16.513 Control Systems

Today, we are going to cover part of Chapter 6
and part of Chapter 8

- – Controllability and Observability
- – State Feedback and State Estimators

Last Time :

- • Controllability
- • Observability
- • Canonical decomposition
    - – Controllable/uncontrollable
    - – Observable/unobservable

Controllability: Definition

Consider the system

$$\dot{x} = Ax + Bu, \quad x \in R^n; \ u \in R^p.$$

Controllability is a relationship between state and input.

Definition: The system, or the pair (A,B), is said to be
controllable if for any initial state $x(0)=x_0$ and any final
state x_d, there exist a finite time $T > 0$ and an input u(t),
$t \in [0,T]$ such that

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau = x_d \qquad (1)$$

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$
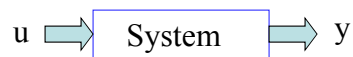
Equivalent conditions for controllability:

1) $W_c(t)$ is nonsingular for any $t > 0$.
2) The matrix $G^c = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ has full row rank, i.e., $\rho(G^c) = n$.
3) The matrix $G^c_{n-p+1} = [B \ AB \ A^2B \ \dots \ A^{n-p}B]$ has full row rank $(p = \text{rank}(B))$
4) $M(\lambda) = [A-\lambda I \ B]$ has full row rank at every eigenvalue of A.

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Observability: A dual concept

Consider an n-dimensional, p-input, q-output system:

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

u ⟹ [System] ⟹ y

Assume that we know the input and can measure the output, but have no access to the state.

Definition: The system, is said to be observable if for any unknown initial state $x(0)$, there exists a finite $t_1 > 0$ such that $x(0)$ can be exactly evaluated over $[0,t_1]$ from the input u and the output y. Otherwise the system is said to be unobservable.

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Equivalent conditions for observability:

1) $W_o(t)$ is nonsingular for some $t > 0$.
2) The observability matrix

$$G^o_{n\text{-}p+1} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-p} \end{bmatrix} \quad \text{or} \ (G^o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix})$$

has full column rank, $\rho(G^o_{n\text{-}p+1}) = n$. (p=rank(C))

3) The matrix

$$M^o(\lambda) = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

has full column rank at every eigenvalue of A.

Controllability decomposition

Theorem: Suppose that $\rho(G^c) = n_1 < n$. Let Q be a nonsingular matrix whose first n_1 columns are LI columns of G^c. Let $P=Q^{-1}$ and $z=Px$. Then

$$\overline{A} = PAP^{-1} = \begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ 0 & \overline{A}_{\bar{c}} \end{bmatrix}, \quad \overline{B} = PB = \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix}, \quad \overline{A}_c \in R^{n_1 \times n_1}, \overline{B}_c \in R^{n_1 \times p}$$

$$\overline{C} = \begin{bmatrix} \overline{C}_c & \overline{C}_{\bar{c}} \end{bmatrix}$$

Moreover, the pair $(\overline{A}_c, \overline{B}_c)$ is controllable and
$\overline{C}_c(sI - \overline{A}_c)^{-1}\overline{B}_c + D = C(sI - A)^{-1}B + D$

$\dot{z}_1 = \overline{A}_c z_1 + \overline{A}_{12}z_2 + \overline{B}_c u$

$\dot{z}_2 = \overline{A}_{\bar{c}}z_2$

$y = \begin{bmatrix} \overline{C}_c & \overline{C}_{\bar{c}} \end{bmatrix}z$

$\dot{z}_1 = \overline{A}_c z_1 + \overline{B}_c u$

$y = \overline{C}_c z_1$

is controllable and has the same transfer function as the original system.

The state z_2 is uncontrollable

$\dot{z}_2 = \overline{A}_{\bar{c}}z_2$

The control has no effect on it

Observability decomposition (follows from duality)

Theorem: Suppose that $\rho(G^o) = n_1 < n$. Let P be a nonsingular matrix whose first n_1 rows are LI rows of G^o. Then

$$\overline{A} = PAP^{-1} = \begin{bmatrix} \overline{A}_o & 0 \\ \overline{A}_{21} & \overline{A}_{\bar{o}} \end{bmatrix}, \quad \overline{B} = PB = \begin{bmatrix} \overline{B}_o \\ \overline{B}_{\bar{o}} \end{bmatrix}, \quad \overline{A}_o \in R^{n_1 \times n_1}, \overline{B}_o \in R^{n_1 \times p}$$

$$\overline{C} = \begin{bmatrix} \overline{C}_o & 0 \end{bmatrix}, \qquad\qquad\qquad \overline{C}_o \in R^{q \times n_1}$$

Moreover, the pair $(\overline{A}_o, \overline{C}_o)$ is observable and
$$\overline{C}_o(sI - \overline{A}_o)^{-1}\overline{B}_o + D = C(sI - A)^{-1}B + D$$

