16.513 Control Systems

Summary of Results From Last Lecture:

Consider the system: $\dot{x} = Ax + Bu; \quad y = Cx + Du$

Given x(0) and u(t) for $t \geq 0$. The solution is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau;$$
$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

The main problem involved is to compute e^{At}.

The system is internally stable iff all the eigenvalues of A have negative real parts: $\text{Re}\,\lambda_i(A) < 0$ for all i

$\Rightarrow x(t) \rightarrow 0$ if u(t)=0

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Solution of Discrete-time Equations

The DT system: $x[k+1] = Ax[k] + Bu[k]$
$y[k] = Cx[k] + Du[k]$

Given x[0] and u(k), $k \geq 0$, the solution is:

$$x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-m-1}Bu[m]$$
$$y[k] = CA^k x[0] + \sum_{m=0}^{k-1} CA^{k-m-1}Bu[m] + Du[k]$$

The main problem involved is to compute A^k.

The system is internally stable iff all the eigenvalues of A are within the unit disk: $|\lambda_i(A)| < 1$ for all i

$\Rightarrow x[k] \rightarrow 0$ if u[k]=0

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Today: we will address some miscellaneous problems about LTI systems

- How to deal with complex eigenvalues
- Realization of a transfer function
- Simulation of systems by using Simulink
  - Course project

To prepare for new topics in this course, we will also study

- Quadratic functions and positive-definiteness

Next Time: We start another topic (Chapter 6)

- Controllability and observability

Deal with complex eigenvalues

Typically, we would like to transform a matrix $A \in R^{n\times n}$ into a diagonal form through equivalent transformation

$$\overline{A} = Q^{-1}AQ \qquad \overline{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Suppose that A has complex eigenvalues.

- Some λ_i will be complex
- The transformation matrix Q^{-1} will have complex entries
- What does the new state $z = Q^{-1}x$ stand for?

Physically, it is meaningless. Numerically, it may render analysis or design results invalid, such as a feedback law with complex numbers

Since A is a real matrix, if A has complex eigenvalues,

- The complex eigenvalues appear as conjugate pairs:

  $\alpha_i + j\beta_i,\ \alpha_i - j\beta_i$.

- The eigenvectors also appear as conjugate pairs

  $v_i + jw_i,\ v_i - jw_i$.

There is a way to avoid complex numbers. Assume

$$A(v + jw) = (\alpha + j\beta)(v + jw) = \alpha v - \beta w + j(\beta v + \alpha w);$$
$$A(v - jw) = (\alpha - j\beta)(v - jw) = \alpha v - \beta w - j(\beta v + \alpha w);;$$

Add the two: $Av = \alpha v - \beta w$

Subtract: $Aw = \beta v + \alpha w$

$$\Rightarrow \quad A[v\ \ w] = [v\ \ w]\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$A[v\ \ w] = [v\ \ w]\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

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Suppose we have two real eigenvalues and two pairs of complex eigenvalues, all distinct,

$$\lambda_1, \lambda_2, \alpha_1 + j\beta_1,\ \alpha_1 - j\beta_1,\ \alpha_2 + j\beta_2,\ \alpha_2 - j\beta_2,$$

with corresponding eigenvectors;

$$q_1, q_2, v_1 + jw_1,\ v_1 - jw_1,\ v_2 + jw_2,\ v_2 - jw_2,$$

$$Aq_i = \lambda_i q_i,\quad A[v_i\ \ w_i] = [v_i\ \ w_i]\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$$

In matrix form;

$$A\underbrace{[q_1\ \ q_2\ \ v_1\ \ w_1\ \ v_2\ \ w_2]}_{Q} = \underbrace{[q_1\ \ q_2\ \ v_1\ \ w_1\ \ v_2\ \ w_2]}_{Q}\underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}}_{\bar{A}}$$

$$\bar{A} = Q^{-1}AQ$$

Not strictly a diagonal form but real: a block diagonal form

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Example: $A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 2 \\ -2 & -1 & 3 \end{bmatrix}$ Find the block diagonal form and the transformation matrix

Use matlab [V,D]=eig(A), you get:

V = complex eigenvectors

0.3162 - 0.3162i	0.3162 + 0.3162i	0.7071
0.6325	0.6325	0.0000
0.6325	0.6325	0.7071

D = diagonal form

1.0000 + 1.0000i	0	0
0	1.0000 - 1.0000i	0
0	0	1.0000

Where inv(V)*A*V=D

Let q1=real(V(:,1));
q2=imag(V(:,1)); q3=V(:,3)
Form Q=[q1 q2 q3], then

Q =

0.3162	-0.3162	0.7071
0.6325	0	0.0000
0.6325	0	0.7071

>> inv(Q)*A*Q

ans =

1.0000	1.0000	0.0000
-1.0000	1.0000	-0.0000
-0.0000	0	1.0000

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What is e^{At} with $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$?

Use the first definition for a matrix function.

