu(kT))T$$

$$x((k+1)T) \approx x(kT) + ATx(kT) + BTu(kT) = (I + AT)x(kT) + BTu(kT)$$
$$y(kT) \approx Cx(kT) + Dy(kT)$$

$$x[k] := x(kT); \quad u[k] := u(kT) \Rightarrow x[k+1] = (I + AT)x[k] + BTu(k), \quad y[k] = Cx[k] + Du[k]$$

Simple but not accurate.

Approach 2:

Real situation: control u implemented by computer and a digital-analog converter. During a holding period,

$$u(t) = u(kT) \text{ for all } t \in [kT, (k+1)T), k=0,1,2,\dots$$

Solution at kT and (k+1)T,

$$x[k] := x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$$

$$x[k+1] = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

$$= e^{AT}\left[e^{AkT}x(0) + \int_0^{(k+1)T} e^{A(kT-\tau)}Bu(\tau)d\tau\right]$$

$$= e^{AT}\left[e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau\right] + \int_{kT}^{kT+T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

$$= e^{AT}x[k] + \int_0^{T} e^{A(T-\tau)}Bu[k]d\tau$$

$$= e^{AT}x[k] + \left(\int_0^{T} e^{A(T-\tau)}d\tau\right)Bu[k] \;\; =: A_d x[k] + B_d u[k]$$

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The discretized system:

$$x[k+1] = A_d x[k] + B_d u[k]$$

$$y[k] = C_d x[k] + D_d u[k]$$

where $A_d = e^{AT}$, $B_d = \left(\int_0^T e^{A(T-\tau)}d\tau\right)B$, $C_d = C$, $D_d = D$

- This exactly describes the input-state, input-output relationship at instants T, 2T, … , kT, …

For B_d, notice that

$$\int_0^T e^{A(T-\tau)}d\tau = e^{AT}\int_0^T e^{-A\tau}d\tau = -e^{AT}A^{-1}\int_0^T(-Ae^{-A\tau})d\tau$$

$$= -e^{AT}A^{-1}\int_0^T de^{-A\tau} = -e^{AT}A^{-1}\left[e^{-A\tau}\right]_0^T$$

$$= -e^{AT}A^{-1}\left[e^{-AT} - I\right] = A^{-1}[A_d - I] \;\Rightarrow\; \boxed{B_d = A^{-1}[A_d - I]B}$$

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From CT sys. to DT sys.

$$\dot{x} = Ax + Bu \qquad x[k+1] = A_d x[k] + B_d u[k]$$
$$y = Cx + Du \quad \Rightarrow \quad y[k] = C_d x[k] + D_d u[k]$$

Let the sampling period be T. Then

$$A_d = e^{AT}, \quad B_d = A^{-1}[A_d - I]B, \quad C_d = C, \quad D_d = D$$

Example: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad T = 0.1$

Use matlab: Ad=expm(A*T); Bd=inv(A)*(Ad-eye(3))*B;

Ad			Bd
0.9998	0.0997	0.0045	0.0002
-0.0045	0.9908	0.0861	0.0045
-0.0861	-0.1767	0.7325	0.0861

Solution of Discrete-time Equations

The DT system:

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

The solution is derived in a straightforward way:

$x[1]=Ax[0]+Bu[0]$

$x[2]=Ax[1]+Bu[1]=A(Ax[0]+Bu[0])+Bu[1]$

$=A^2x[0]+ABu[0]+Bu[1]$

$x[3]=Ax[2]+Bu[2]=A^3x[0]+A^2Bu[0]+ABu[1]+Bu[2]$

$$x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-m-1} Bu[m]$$
$$y[k] = CA^k x[0] + \sum_{m=0}^{k-1} CA^{k-m-1} Bu[m] + Du[k]$$

Some properties about the zero-input response

$x[k] = A^k x_0$

Consider a Jordan block

$$A^k = \begin{bmatrix} \lambda^k & k\lambda^k & k(k-1)\lambda^k/2! & k(k-1)(k-2)\lambda^k/3! \\ 0 & \lambda^k & k\lambda^k & k(k-1)\lambda^k/2! \\ 0 & 0 & \lambda^k & k\lambda^k \\ 0 & 0 & 0 & \lambda^k \end{bmatrix}$$

For a general A, the terms of A^k are linear combinations of

$$\lambda_i^k, k\lambda_i^k, k(k-1)\lambda_i^k, \cdots, \qquad i = 1,2,\cdots,m$$