Discussion: After state transformation, the equivalent system is

$$\dot{z}_1 = \overline{A}_o z_1 + \overline{B}_o u \qquad\qquad z_2 \text{ may be affected by } z_1$$
$$\dot{z}_2 = \overline{A}_{21} z_1 + \overline{A}_{\bar{o}} z_2 + \overline{B}_{\bar{o}} u, \qquad \text{but has no effect on y or } z_1,$$
$$y = \overline{C}_o z_1 + Du \qquad\qquad\qquad z_2 \text{ not observable}$$

$$\dot{z}_1 = \overline{A}_o z_1 + \overline{B}_o u \qquad \text{has the same transfer function as the original system and}$$
$$y = \overline{C}_o z_1 + Du \qquad \text{is obervable.} \qquad\qquad\qquad 7$$

Today:

- Controllability and observability continued
  - Controllability/observability decomposition
  - Minimal realization
  - Conditions for Jordan form conditions
  - Parallel results for discrete-time systems
  - Controllability after sampling
- State feedback design – Pole assignment
  - Using controllable canonical form
  - By solving matrix equation

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Controllability-Observability decomposition

Theorem: All LTI system can be transformed via equivalent transformation into the following form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \overline{A}_{co} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{c\bar{o}} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{\bar{c}o} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \overline{C}_{co} & 0 & \overline{C}_{\bar{c}o} & 0 \end{bmatrix} z + Du$$

where $\left(\begin{bmatrix} \overline{A}_{co} & 0 \\ \overline{A}_{21} & \overline{A}_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\bar{o}} \end{bmatrix} \right)$ is controllable

$(\overline{A}_{co}, \overline{B}_{co})$ is controllable and $(\overline{A}_{co}, \overline{C}_{co})$ is observable.
$(\overline{A}_{\bar{c}o}, \overline{C}_{\bar{c}o})$ is observable.

Moreover, $\quad \overline{C}_{co}(sI - \overline{A}_{co})^{-1}\overline{B}_{co} + D = C(sI - A)^{-1}B + D$

$\dot{z}_1 = \overline{A}_{co}z_1 + \overline{B}_{co}u \quad$ is a controllable and observable realization
$y = \overline{C}_{co}z_1 + Du \quad$ It has the same transfer function as the original system 9

Minimal realization of a transfer matrix

Observation:
Let G(s) be a proper rational transfer matrix.
We learned earlier that there exists (A,B,C,D) such that
$\qquad$ G(s)=C(sI-A)$^{-1}$B+D
The realization is not unique.

Definition: A realization (A,B,C,D) of G which has the minimal dimension of state space is called a minimal realization of G.

Question:
Which one is a minimal realization? How to obtain a minimal realization?

Theorem: (A,B,C,D) is a minimal realization iff (A,B) is controllable and (A,C) is observable.

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Procedure to obtain a minimal realization:

An earlier result: For a strictly proper and rational matrix G(s),

- Let $d(s) = s^r + a_1 s^{r-1} + a_2 s^{r-2} + \ldots + a_{r-1} s + a_r$ be the least common denominator of all entries

- Then G(s) can be expressed as (assume G is q×p)

$$G(s) = \frac{1}{d(s)} \left[N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1} s + N_r \right], \quad N_i \in R^{q \times p}$$

- A realization of G(s) is given as:

$$A = \begin{bmatrix} -a_1 I_p & -a_2 I_p & \cdots & -a_{r-1} I_p & -a_r I_p \\ I_p & 0 & \cdots & 0 & 0 \\ 0 & I_p & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I_p \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} N_1 & N_2 & \cdots & N_{r-1} & N_r \end{bmatrix}$$

- Use equivalence transformation z = Px such that

$$\overline{A} = PAP^{-1} = \begin{bmatrix} \overline{A}_{co} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{c\bar{o}} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{\bar{c}o} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{\bar{c}\bar{o}} \end{bmatrix}, \quad \overline{B} = PB = \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$\overline{C} = CP^{-1} = \begin{bmatrix} \overline{C}_{co} & 0 & \overline{C}_{\bar{c}o} & 0 \end{bmatrix}$$

$(\overline{A}_{co}, \overline{B}_{co})$ is controllable and $(\overline{A}_{co}, \overline{C}_{co})$ is observable.

- Then $\overline{C}_{co} (sI - \overline{A}_{co})^{-1} \overline{B}_{co} = C(sI - A)^{-1} B = G(s)$ and $(\overline{A}_{co}, \overline{B}_{co}, \overline{C}_{co})$ is a minimal realization of G(s).

- If G(s) is not strictly proper, we can first decompose it as $G(s) = G_{sp}(s) + D$ where $G_{sp}(s)$ is strictly proper.

Conditions for Jordan form equations

- Equivalence transformations do not change controllability and observability
- These properties are easy to see from Jordan form.

Theorem: Assume that A has m distinct eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_m$ and has a Jordan form arranged by the eigenvalues with blocks

$$J = \mathrm{diag}[\underbrace{J_{11},J_{12},\ldots}_{\lambda_1} \vdots \underbrace{J_{21},J_{22},\ldots}_{\lambda_2} \vdots \ldots\ldots, \vdots \underbrace{J_{m1},J_{m2},\ldots}_{\lambda_m}]$$

Let the row of B corresponding to the last row of J_{ij} be $\underline{b}_{ij}$.
Let the columns of C corresponding to the first column of J_{ij} be $\underline{c}_{ij}$
Then the system is controllable iff for each i , the rows
$\{\underline{b}_{i1}, \underline{b}_{i2},\ldots,\}$ are LI.
The system is observable iff for each i, the columns
$\{\underline{c}_{i1},\underline{c}_{i2},\ldots,\}$ are LI.