- Let $f(\lambda)=e^{\lambda t}$. Find $g(\lambda)=k_0+k_1\lambda$.
- The eigenvalues of A: $\lambda_1 = \alpha + j\beta, \quad \lambda_2 = \alpha - j\beta,$

$$\begin{aligned} g(\lambda_1) &= f(\lambda_1) \\ g(\lambda_2) &= f(\lambda_2) \end{aligned} \Rightarrow \begin{aligned} k_0 + k_1(\alpha + j\beta) &= e^{(\alpha+j\beta)t} = e^{\alpha t}(\cos\beta t + j\sin\beta t) \\ k_0 + k_1(\alpha - j\beta) &= e^{(\alpha-j\beta)t} = e^{\alpha t}(\cos\beta t - j\sin\beta t) \end{aligned}$$

$$\Rightarrow \quad k_1 = \frac{e^{\alpha t}\sin\beta t}{\beta}, \quad k_0 = e^{\alpha t}(\cos\beta t - \frac{\alpha}{\beta}\sin\beta t)$$

$$e^{At} = g(A) = e^{\alpha t}(\cos\beta t - \frac{\alpha}{\beta}\sin\beta t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{e^{\alpha t}}{\beta}\sin\beta t\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$$= e^{\alpha t}\begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}$$

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In summary:

With $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, $e^{At} = e^{\alpha t}\begin{bmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{bmatrix}$,

For the system: $\dot{x} = Ax + Bu$

The zero-input solution is

$$x(t) = e^{At}x(0) = e^{\alpha t}\begin{bmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{bmatrix} x(0)$$

The real part α (of the eigenvalue) determines the stability of the system;
The imaginary part β determines the frequency of the oscillation.

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For a matrix

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

$$e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\alpha_1 t}\cos\beta_1 t & e^{\alpha_1 t}\sin\beta_1 t & 0 & 0 \\ 0 & 0 & -e^{\alpha_1 t}\sin\beta_1 t & e^{\alpha_1 t}\cos\beta_1 t & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\alpha_2 t}\cos\beta_2 t & e^{\alpha_2 t}\sin\beta_2 t \\ 0 & 0 & 0 & 0 & -e^{\alpha_2 t}\sin\beta_2 t & e^{\alpha_2 t}\cos\beta_2 t \end{bmatrix}$$

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Realization of a periodic signal, e.g., with two harmonics:

$$v(t) = v_{dc} + V_1 \sin(\beta\, t + \theta_1) + V_2 \sin(2\beta t + \theta_2)$$

Can be realized as the output of a 5th-order system:

$$\dot{x} = Ax; \quad v = \Gamma\, x, \qquad x(0) = x_0 = [a\; b\; c\; d\; e]^T$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta \\ 0 & 0 & 0 & -2\beta & 0 \end{bmatrix}, \quad \Gamma = [1 \quad 1 \quad 0 \quad 1 \quad 0] \qquad v(t) = \Gamma e^{At} x_0$$

$$v(t) = [1 \quad 1 \quad 0 \quad 1 \quad 0] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\beta t & \sin\beta t & 0 & 0 \\ 0 & -\sin\beta t & \cos\beta t & 0 & 0 \\ 0 & 0 & 0 & \cos 2\beta t & \sin 2\beta t \\ 0 & 0 & 0 & -\sin 2\beta t & \cos 2\beta t \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix},$$

$a = v_{dc}$
$b = V_1 \sin\theta_1$
$c = V_1 \cos\theta_1$
$d = V_2 \sin\theta_2$
$e = V_2 \cos\theta_2$

Will learn how to build an observer to reconstruct a periodic signal

Today: some miscellaneous problems about LTI systems

- How to deal with complex eigenvalues
- Realization of a transfer function
- Simulation of systems by using Simulink
- Course Project

And more from linear algebra

- Quadratic functions and positive-definiteness

State-space realizations of transfer functions

Given state equations

$$\dot{x} = Ax + Bu; \quad y = Cx + Du \qquad (*)$$

The transfer function is: $G(s) = C(sI - A)^{-1}B + D$

Now, given G(s), how to find (A,B,C,D)?

Background:

- Sometimes it is hard to obtain a state-space description.
- But you can identify the transfer function using frequency response.
- We have more advanced design methods for state-space models.

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Example: $G(s) = \dfrac{4s^2 + 5s + 6}{s^3 + s^2 + 2s + 3}$

It can be verified that $G(s) = C(sI-A)^{-1}B + D$ with

$$A = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}, \quad D = 0$$

First, $|sI - A| = \begin{vmatrix} s+1 & 2 & 3 \\ -1 & s & 0 \\ 0 & -1 & s \end{vmatrix} = s^3 + s^2 + 2s + 3$

Then, $(sI - A)^{-1} = \dfrac{1}{s^3 + s^2 + 2s + 3} \begin{bmatrix} s^2 & * & * \\ s & * & * \\ 1 & * & * \end{bmatrix}$

The adjunct matrix

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$$A=\begin{bmatrix}-1 & -2 & -3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ B=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad C=\begin{bmatrix}4 & 5 & 6\end{bmatrix},\quad D=0$$

$$(sI-A)^{-1}=\frac{1}{s^3+s^2+2s+3}\begin{bmatrix}s^2 & * & *\\ s & * & *\\ 1 & * & *\end{bmatrix}$$

$$C(sI-A)^{-1}B+D=\frac{\begin{bmatrix}4 & 5 & 6\end{bmatrix}}{s^3+s^2+2s+3}\begin{bmatrix}s^2 & * & *\\ s & * & *\\ 1 & * & *\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$=\frac{4s^2+5s+6}{s^3+s^2+2s+3}\quad = G(s)$$

- We say that (A,B,C,D) is a realization of G(s)

If there exists (A,B,C,D) such that $G(s)=C(sI-A)^{-1}B+D$ then we say that G(s) is realizable.