- $|\lambda_i| < 1$, for all .i, then as $k \to \infty$, all terms converges to 0, $A^k \to 0$, x[k] always converges to 0. $\to$ Stable system.
- $|\lambda_i| > 1$, for some .i, then as $k \to \infty$, some terms diverge. There exist x_0 such that x[k] grows unbounded. Unstable
- $|\lambda_i| \le 1$ for all .i, all eigenvalues with unit magnitude are simple, A^k is bounded for all k but not converge to 0. Critical case
- $|\lambda_i| \le 1$ for all .i, some eigenvalues with unit magnitude are repeated, A^k unbounded; x[k] unbounded for some x_0 Unstable

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An Earlier Example: Interest and Amortization

- How to describe paying back a car loan over four years with initial debt D, interest r, and monthly payment p?
  - Let x[k] be the amount you owe at the beginning of the kth month. Then

    $$x[k+1] = (1 + r)\, x[k] - p$$

  - Initial and terminal conditions: x[0] = D and final condition x[48] = 0
    - How to find p?
- By solving the system, $x[48]=a_1 D+a_2 p \;\to p$

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The system:

$$x[k+1] = \underbrace{(1+r)}_{A}\, x[k] + \underbrace{(-1)}_{B}\, \underbrace{p}_{u}$$

Solution:

$$x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-m-1} Bu[m]$$

$$= (1+r)^k x[0] + \sum_{m=0}^{k-1} (1+r)^{k-m-1} (-1)p$$

$$= (1+r)^k D - \left(\sum_{m=0}^{k-1} (1+r)^{k-m-1} \right) p = (1+r)^k D - \frac{(1+r)^k - 1}{r} p$$

Given D=20000; r=0.004; x[48]=0;

Your monthly payment

$$0 = (1+0.004)^{48}\, 20000 - \frac{(1+0.004)^{48} - 1}{0.004} p$$

p=458.7761

Today:

- We will compute e^{At};
- Some of its properties;
- Solution to a continuous-time system

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- Solution to the discrete-time system

$$x[k+1] = A[k] + Bu[k]; \quad y[k] = Cx[k] + Du[k]$$

➢ Equivalent state equations

Equivalent state equations

Given state-space description:

$$\dot{x} = Ax + Bu; \quad y = Cx + Du \qquad (*)$$

Let P be a nonsingular matrix.

Define $\bar{x} = Px$, then $x = P^{-1}\bar{x}$

$$\dot{\bar{x}} = P\dot{x} = PAx + PBu = PAP^{-1}\bar{x} + PBu$$

$$y = Cx + Du = CP^{-1}\bar{x} + Du$$

Denote $\bar{A} = PAP^{-1}$, $\bar{B} = PB$, $\bar{C} = CP^{-1}$, $\bar{D} = D$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u; \quad y = \bar{C}\bar{x} + \bar{D}u \qquad (**)$$

- (*) and (**) are said to be equivalent to each other and the procedure from (*) to (**) is called an equivalent transformation

Note: For DT systems, the equivalent transformation is the same.

Recall: $\bar{A} = PAP^{-1}$ and A are similar to each other

- They have same eigenvalues. Same stability perf.

What do we expect from the two transfer functions:

$$G(s) = C(sI - A)^{-1}B + D \quad \text{and} \quad \bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

$$\Rightarrow \boxed{G(s) = \bar{G}(s)}$$

To verify,

$$\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

$$= CP^{-1}(sPP^{-1} - PAP^{-1})^{-1}PB + D$$

$$= CP^{-1}(P(sI - A)P^{-1})^{-1}PB + D \qquad \boxed{(XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}}$$

$$= CP^{-1}P(sI - A)^{-1}P^{-1}PB + D = C(sI - A)^{-1}B + D$$

Example: Given a state equation

$$\dot{x} = Ax + Bu, \qquad A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Let $Q = [B \quad A^2B \quad AB \quad A^3B]$ (the inverse exist). Define $z = Q^{-1}x$

Compute $\bar{A}$ and $\bar{B}$ such that $\dot{z} = \bar{A}z + \bar{B}u$

Solution: $Q^{-1}AQ = \bar{A} \Leftrightarrow AQ = Q\bar{A}$, $Q^{-1}B = \bar{B} \Leftrightarrow B = Q\bar{B}$. Let $\bar{A} = [a_1 \quad a_2 \quad a_3 \quad a_4]$.