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Example:

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} * \\ \underline{b}_{11} \\ \underline{b}_{12} \\ \underline{b}_{13} \\ * \\ * \\ \underline{b}_{21} \end{bmatrix}$$

$$C = \begin{bmatrix} \underline{c}_{11} & * & \underline{c}_{12} & \underline{c}_{13} & \underline{c}_{21} & * & * \end{bmatrix}$$

$\lambda_1 \neq \lambda_2$.

(A,B) is controllable iff $\{\underline{b}_{11},\underline{b}_{12},\underline{b}_{13}\}$ is LI and $\underline{b}_{21}\neq 0$

(A,C) is observable iff $\{\underline{c}_{11},\underline{c}_{12},\underline{c}_{13}\}$ is LI and $\underline{c}_{21}\neq 0$

The columns of C and the rows of B marked by "*" have no effect on controllability or observability.

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Example:

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} * & & \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ * & & \\ * & & \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ b_2 \\ b_3 \\ b_4 \\ \\ \\ b_7 \end{matrix}$$

$$C = \begin{bmatrix} 1 & & 2 & 0 & 1 & & \\ 2 & * & 1 & 0 & 0 & * & * \\ 1 & & 2 & 1 & 0 & & \end{bmatrix}$$
$$\quad\; c_1 \qquad\; c_3 \;\; c_4 \;\; c_5$$

Case 1: $\lambda_1 \neq \lambda_2$.

Case 2: $\lambda_1 = \lambda_2$.

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Example:
$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Case 1: $\lambda_1 \neq \lambda_2$. (A,B) is controllable iff $b_2 \neq 0$, and $b_4 \neq 0$
(A,C) observable iff $c_1 \neq 0$, and $c_3 \neq 0$

Case 2: $\lambda_1 = \lambda_2$, then what?

Theorem: For a single input system,
It is controllable iff for each distinct eigenvalue, there is only
one Jordan block and each element of B corresponding to the
last row of a Jordan block is nonzero;
It is observable iff for each distinct eigenvalue, there is only
one Jordan block and each element of C corresponding to the
first column of a Jordan block is nonzero.

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Discrete-Time Systems

The system described by difference equations:

$$x[k+1] = Ax[k]+Bu[k]$$
$$y[k]= Cx[k]+Du[k]$$

Results on controllability and observability are quite similar to those for continuous-time systems.

Definitions

Consider the difference equation
$$x[k +1] = Ax[k]+ Bu[k]$$
$$y[k]= Cx[k]+ D[k]$$
where $x \in R^n$, $u \in R^p$.

Definition 1: The system, or the pair (A,B), is said to be controllable if for any initial state $x(0)=x_0$ and any final state x_d, there exist an integer $k_1 > 0$ and a sequence of input u[k], $k \in [0,k_1]$ such that

$$x[k_1] = A^{k_1} x_0 + \sum_{m=0}^{k_1-1} A^{k_1-1-m} u[m] = x_d \qquad (1)$$

Definition 2 : The system, or the pair (A,C), is said to be observable if for any unknown initial state x(0), there exists a finite $k_1 > 0$ such that x(0) can be exactly evaluated over $[0,k_1]$ from the input u and the output y.   Otherwise the system is said to be unobservable.

Equivalent conditions for controllability:

The following are equivalent conditions for the pair (A,B) to be controllable:

1) The matrix $G^c = [B \; AB \; A^2B \ldots A^{n-1}B]$ has full row rank i.e., $\rho(G^c) = n$.
2) The matrix $M^c(\lambda) = [A - \lambda I \; B]$ has full row rank at every eigenvalues of A.
3) The following $n \times n$ matrix is nonsingular

$$W_{dc}[n-1] = \sum_{m=0}^{n-1} A^m BB'(A^m)'$$

Note: There may exist an integer $n_1 < n$ such that $W_{dc}(n_1-1)$ is nonsingular.

Equivalent conditions for observability:

1) The observability matrix

$$G^o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank.

2) The matrix $\quad M^o(\lambda) = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$

has full column rank at every eigenvalue of A.

3) The following $n \times n$ matrix is nonsingular

$$W_{do}[n-1] = \sum_{m=0}^{n-1} (A^m)'C'CA^m$$

Controllability after sampling

A continuous-time system

$$\dot{x} = Ax + Bu;$$

Let the sampling period be T. During the sampling period,

$$u(t) = u(kT) \text{ for all } t \in [kT, (k+1)T), k=0,1,2,\dots$$

Define u[k]:= u[kT]; x[k]=x[kT]. The relation between u[k] and x[k] is governed by the difference equation:

$$x[k+1] = A_d x[k] + B_d u[k]$$

where $A_d = e^{AT}, \quad B_d = A^{-1}[A_d - I]B$

Question: Is controllability retained after discretization?