Theorem: A transfer matrix G(s) is realizable with LTI state equation if and only if it is a proper rational matrix.

- First observe that $C(sI-A)^{-1}B+D$ is always proper and rational (necessity proved).
- A proper rational matrix can be decomposed as the sum of a constant matrix and a strictly proper rational matrix: $G(s)=G_{sp}(s)+D$, $D=G(\infty)$
- Let $d(s)=s^r+a_1s^{r-1}+a_2s^{r-2}+\ldots+a_{r-1}s+a_r$ be the least common denominator of all entries of $G_{sp}(s)$
- Then $G_{sp}(s)$ can be expressed as (assume G is q×p)

$$G_{sp}(s)=\frac{1}{d(s)}\left[N_1s^{r-1}+N_2s^{r-2}+\cdots+N_{r-1}s+N_r\right],\quad N_i\in R^{q\times p}$$

With

$$G_{sp}(s)=\frac{1}{d(s)}\left[N_1 s^{r-1}+N_2 s^{r-2}+\cdots+N_{r-1}s+N_r\right],\ N_i\in R^{m\times p}$$

$$d(s)=s^r+a_1 s^{r-1}+\cdots+a_{r-1}s+a_r$$

The realization of $G_{sp}(s)$ is given as:

$$A=\begin{bmatrix} -a_1 I_p & -a_2 I_p & \cdots & -a_{r-1}I_p & -a_r I_p \\ I_p & 0 & \cdots & 0 & 0 \\ 0 & I_p & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 \end{bmatrix},\quad B=\begin{bmatrix} I_p \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C=\begin{bmatrix} N_1 & N_2 & \cdots & N_{r-1} & N_r \end{bmatrix}$$

Another form of realization: Problem 4.9

Example: $$G(s)=\begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

Step 1: break it into a constant part and a strictly proper part

$$G(s)=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} \frac{-6}{s+0.5} & \frac{3}{s+2} \\ \frac{0.5}{(s+0.5)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix},\quad \Leftarrow \frac{4s-10}{2s+1}=\frac{4s+2}{2s+1}+\frac{-12}{2s+1}$$

$G_{sp}(s)$

Step 2: the monic least common denominator

$$d(s)=(s+0.5)(s+2)(s+2)=s^3+\underset{a_1}{4.5}s^2+\underset{a_2}{6}s+\underset{a_3}{2}$$

Step 3: $$G_{sp}(s)=\frac{1}{(s+0.5)(s+2)^2}\begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{bmatrix}$$

$$=\frac{1}{d(s)}\begin{bmatrix} -6s^2-24s-24 & 3s^2+6s+3 \\ 0.5s+1 & s^2+1.5s+0.5 \end{bmatrix}$$

$$d(s)=s^3 + 4.5s^2 + 6s + 2 = a_1 s^2 + a_2 s + a_3$$

$$\text{From step 3: } G_{sp}(S) = \frac{1}{d(s)}\begin{bmatrix} -6s^2-24s-24 & 3s^2+7.5s+3 \\ 0.5s+1 & s^2+1.5s+0.5 \end{bmatrix}$$

$$= \frac{1}{d(s)}\left\{ \underbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}_{N_1} s^2 + \underbrace{\begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix}}_{N_2} s + \underbrace{\begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix}}_{N_3} \right\}$$

Step 4: ($a_1 I_p$, $a_2 I_p$, $a_3 I_p$)

$$A = \begin{bmatrix} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

N_1 N_2 N_3

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Another realization of the same system is given in p.105 Example 4.7, where the dimension is only 4.

Discussion:

- The realization (A,B,C,D) for a particular G(s) is not unique;
- All the equivalence transformations are also valid realizations;
- With different methods, the dimensions of the resulting systems, i.e., the number of state variables, may be different. There exist a minimal-order realization
- We will learn later how to reduce the order of a realization to the minimal number.