$$AQ = A[B \quad A^2B \quad AB \quad A^3B] = [AB \quad A^3B \quad A^2B \quad A^4B]$$

$$Q\bar{A} = [Qa_1 \quad Qa_2 \quad Qa_3 \quad Qa_4]$$

$$AB = Qa_1 = [B \quad A^2B \quad AB \quad A^3B]a_1; \quad A^3B = Qa_2 = [B \quad A^2B \quad AB \quad A^3B]a_2;$$

$$A^2B = Qa_3 = [B \quad A^2B \quad AB \quad A^3B]a_3; \quad A^4B = Qa_4 = [B \quad A^2B \quad AB \quad A^3B]a_4$$

Immediately, $a_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$ How to get a_4?

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a_4 has to satisfy

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$A^4B = [B \quad A^2B \quad AB \quad A^3B]a_4 \qquad (*)$$

Let $a_4 = [k_1 \ \ k_2 \ \ k_3 \ \ k_4]'$, (*) can be written as

$$A^4B = k_1B + k_2A^2B + k_3AB + k_4A^3B \qquad (**)$$

From Cayley-Hamilton's theorem: $\Delta(A)=0$.

$$\Delta(s) = |sI - A| = (s^2+1)(s^2 - s - 2) = s^4 - s^3 - s^2 - s - 2$$

$$\Delta(A) = A^4 - A^3 - A^2 - A - 2I = 0 \Rightarrow A^4B - A^3B - A^2B - AB - 2B = 0$$

$$\Rightarrow A^4B = 2B + A^2B + AB + A^3B \Rightarrow k_1 = 2,\ k_2 = k_3 = k_4 = 1$$

$a_4 = [2 \ \ 1 \ \ 1 \ \ 1]'$,

$$\Downarrow \quad \bar{A} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

For $\bar{B}$, it satisfies

$$B = [B \quad A^2B \quad AB \quad A^3B]\bar{B} \Rightarrow \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Next Time:

- How to deal with complex eigenvalues
- Realization of a transfer function
- Simulation of systems by using Simulink
  - Course project

And more from linear algebra

- Quadratic functions and positive-definiteness

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Problem Set #7

1. The system:

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix} x, \; x(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Compute x(t) for $t \geq 0$.

2. For the LTI system

$$\dot{x}(t) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t); \quad y = [-1 \quad 1]x$$

a) Given x(0)=[1 1]', compute the zero-input response y(t);
b) Given u(t)=1 for $t \geq 0$, compute the zero-state response y(t);
c) Let the sampling period be T=0.1. Use matlab to compute the discretized system matrices A_d,B_d.

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Midterm Review (Lecture #1-Lecture #6)

- Modeling of LTI systems
- Linear algebra
  - Vector spaces: LI, LD, basis, inner product, orthogonal
  - Linear algebraic equation: range space, null space, conditions for the existence of solution, all solutions
  - Eigenvalues, eigenvectors, diagonal form
  - Generalized eigenvectors, Jordan from
  - Polynomial functions of a matrix
  - e^{At}

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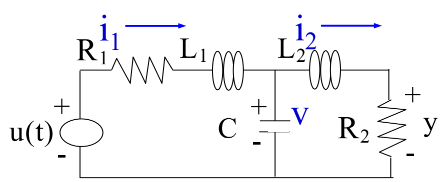
Model of a circuit:



- State variables?
  - i_1, i_2, and v,
- State and output equations?

$$L_1 \frac{di_1}{dt} = v_{L_1} = u - R_1 i_1 - v$$

$$L_2 \frac{di_2}{dt} = v_{L_2} = v - R_2 i_2$$

$$C \frac{dv}{dt} = i_C = i_1 - i_2$$

$$\Rightarrow$$

$$\frac{di_1}{dt} = -\frac{R_1}{L_1} i_1 - \frac{1}{L_1} v + \frac{1}{L_1} u$$

$$\frac{di_2}{dt} = -\frac{R_2}{L_2} i_2 + \frac{1}{L_2} v$$

$$\frac{dv}{dt} = \frac{1}{C} i_1 - \frac{v}{C} i_2$$

$$\underbrace{\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dv}{dt} \end{bmatrix}}_{\dot{x}} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ v \end{bmatrix}}_{x} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = R_2 i_2 = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

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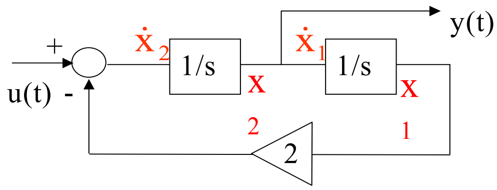
Integrators + amplifiers