Summary of results from §6.7

- If the pair (A,B) is uncontrollable, then (A_d,B_d) is also uncontrollable for any sampling time T.
- If all the eigenvalues of A is real, then (A,B) controllable implies that (A_d,B_d) is controllable.
- If A has complex eigenvalues, controllability maybe lost for some special sampling period T.

We use Re[x] and Im[x] to denote the real part and the imaginary part of a complex number x. Suppose (A,B) is controllable. A sufficient condition for (A_d,B_d) to be controllable is that $|Im[\lambda_i - \lambda_j]| \neq 2\pi m/T$ for m=1,2,…, whenever Re[λ_i-λ_j]=0. The condition is to ensure that the number of Jordan blocks will not increase for a particular eigenvalue. Note that if λ_i is an eigenvalue of A, then $e^{\lambda_i T}$ is an eigenvalue of A_d. If λ_i and λ_j have same real parts, $e^{\lambda_i T}$ and $e^{\lambda_j T}$ may be the same.

Example:
$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$G_c = [B \quad AB] = \begin{bmatrix} 1 & \alpha \\ 0 & \beta \end{bmatrix},$$

$$\det G^c = \beta$$

The CT system is controllable if $\beta \neq 0$. Now suppose $\beta \neq 0$. Let the sampling period be T.

$$x[k+1] = A_d x[k] + B_d u[k],$$

$$A_d = e^{AT} = e^{\alpha T} \begin{bmatrix} \cos\beta T & -\sin\beta T \\ \sin\beta T & \cos\beta T \end{bmatrix}, \quad B_d = \begin{bmatrix} \alpha\cos\beta T - \alpha + \beta\sin\beta T \\ -\beta\cos\beta T + \beta + \alpha\sin\beta T \end{bmatrix}$$

What happens when $T = \pi/\beta$?

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So far, we have studied controllability and observability

Main Problems of the Course

– Analysis: Solutions to LTI systems, stability etc.
– Controllability and observability;
– Feedback design and construction of observers
– Optimal control

Next, we will start to address design problems

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Stabilization problems

Given a LTI system

$$\dot{x} = Ax + Bu.$$

- A typical control problem is to bring the state x from any initial condition to the origin and keep it there.

  If A is stable, we only need to set u=0 and x(t) will converge to the origin asymptotically.

- Another problem is to bring x to a desirable point x_d as fast as possible and keep it there.
- Both of these problems are about stabilization at an equilibrium point. The second problem can be transformed into the first one.

For example, given an LTI system:

$$\dot{z} = Az + Bv; \quad y = Cz + Dv$$

Suppose that A is nonsingular and $v = u + u_e$. (u_e a given constant). We have

$$\dot{z} = Az + Bu_e + Bu;$$

Let $z_e = -A^{-1}Bu_e$ and define $x = z - z_e$. Then

$$\dot{x} = \dot{z} = Az + AA^{-1}Bu_e + Bu = A(z - z_e) + Bu = Ax + Bu;$$
$$\Rightarrow \quad \dot{x} = Ax + Bu.$$

Suppose that z_e is a desirable point where we would like to keep z there. If A is stable, then by setting $u = 0$, x(t) will converge to 0 from any initial x_0 and will stay there.
$\Rightarrow z(t) = x(t) + z_e$ converges to z_e and stay there.

Question: What if A is not stable?
What if A is stable but the convergence rate is too slow?

For the equation
$$\dot{x} = Ax + Bu.$$

Recall that if (A,B) is controllable, then the following control
$$u(t) = -B'e^{A'(t_1-t)}W^{-1}(t_1)[e^{At_1}x_0 - x_d] \qquad (*)$$

can bring x from any initial condition x_0 to any final destination x_d. The time duration $[0,t_1]$ can be arbitrarily small. And the control is of minimal energy.

However, this control strategy is not used in practice.

Reasons:
- Very sensitive to parameter changes and implementation error;
- Even if the state is at the origin, disturbances keep driving it away from the origin.
- Not easy to compute.
- In summary: not reliable, complicated and frustrating.

A practical and effective solution: state feedback

For the system
$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

If we let u = r – K x. Then
$$\dot{x} = (A - BK)x + Br.$$

- If A is unstable but (A,B) is controllable, we can make A – BK stable by choosing K properly;
- If A is stable but the convergence rate is too slow, we can improve the convergence property by designing K properly.

➢ The feedback law u = r –Kx is simple for implementation but very effective.

We shall find out how to design a state feedback law.

An additional tool: State estimation

What if the state cannot be obtained through measurement?

Assume that all the information that can be measured is
$y = Cx + Du$.

If the system is observable, we shall use a state-estimator,
called an observer to estimate the state from the measurement
y and the input u.