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Today: some miscellaneous problems about LTI systems

- How to deal with complex eigenvalues
- Realization of a transfer function
- ➢ Simulation of systems by using Simulink
- Course project

And more from linear algebra

- Quadratic functions and positive-definiteness

A Tool for System Simulation: SIMULINK

Can be used for simulation of various systems:

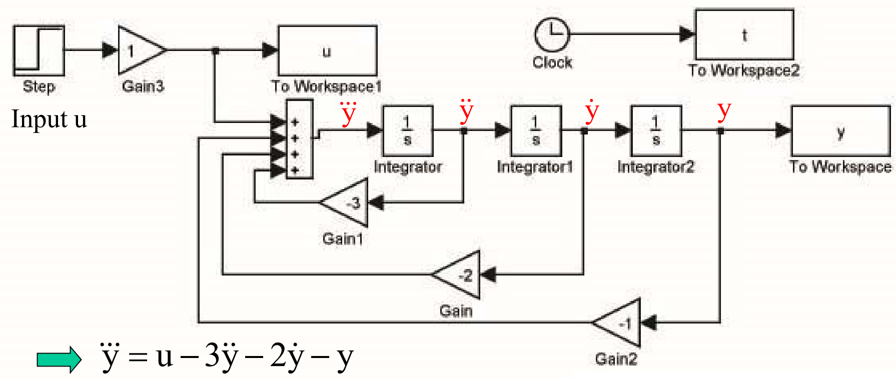
- Linear, CT or DT,
- Nonlinear;
- Switched;
- Hybrid: CT + DT components, signals;

Input signals can be arbitrarily generated:

- Standard: sinusoidal, polynomial, square, impulse
- Customized: from a function, look-up table

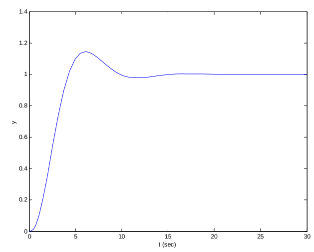
Output signals can be stored or demonstrated in different ways.

Example:



$\Rightarrow\ \dddot{y} = u - 3\ddot{y} - 2\dot{y} - y$

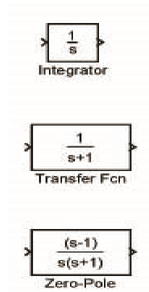
$\Rightarrow\ \dddot{y} + 3\ddot{y} + 2\dot{y} + y = u$



Click simulation and use plot(t,y), you will get a time response of y

- The parameters can be easily changed;
- The initial condition can be easily changed.

Simulink for linear systems



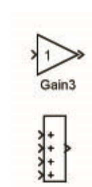
Main components with dynamics:

- integrators,
- state-space description (A,B,C,D)
- transfer function
- derivative (rarely used)

The first two components need initial conditions

Math components:

- addition (a+b+c); product (a×b);
- dot (inner) product <x,y>;
- gain (amplifier) kx : x a scalar
- matrix gain Kx: x a vector



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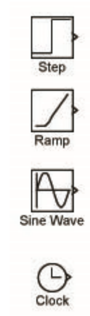
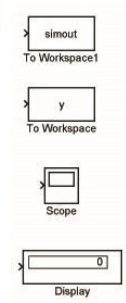
Sources: input signals
- constant, step, ramp
- pulse, sine wave, square wave
- from data file
- signal generator



Sinks: for output demonstration or storage
- digital display
- scope
- save to file
- export to workspace
- XY graph



Nonlinear: functions and operations
- saturation, deadzone, switch

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Signals and systems:

- Demux: input a vector signal and output all the components

- Mux: input a bunch of scalar signals and output a vector signal

Functions and tables:
- input u → output y: y=f(u); f composed from available functions or operations; e.g, y=sin(u1)+u1*u2
- matlab function: y=f(u); f written by a matlab file
- look-up table.

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Example: Find the solution to the LTI systems

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

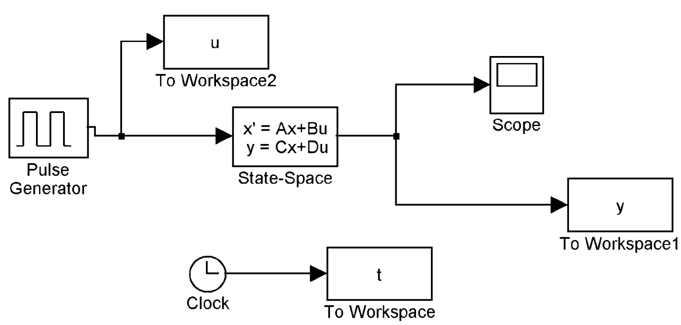
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

where x(0)=0; u(t) is a square wave.

Steps:

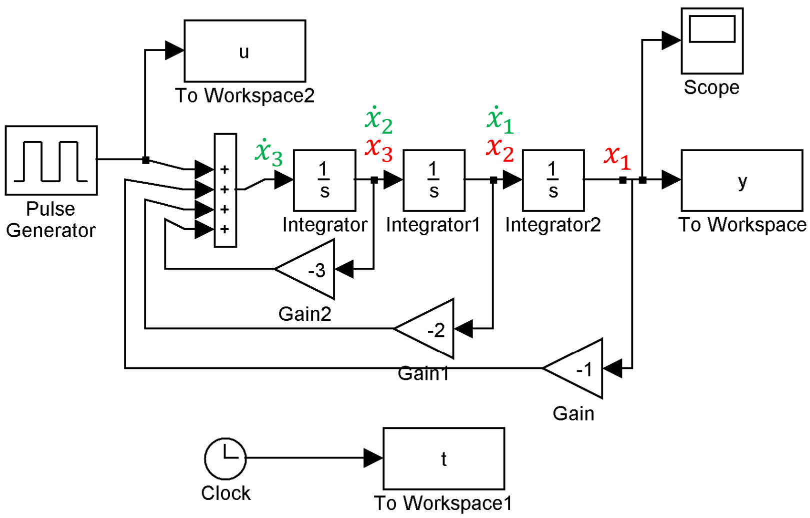
1. Open matlab workspace
2. type simulink and return
   - simulink library browser window is open
3. Click file and choose new then choose model
   - a blank window is open
4. Open one of the commonly used blocks and drag and drop whatever you need to the blank window.
5. Connect the components by arrows.

First approach: use state-space description:

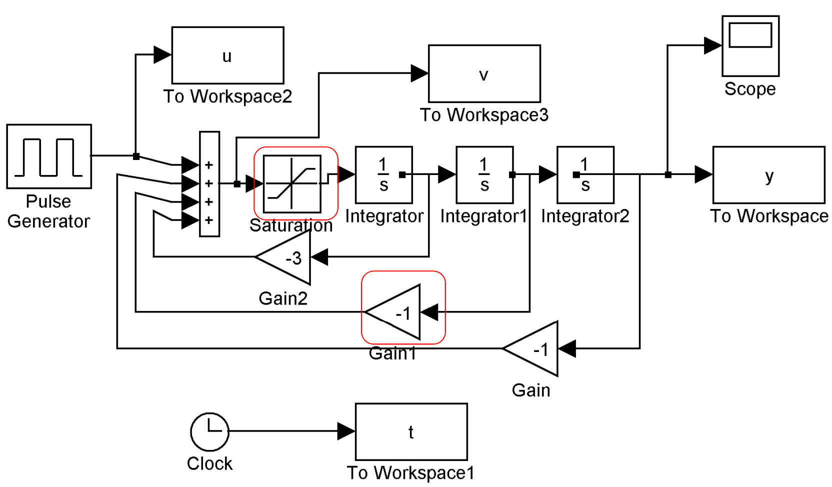


- Click each component to setup the parameters properly
  sinks labeled "t", "u", "y": choose "array" for save format
  sampling time can be a parameter inputted from workspace
- When ready, click simulation and choose configuration parameters to setup simulation time. Finally, click simulation and choose start
- When finished, type plot(t,y,t,u) to plot the input and output

Second approach: use integrators and amplifiers:

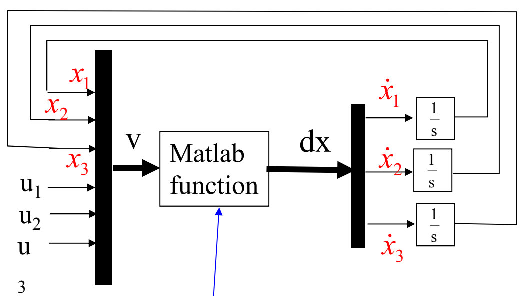


You can make any kind of changes to the model:
Change the parameters, the sampling time, add some
nonlinear component such as a saturation:



Simulation for nonlinear system:

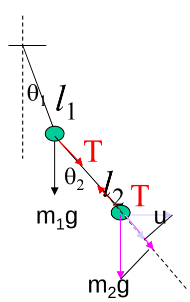
$$\dot{x} = f(x,u) \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$



Click on matlab function to choose fun1

```
function dx=fun1(v)
 x1=v(1);
 x2=v(2);
 xn=v(n)
 u1=v(n+1);
 u2=v(n+2);
 um=v(n+m);
 dx(1)=f1(x1,...,u1,…)
 dx(2)=f2(x1,…,u1,…)
 dx(n)=f3(x1,…,u1,…)
```


Simulation for a two-link pendulum

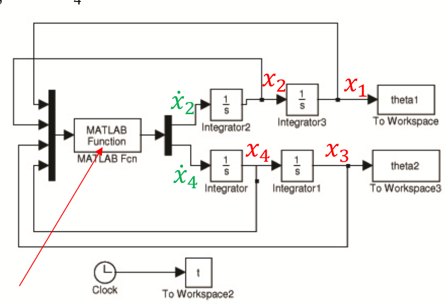


$$\dot{x}_1 = x_2;$$

$$\dot{x}_2 = -\frac{g}{l_1}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) - 0.2x_2$$

$$\dot{x}_3 = x_4;$$

$$\dot{x}_4 = -\frac{g}{l_2}\sin x_3 - 0.2x_4$$




```
function dx=ff(x)
g=9.8; m1=1;m2=1;a1=1;a2=1;
x1=x(1);x2=x(2);
x3=x(3);x4=x(4);

dx2=-(g/a1)*sin(x1)+(m2*g/(m1*a1))*cos(x3)*sin(x3-x1)-0.2*x2;
dx4=-(g/a2)*sin(x3)-0.2*x4;
dx=[dx2;dx4]
```

At this point, it is time to give a summary on
what we have achieved and
what will be studied

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Main Problems of the Course

- Analysis: Solutions to LTI systems, stability etc.
- Controllability and observability;
- Feedback design and construction of observers
- Optimal control
- Lyapunov stability

Course project will involve feedback design of
an inverted pendulum system.