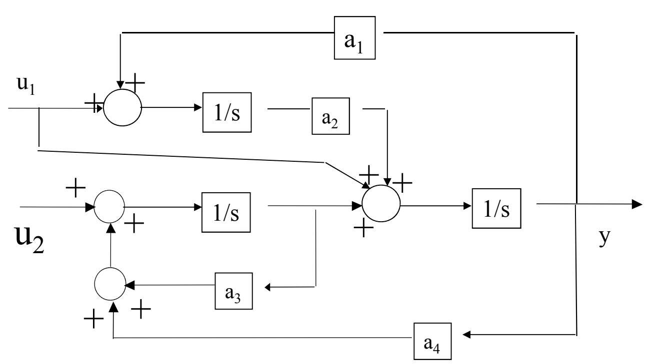
- What are the state variables?
- Select output of integrators as SVs
- What are the state and output equations?

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u - 2x_1 \qquad y = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{0}_{D} u$$

Example



Linear Algebra Basics

- Elementary operations that preserve determinant
- Elementary operations that preserve rank
- Use elementary operation to transform a matrix into upper or lower triangular form

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Linear Independence

- A set of vectors $\{x_1, x_2, .., x_m\}$ in R^n is LD if
  $\exists \{\alpha_1, \alpha_2, .., \alpha_m\}$ in R, not all zero, s.t.
  $\alpha_1 x_1 + \alpha_2 x_2 + .. + \alpha_m x_m = 0$ (*)
- If the only set of $\{\alpha_i\}_{i=1 \text{ to } m}$ s.t. the above holds is
  $\alpha_1 = \alpha_2 = .. = \alpha_m = 0$
  then $\{x_i\}_{i=1 \text{ to } m}$ is said to be LI

- Given $\{x_1, x_2, .., x_m\}$, form $A = [x_1 \quad x_2 \quad \ldots \quad x_m]$

If $A\alpha=0$ has a unique solution, LI;
If $A\alpha=0$ has nonunique solution, LD.

If rank(A)=m, the solution is unique $\Rightarrow$ LI
If rank(A)<m, the solution is not unique $\Rightarrow$ LD.

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Examples: are the following sets of vectors LI, or LD?

$$\{[1],[2]\},\quad \left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\a\end{bmatrix},\begin{bmatrix}a\\-1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}a\\2\\0\end{bmatrix},\begin{bmatrix}b\\c\\3\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}a\\2\\f\end{bmatrix},\begin{bmatrix}b\\d\\3\end{bmatrix},\begin{bmatrix}c\\e\\4\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1\\a\\0\end{bmatrix},\begin{bmatrix}a\\2\\0\end{bmatrix},\begin{bmatrix}b\\c\\3\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}0\\1\\3\end{bmatrix}\right\}$$

- LI if the rank of the matrix equals the number of columns

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Basis and Representations

- A set of LI vectors $\{e_1, e_2, .., e_n\}$ of R^n is said to be a basis of R^n if every vector in R^n can be expressed as a unique linear combination of them
  - For any $x \in R^n$, there exist unique $\{\beta_1, \beta_2, .., \beta_n\}$ s.t.

$$x = \beta_1 e_1 + \beta_2 e_2 + .. + \beta_n e_n = \sum_{i=1}^{n} \beta_i e_i$$

$$x = \begin{bmatrix} e_1 & e_2 & \ldots & e_n \end{bmatrix} \begin{bmatrix}\beta_1\\ \beta_2\\ \vdots \\ \beta_n\end{bmatrix}$$

$$x = \begin{bmatrix} e_1 & e_2 & \ldots & e_n \end{bmatrix} \beta$$

  - β: Representation of x with respect to the basis

Theorem: In an n-dimensional vector space (or subspace), any set of n LI vectors qualifies as a basis

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Change of basis:

- Given a basis $\left(e_1 \quad e_2 \quad \ldots \quad e_n\right)$;
- Let the new basis be:

$$\left(\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n\right) = \left(e_1 \quad e_2 \quad \ldots \quad e_n\right)Q$$

Then,

$$\left(e_1 \quad e_2 \quad \ldots \quad e_n\right) = \left(\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n\right)Q^{-1}$$