The observer is also an LTI system with input as u and y,
and its output is the estimate of the state x:

$$u \longrightarrow \boxed{\text{Observer}} \longrightarrow \hat{x}(t) \text{ - an estimate of } x(t)$$
$$y \longrightarrow$$
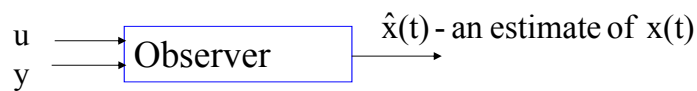
We will learn how to design an observer.

$$\boxed{\text{We Start with State Feedback Design}}$$

State feedback design: single input case

A single input single output system,

$$\dot{x} = Ax + bu, \quad y = cx \qquad \text{(assume D=0 for simplicity)}$$

where $A \in R^{n \times n}$, $b \in R^{n \times 1}$ has only one column and $c \in R^{1 \times n}$ has
one row. p=q=1.

Let $k \in R^{1 \times n}$ be a row vector. Then $kx \in R$. With state feedback
u= r – kx, we have

$$\dot{x} = (A - bk)x + br, \quad y = cx$$

Theorem: The pair (A-bk, b) is controllable iff (A,b) is
controllable. (see page 232 for proof.)

Comment: state-feedback does not change controllability property.
However, the observability of (A-bk,c) might be different from
that of (A,c).

- What can be gained from using state feedback?

    The original system: $\dot{x} = Ax + bu, \ y = cx$

    With state feedback we have: $\dot{x} = (A - bk)x + br, \ y = cx$

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    A result to be shown later: if (A,b) is controllable, then the eigenvalues of $A-bk$ can be placed anywhere by choosing k properly.

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- Example:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ A - bk = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 - k_1 & 3 - k_2 \\ 3 & 1 \end{bmatrix}$$

  – Eigenvalues of A: $\lambda_1 = 4, \lambda_2 = -2, \Rightarrow$ unstable.

  – Characteristic polynomials for A-bk is
    $$\Delta(s) = s^2 + (k_1-2)s + (3k_2 - k_1 - 8) = s^2 + a_1 s + a_0$$

➢ The two coefficients a_1 and a_0 can take any values.

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Controllable Canonical Form

For simplicity, we consider a 4th-order system. The results for the general case can be easily extended from the pattern.

Theorem: Suppose that (A,b) is controllable and

$$\det(sI - A) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$$

Let $Q := P^{-1} = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

With the state transformation $z = Px$, we have

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$$\bar{A} = PAP^{-1} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{b} = Pb = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{c} = cP^{-1} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}$$

</div>

Controllable Canonical form

Furthermore, $\bar{c}(sI - \bar{A})^{-1}\bar{b} = \dfrac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$

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Proof: We can break the transformation into two steps:

$x \rightarrow P_1 x \rightarrow P_2 P_1 x$, where

$$P_1^{-1} = Q_1 = \begin{bmatrix} b & Ab & A^2 b & A^3 b \end{bmatrix}, \quad P_2^{-1} = Q_2 = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With the first transformation, we obtain

$$\overline{A}_1 = Q_1^{-1} A Q_1 = \begin{bmatrix} 0 & 0 & 0 & -\alpha_4 \\ 1 & 0 & 0 & -\alpha_3 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \end{bmatrix}, \quad \overline{B}_1 = Q_1^{-1} B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

With the second transformation, we obtain

$$\overline{A} = Q_2^{-1} \overline{A}_1 Q_2 = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \overline{B} = Q_2^{-1} \overline{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we can verify that $\quad Q_2 \overline{A} = \overline{A}_1 Q_2, \quad Q_2 \overline{B} = \overline{B}_1$ 33

Exact pole assignment

Theorem: Suppose that (A,b) is controllable. Then the eigenvalues of A-bk can be arbitrarily assigned provided that complex conjugate eigenvalues are assigned in pairs.

Proof: Let z = Px be the state transformation that transforms the equations into controllable canonical form:

$$\overline{A} = PAP^{-1} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \overline{b} = Pb = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

With $\overline{k} = \begin{bmatrix} \overline{k}_1 & \overline{k}_2 & \overline{k}_3 & \overline{k}_4 \end{bmatrix}$, we have

$$\overline{A} - \overline{b}\overline{k} = \begin{bmatrix} -\alpha_1 - \overline{k}_1 & -\alpha_2 - \overline{k}_2 & -\alpha_3 - \overline{k}_3 & -\alpha_4 - \overline{k}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\det(sI - \overline{A} + \overline{b}\overline{k}) = s^4 + (\alpha_1 + \overline{k}_1)s^3 + (\alpha_2 + \overline{k}_2)s^2 + (\alpha_3 + \overline{k}_3)s + \alpha_4 + \overline{k}_4 \quad \text{34}$$

$$\det(sI - \overline{A} + \overline{bk}) = s^4 + (\alpha_1 + \overline{k}_1)s^3 + (\alpha_2 + \overline{k}_2)s^2 + (\alpha_3 + \overline{k}_3)s + \alpha_4 + \overline{k}_4$$

This means that the eigenvalues of $\overline{A} - \overline{bk}$ can be arbitrarily assigned. How about A - bk?

If we let $k = \overline{k}Q^{-1} = \overline{k}P$,

then $A - bk = Q\overline{A}Q^{-1} - Q\overline{bk}Q^{-1} = Q(\overline{A} - \overline{bk})Q^{-1}$,

$\Rightarrow$ The eigenvalues of A - bk are the same as those of $\overline{A} - \overline{bk}$

From the proof, a procedure to design the feedback gain k can be derived.