- Design a feedback law through the linearized system
- Apply the feedback law to the nonlinear system
- Use simulink to check if desirable performance requirements are satisfied.

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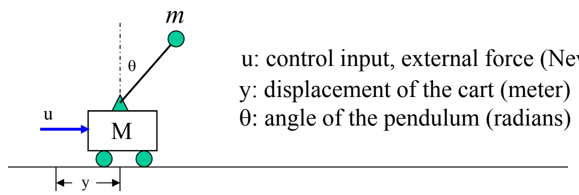
Course Project

A cart with an inverted pendulum (page 22, Chen's book)

State:



u: control input, external force (Newton)
y: displacement of the cart (meter)
θ: angle of the pendulum (radians)

$$x = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

The control problems are

1: Stabilization: Design a feedback law u=Fx such that $x(t) \to 0$ for x(0) close to the origin.
2: For $x(0)=(0,0,-\pi,0)$, apply an impulse force ($u(t)=u_{max}$ for $t \in [0,t_0]$) to bring θ to a certain range and then switch to the linear controller so that $x(t) \to 0$.

Assume that there is no friction or damping. The nonlinear model is as follows.

$$\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + ml\dot{\theta}^2 \sin\theta \\ g\sin\theta \end{bmatrix}$$

$m = 1kg$: mass of the pendulum
$l = 0.2m$: length of the pendulum
$M = 5kg$: mass of the cart, $g = 9.8$ 35

Linearize the system at x=0;

$$\begin{bmatrix} M+m & ml \\ 1 & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u \\ g\theta \end{bmatrix} \qquad x = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

The state space description for the linearized system

$$\dot{x} = Ax + Bu$$

Problems:

1. Find matrices A, B for the state space equation.
2. Design a feedback law $u=F_1x$ so that $A+BF_1$ has eigenvalues at $-1 \pm j1; -2.5$ and -5. Build a simulink model for the closed-loop linear system. Plot the response under initial condition x(0)=[1.5,0,1,-3].
3. Build a simulink model for the original nonlinear system, verify that stabilization is achieved by $u=F_1x$ when x(0) is close to the origin. Find the maximal θ_0 so that the nonlinear system can be stabilized from $x(0)=(0,0,\theta_0,0)$.
4. For $x(0)=(0,0,\pi/5,0)$, compare the response y(t) and θ(t) for the linearized system and the nonlinear system under the same feedback $u=F_1x$.

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5. Assume that the initial condition is $x(0)=(0,0,-\pi,0)$.
For the nonlinear system, construct a switching law to bring the pendulum upward and stabilized at $x=0$. (cart still at $y=0$, pendulum inverted, $\theta=0$).
An initial impulse control is applied with $u(t)= u_{max}$ for $t\in(0,t_0]$ and $u(t)=0$ for $t \geq t_0$. After the angle is within a small range, i.e., $|\theta|\leq\theta_d$, switch to a linear controller $u=F_2x$. Find u_{max}, t_0, θ_d, and F_2 so that the following requirements are satisfied:
1) $|y(t)|\leq 1$ for all $t>0$ or keep the maximal y as small as possible.
2) $|y(t)| \leq 0.02$ for $t > 2.5$.
3) $|u|\leq 150$ for all $t>0$.

Note:
In all the simulation, please choose a fixed sampling period: 0.001second

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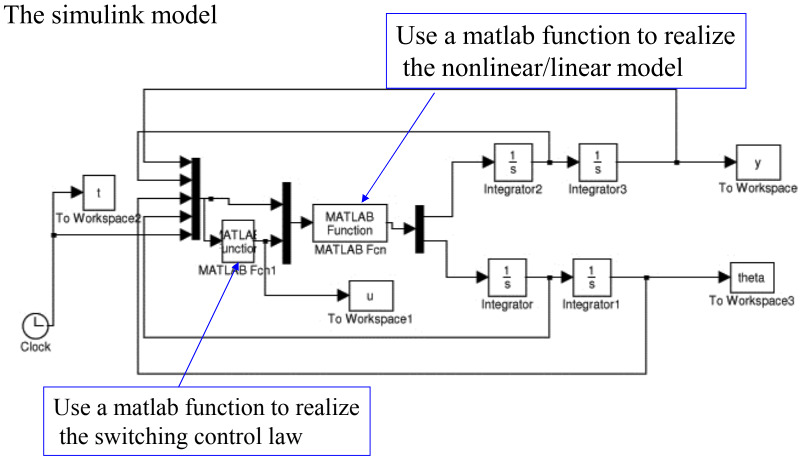
Some guidelines:

The simulink model



You may use a not so good control law to check if your simulink model is built correctly.