- For x such that

$$x = \left(e_1 \quad e_2 \quad \ldots \quad e_n\right)\beta$$

We have $x = \left(\bar{e}_1 \quad \bar{e}_2 \quad \ldots \quad \bar{e}_n\right)Q^{-1}\beta$

Linear algebraic equation Ax = y

– If $\rho(A) \neq \rho([A : y])$ (i.e., $y \notin \mathcal{R}(A)$), then the equations are inconsistent, and there is no solution

– If $\rho(A) = \rho([A : y])$, then $\exists$ at least one solution

- If $\rho(A) = \rho([A : y]) < n$ (i.e., $\nu(A) > 0$), then there are infinite number of solutions
- If $\rho(A) = \rho([A : y]) = n$ (i.e., $\nu(A) = 0$), then there is a unique solution

– For an $n \times n$ matrix, $Ax = y$ has a unique solution $\forall\, y \in R^m$ iff A^{-1} exists, or $|A| \neq 0$

Key concepts: Assume $A \in R^{m \times n}$.

Range space R(A): $\{y \in R^m: \text{ exists } x \in R^n \text{ s.t. } y=Ax\}$

- subspace of R^m,
- dimension = $\rho(A)$, rank of A
- basis: formed by the maximal number of LI columns of A

Null space N(A): $\{x \in R^n: Ax=0\}$

- subspace of R^n,
- dimension $\nu(A)=n-\rho(A)$
- basis: formed by $\nu(A)$ LI solutions to $Ax=0$.

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Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\quad a_1 \quad a_2 \quad a_3 \quad a_4$

The range space R(A) is spanned by $\{a_1, a_2, a_3, a_4\}$
What is the relationship among the vectors?
What are $\rho(A)$, $\nu(A)$? The dimension of R(A)?
The basis of R(A)? The null spaces?

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Parameterization of all solutions

Theorem: Given m×n matrix A and a m×1 vector y.

- Let x_p be a solution to Ax = y.
- Let ν(A)=k.
- Suppose k>0 and the null space is spanned by $\{n_1, n_2, \ldots n_k\}$

➢ The set of all solutions is given by
$\{x = x_p + \alpha_1 n_1 + \alpha_2 n_2 + \ldots + \alpha_k n_k : \alpha_i \in R\}$

Eigenvalues, eigenvectors and diagonal form

A scalar λ is called an eigenvalue of $A \in C^{n\times n}$ if ∃ a nonzero $x \in C^n$, such that $Ax = \lambda x$ and x is the eigenvector associated with λ.

Case 1: All eigenvalues are distinct

Theorem: the sets of eigenvectors $\{v_1, v_2, \ldots, v_n\}$ is LI.
Let $Q=[v_1\ v_2\ \ldots\ v_n]$, then

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & .. & 0 \\ : & : & & : \\ 0 & 0 & .. & \lambda_n \end{bmatrix}$$

Definition. A vector v is a generalized eigenvector of grade k associated with λ if

$$(A-\lambda I)^k v = 0, \quad \text{but } (A-\lambda I)^{k-1} v \neq 0$$

Denote $v_k \equiv v,$

$v_{k-1} \equiv (A-\lambda I)v = (A-\lambda I)v_k,$ $\qquad Av_k = v_{k-1} + \lambda v_k$

$v_{k-2} \equiv (A-\lambda I)^2 v = (A-\lambda I)v_{k-1},$ $\qquad Av_{k-1} = v_{k-2} + \lambda v_{k-1}$

$v_1 \equiv (A-\lambda I)^{k-1} v = (A-\lambda I)v_2,$ $\qquad Av_2 = v_1 + \lambda v_2$

$(A-\lambda I)v_1 = (A-\lambda I)^k v = 0,$ $\qquad Av_1 = \lambda v_1$

– What is the new representation w.r.t. $\{v_1, v_2, ., v_k\}$? i.e.,

$A[v_1\ v_2 \ldots v_k] = [v_1\ v_2 \ldots v_k]\bar{A}$

$$\bar{A} = \begin{bmatrix} \lambda & 1 & .. & 0 \\ 0 & \lambda & .. & 0 \\ : & : & & 1 \\ 0 & 0 & .. & \lambda \end{bmatrix}$$

λ 0……0
1 λ. …..0

A Jordan block

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- Polynomial functions of a square matrix.
- Computation of e^{At} .

Today's material will not be included in the midterm.

Questions?

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Midterm exam: 6.30-9:30 pm, Oct 20 (Thursday)

Open book, open notes

No calculator, No Laptop

Good luck!

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