Procedure for assigning the eigenvalues of A-bk.

Step 1. Choose the desired eigenvalue set $\{\lambda_i, i=1,2,\ldots n\}$ which contains conjugate complex pairs, e.g., $\lambda_i = -1+j2$ and $\lambda_{i+1} = -1-j2$ and obtain the coefficients of

$$\Delta_d(s) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda^n) = s^n + \overline{\alpha}_1 s^{n-1} + \cdots + \overline{\alpha}_{n-1}s + \overline{\alpha}_n$$

Step 2. Compute the characteristic polynomial of A

$$\Delta(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

and the transformation matrix, e.g., for n = 4

$$Q := P^{-1} = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Then $\bar{A} = PAP^{-1} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\quad \bar{b} = Pb = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$\bar{A} - \bar{b}\bar{k} = \begin{bmatrix} -\alpha_1 - \bar{k}_1 & -\alpha_2 - \bar{k}_2 & -\alpha_3 - \bar{k}_3 & -\alpha_4 - \bar{k}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Step 3: $\boxed{\text{Choose } \bar{k}_i = \bar{\alpha}_i - \alpha_i}$

Then $\bar{A} - \bar{b}\bar{k} = \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Step 4: $\boxed{\text{Compute } k = \bar{k}\, P.}$

then $A - bk = Q(\bar{A} - \bar{b}\bar{k})Q^{-1}$ has the desired eigenvalues $\{\lambda_i, i = 1,2,...,n\}$

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Example:

$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\quad$ eigenvalues: $0, 1, -2$, unstable

$G^c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ nonsingular, (A, B) controllable.

Step 1: The desired eigenvalues $-1, -2+j2, -2-j2$

$\Delta_d(s) = (s+1)(s+2+j2)(s+2-j2) = s^3 + \underset{\bar{\alpha}_1}{5}s^2 + \underset{\bar{\alpha}_2}{12}s + \underset{\bar{\alpha}_3}{8}$

Step 2: Characteristic polynomial of A

$\det(sI - A) = s^3 + \underset{\alpha_1}{1}s^2 - \underset{\alpha_2}{2}s + \underset{\alpha_3}{0}$

$Q = G^c \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\quad P = Q^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

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$\bar{k}_1 = \bar{\alpha}_1 - \alpha_1 = 4; \quad \bar{k}_2 = \bar{\alpha}_2 - \alpha_2 = 14; \quad \bar{k}_3 = \bar{\alpha}_3 - \alpha_3 = 8;$

$\Rightarrow \bar{k} = \begin{bmatrix} 4 & 14 & 8 \end{bmatrix}$

$k = \bar{k}P = \begin{bmatrix} 4 & 14 & 8 \end{bmatrix} \dfrac{1}{2} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 \end{bmatrix}$

Verify:

$A - bk = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -9 \\ 0 & -4 & -8 \\ 0 & 1 & 0 \end{bmatrix}$

$\det(sI - A) = \begin{bmatrix} s+1 & 2 & 9 \\ 0 & s+4 & 8 \\ 0 & -1 & s \end{bmatrix} = (s+1)(s^2 + 4s + 4 + 4)$

$= (s+1)((s+2)^2 + 2^2)$

Eigenvalues of A - bk : -1, -2 + j2, -2 - j2   ✓   39

Transfer function of the feedback system:

The original system $\dot{x} = Ax + bu, \ y = cx$

$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}$

Transfer function from u to y: $c(sI - A)^{-1}b = \dfrac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$

The system with state feedback, $\dot{x} = (A - bk)x + br, \ y = cx$

$A - bk = \begin{bmatrix} -\alpha_1 - k_1 & -\alpha_2 - k_2 & -\alpha_3 - k_3 & -\alpha_4 - k_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Transfer function from r to y:

$c(sI - A + bk)^{-1}b = \dfrac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + (\alpha_1 + k_1)s^3 + (\alpha_2 + k_2)s^2 + (\alpha_3 + k_3)s + \alpha_4 + k_4}$ 40

Compare:

$$c(sI-A)^{-1}b = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

$$c(sI-A+bk)^{-1}b = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + (\alpha_1+k_1)s^3 + (\alpha_2+k_2)s^2 + (\alpha_3+k_3)s + \alpha_4 + k_4}$$

Conclusion:
- State feedback does not change the zeros of the system.
- If (A,b) is controllable, the poles can be arbitrarily assigned.
- The feedback gain k that assign the eigenvalues is unique.
  - (Not unique if the system has multiple inputs).
- If a new pole is the same as one of the zeros, the order of the closed-loop system can be reduced. $\Rightarrow$ must be unobservable. (since the controllability is the same).

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Desirable eigenvalue region

At the first step of the procedure, we need to choose the desirable eigenvalues. How to do this?

There are some general rules, depending on the performance specs. Such as the overshoot, rise time, settling time (convergence rate).