$$u = [0.7071 \quad 3.1831 \quad 125.5455 \quad 16.5057]x$$
$$= 0.7071y + 3.1831\dot{y} + 125.5455\,\theta + 16.5057\dot{\theta}$$

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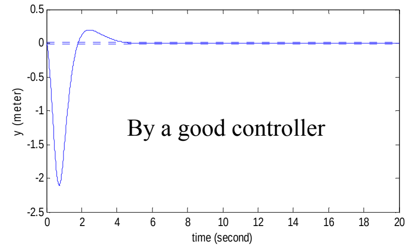
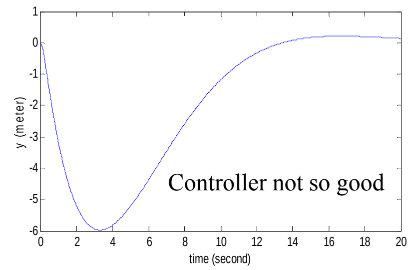
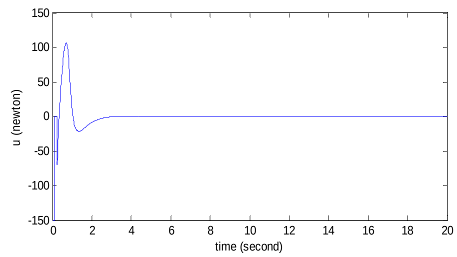
Sample design results:







An animation code will be provided

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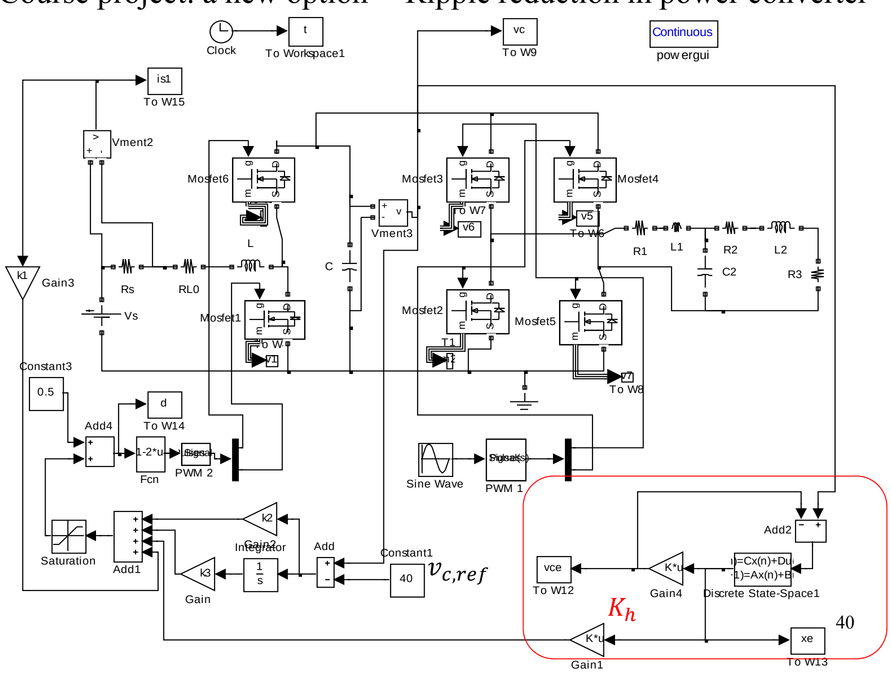
Course project: a new option-- Ripple reduction in power converter



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Everything outside the red box is provided:

```
Vs=24;hh=0.000002;%(hh is sampling time for simulation)
%boost converter parameters
Rs=0.1;RL0=0.067;L=0.00033;C=0.00013;
%inverter parameters;
R1=0.056;L1=0.0001;R2=0.15;L2=0.00068;
C2=0.00002;R3=50;
%Mosfet parameters
Ron=0.007;Rd=1e-4;Vf=0.1;Rsn=1e6;(snubber resistance);Cs=1e-6;
k1=-2.4;k2=-0.06;k3=-1;
```

PWM frequency=12000Hz, sinusoidal function frequency = 60Hz

Duty cycle for the boost converter 0< d <0.75

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The boost converter is designed so that the dc-link voltage vc will track a reference voltage, $v_{c,ref}$. In the figure, $v_{c,ref} = 40V$. When an inverter is connected as a load, there will be ripples in vc, with frequency twice that of the inverter output voltage.

Objective:

1. Design a 5th-order discrete-time observer, to estimate the first order and second order harmonics of vc. (see slide 12).
   For computer implementation, the observer is discretized with sampling frequency 12000Hz (same as PWM frequency).
   Denote the estimated state as vce.
2. Use the first order harmonics as feedback to reduce the ripple of vc
   The gain Kh will be Kh=[0 k4 k5 0 0].
   Choose k4, k5 by trial and error to achieve minimal ripple.