Generally,
- Large real parts of eigenvalues $\Rightarrow$ fast convergence, short settling time
- Large imaginary parts of eigenvalues $\Rightarrow$ big oscillations and big overshoots.
- If the ratio between the imag part and the real part is appropriate, we may have small overshoot and fast rise time

Im s

Re s

A typical region for desired eigenvalues

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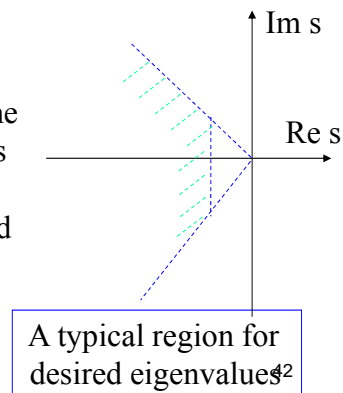
State feedback design: multiple input case

Consider a system,

$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

where $A \in R^{n \times n}$, $B \in R^{n \times p}$, $C \in R^{q \times n}$.

- We can also transform the system into a controllable canonical form.
  - The idea is extended from the single-input case;
  - The canonical form also reveals the structure to see how the poles are moved;
  - However, the procedure can be very complicated. (see §8.6.3)

- Here we will study a quite different approach. It also applies to single input systems.

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State feedback design: By solving matrix equation

- In this approach, we don't transform a system into a controllable canonical form
- How does it work? The main idea is as follows.
  - The problem: Find K s.t. A - BK has a set of desired eigenvalues, say the eigenvalues of F. This is the case if A-BK and F are similar, i.e., there exists a nonsingular matrix T s.t.,

    A - BK = TFT⁻¹

    ~ Similar matrices have same eigenvalues
  - Key: Find both K and T

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The new problem:

Given A, B and F, find K and nonsingular T such that

$$A - BK = TFT^{-1}$$

Multiply both sides from right with T, we obtain

$$AT - BKT = TF$$

Since T is nonsingular, there is a one to one correspondence between KT and K. If we let $K_0 = KT$, then $K = K_0 T^{-1}$. Now,

$$AT - BK_0 = TF \quad \Longleftrightarrow \quad AT - TF = BK_0$$

The procedure: choose $K_0 \in R^{p \times n}$. Solve $AT - TF = BK_0$ for T. If T is nonsingular, let $K = K_0 T^{-1}$. Then A-BK and F are similar. Then A-BK has the desired eigenvalues.

Main concerns:
- How to solve the matrix equation $AT - TF = BK_0$?
- Under what condition is the solution T nonsingular?

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Main concerns:
- How to solve the matrix equation $AT - TF = BK_0$?
- Under what condition is the solution T nonsingular?

Summary of the main points:
- The matrix equation can be transformed into a regular linear algebraic equation with n×n unknowns.
- It has a unique solution iff A and F have no common eigenvalues.
- If (A,B) is controllable, then the solution is generally nonsingular with K_0 arbitrarily chosen.
  - If K_0 is generated by rand(p,n) or randn(p,n), then the probability that T is nonsingular is 1.
- When p = 1, the resulting $K = K_0 T^{-1}$ is unique.
- When p > 1, the resulting $K = K_0 T^{-1}$ is not unique.
- Based on these results, optimization algorithms can be developed for improving other performances while the eigenvalues are at the desired locations.

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Transformation into a regular algebraic equation:

Example: Solve $AT - TF = BK_0$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad BK_0 = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \quad T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

$$S = AT - TF = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} - \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3t_{11} + 3t_{21} & 4t_{12} + 3t_{22} \\ 2t_{21} & 3t_{22} \end{bmatrix} = BK_0 = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$$

- Does it have a solution?
- Recognizing that we have 4 variables and 4 conditions, the above can be converted to:

$$\begin{bmatrix} s_{11} \\ s_{12} \\ s_{21} \\ s_{22} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 & 0 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} t_{11} \\ t_{12} \\ t_{21} \\ t_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \end{bmatrix}$$

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About the solution to $AT - TF = BK_0$

Theorem 1: If A and F have no common eigenvalues, then the equation has a unique solution. (§3.7)

Theorem 2: If A and F have no common eigenvalues, the necessary conditions for T to be nonsingular are that {A, B} is controllable and {F,K_0} is observable. For the single input case (p=1), T is nonsingular iff {A, B} is controllable and {F,K_0} is observable.

Theorem 3: Suppose that A and F have no common eigenvalues and (A,B) is controllable. Then for almost all K_0, T is nonsingular.

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Algorithm

- Select F having desired closed-loop eigenvalues which are different from those of A
- Choose an arbitrary K_0 such that $\{F, K_0\}$ is observable
- Solve $AT - TF = BK_0$ to obtain the unique T.

   The matlab command to solve the equation is

   $$\boxed{T = \text{lyap}(A, -F, -B*K0)}$$

- If T is non-singular, let $K = K_0 T^{-1}$.

   Then A-BK has the desired eigenvalues.