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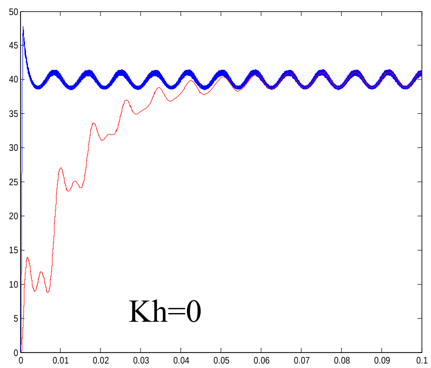
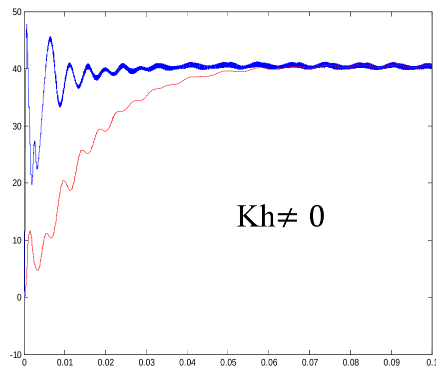
Results:

Blue curve: dc link voltage: vc
Red curve: estimated dc link voltage, vce





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- We are going to learn how to design a good control law.
- Before that, we need to study
  - Controllability and observability;

- We need some background on linear algebra:
  - positive-definiteness of a square matrix.

They are also essential to Lyapunov stability and optimal control.

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Quadratic functions and positive-definiteness (§3.9)

Given a symmetric matrix P=P' ($p_{ij}=p_{ji}$).
A quadratic function can be defined as

$$V(x) = x'Px$$

Example: $V_1(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$

x' P x

$$V_2(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_1x_3 + 2fx_2x_3$$

For higher order vector spaces, $V(x) = x'Px = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}x_ix_j$

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Definition:
A symmetric matrix P is said to be positive definite, denoted by P > 0, if x'Px > 0 for all x≠0. It is said to be positive semidefinite, denoted by P ≥ 0, if x'Px ≥ 0 for all x.

- Under what condition is V(x)=x'Px positive definite?
- This depends on the eigenvalues of P.

Compare the eigenvalues of

$$P_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

{a+jb, a-jb} for P_1, $\dfrac{a + c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$ for P_2

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Theorem: A real symmetric matrix has real eigenvalues.

Proof: Suppose that λ is an eigenvalue, possibly complex, v is the eigenvector such that $Pv = \lambda v$. The complex conjugate transpose of v is v^*, the complex conjugate transpose of P is P'. We have

$$(v^*Pv)^* = v^*P^*v = v^*P'v = v^*Pv$$

v^*Pv must be a real number. Also recall that v^*v is a real number. From $Pv=\lambda v$, we have

$$v^*Pv = \lambda v^*v \Rightarrow \lambda \text{ must be a real number}$$

Theorem: A real symmetric matrix P is always diagonalizable.

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Theorem: A symmetric matrix P is positive definite ($P > 0$) if and only if its eigenvalues are all positive.

Proof: There exist diagonal matrix D and orthogonal matrix U such that $P=UDU'$.

Consider the quadratic form $z'Dz$. Have

$$z'Dz = d_{11}z_1^2 + d_{22}z_2^2 + \ldots + d_{nn}z_n^2 > 0 \text{ for all } z \neq 0$$

Let $z=U'x$. $z=0$ iff $x=0$. Hence

$$x'Px = x'UDU'x = z'Dz > 0 \text{ for all } x \neq 0$$

Theorem: A symmetric matrix P is positive definite iff there exists a nonsingular matrix N such that $P=NN'$.

Proof: ….

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In summary:

Given a symmetric matrix P.

- All the eigenvalues and eigenvectors are real.
- Exists a matrix U, UU'=U'U=I, and a diagonal D, such that P=UDU'.
- P is positive definite iff
  - all eigenvalues are positive;
  - exists nonsingular N such that P=NN';
- P positive semi-definite iff
  - all eigenvalues are non-negative;
  - exists N such that P=NN';
- P negative definite iff
  - all eigenvalues are negative;
  - exists nonsingular N such that P= – NN'

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Today: some miscellaneous problems about LTI systems

- How to deal with complex eigenvalues
- Realization of a transfer function
- Simulation of systems by using Simulink
- Course project
- Quadratic functions and positive-definiteness

Next Time: Chapter 6.

- Controllability and Observability

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Problem set #8

1. Use the first definition of a matrix function to compute e^{At} for

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

2. Find a state space realization for

$$G(s) = \frac{s^3 + 2s^2 + 3s + 4}{s^4 + 3s^3 + 4s^2 + 4s + 2}.$$

Use integrators and amplifiers to construct a Simulink model for it. Let the input be a step signal: u(t)=0 for t<0 and u(t)=2 for t > 0. Choose the sampling time to be T=0.1. Simulate the output under 0 initial condition and plot the output response for t=0 to t=15. (print the model and the output response). You can try different input signals.

3. Construct the simulink model on page 33 (two link pendulum) and run simulation from t=0 to t=20, with initial condition x(0)=(0.5,0,-1,1). Choose sampling time=0.001second. Plot the two outputs theta1 and theta2.