- If T is singular, which is rarely the case,

   choose a different K_0 and try again

- Finally, don't forget to check if A-BK has the desired eigenvalues. You might have typed the wrong numbers.

   $$\boxed{\text{eig}(A-B*K) = ?}$$

About the selection of F:

- First, select the desired eigenvalues with some rules
- If the desired eigenvalues are all real, simply let
  $F = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$
- If the desired eigenvalues has complex conjugate pairs,
  say, $\lambda_1, \alpha_1 + j\beta_1, \alpha_1 - j\beta_1, \alpha_2 + j\beta_2, \alpha_2 - j\beta_2$, choose

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Use T= lyap(A,-F,-B*K0), and K=K0*inv(T)

K0:				K		
1	0	1		5.3621	-2.7414	4.1724
0	1	0		0.2931	2.6379	1.7586
0	1	0		1.4571	0.4857	-2.4286
1	1	1		-3.3714	6.5429	-4.7143
1	2	3		43.5000	-21.5000	-53.0000
4	5	6		72.9000	-35.5000	-85.4000

Observe that some K have small elements, but some may have big elements. In implementation, we like to use small valued K. [51]

Observation: If there are more than one K that assign the eigenvalues of A-BK to the same locations, then there are infinitely many of them.

An interesting and meaningful problem:
Pick one from those K's which assign the eigenvalues such that the spectral norm of K, i.e., $\|K\|_2$ is minimized.
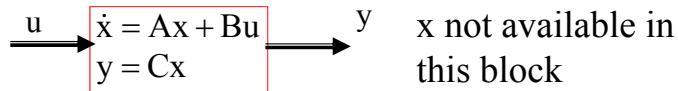
We may also develop algorithms to choose K to optimize or improve other performances, see, e.g.,

T. Hu, Z. Lin and J. Lam, ``A unified gradient approach to performance optimization under pole assignment constraint'', Journal of Optimization Theory and Applications, July, 2004
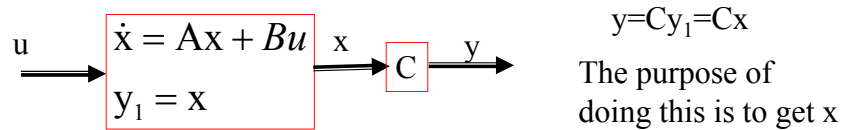T. Hu and J. Lam, ``Improvement of parametric stability margin under pole assignment,'' IEEE Transactions on Automatic Control, Vol.~44, No.~10, pp.~1938-1942, 1999.
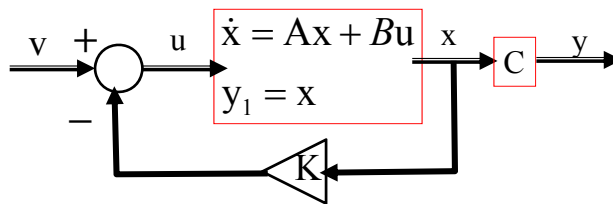52

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How to realize state-feedback in Simulink?

$$u \rightarrow \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}} \rightarrow y$$ x not available in this block

Same system can be equivalently realized with

$$u \rightarrow \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y_1 = x \end{array}} \xrightarrow{x} \boxed{C} \xrightarrow{y}$$

$y = Cy_1 = Cx$

The purpose of doing this is to get x

Under state feedback u=v-Kx,

$$v \xrightarrow{+} \bigcirc \xrightarrow{u} \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y_1 = x \end{array}} \xrightarrow{x} \boxed{C} \xrightarrow{y}$$
$$\underset{-}{\quad} \leftarrow \boxed{K} \leftarrow$$

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Today:

- Controllability and observability continued
  - Controllability/observability decomposition
  - Minimal realization
  - Conditions for Jordan form conditions
  - Parallel results for discrete-time systems
  - Controllability after sampling
- State feedback design
  - Using controllable canonical form
  - By solving matrix equations

Next Time:

- Regulation and tracking
- Robust tracking and disturbance rejection
- Stabilization
- State estimation

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Problem set #10

1. Is the following state equation controllable? Observable?

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} x$$

2. For the following state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

1) Find a state feedback u = r - k x to place the poles at -2,-3,-4.
   Use both methods (via controllable canonical form, via solving matrix
   equation, show all steps) and compare the results.
2) Find a state feedback u = r - f x to place the poles at -3+j3, -3-j3, -8
   Use both methods and compare the results.
3) Use simulink to simulate the closed-loop systems resulting from 1) and 2),
   respectively, under initial condition x(0)=[1 -1 3]' and r(t) =unit step.
   Plot y(t) for the two cases in the same figure.

3. For the following state equation

$$\dot{x} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

1) Find two different state feedback $u = r - K_1 x$ and $u = r - K_2 x$ to place the poles at -3+j3, -3-j3, -6. Try to find K_1 and K_2 such that one has relatively larger elements and the other one has relatively small elements.

2) Use simulink to simulate the closed-loop systems resulting from

Case 1: $u = r - K_1 x$, $x(0) = [1\ 2\ 3]'$ and $r(t) = 0$.

Case 2: $u = r - K_2 x$, $x(0) = [1\ 2\ 3]'$ and $r(t) = 0$.
Plot y(t) for the two cases in the same figure.
Plot $u_1(t)$ for the two cases in the same figure.
Plot $u_2(t)$ for the two cases in the same figure.

Note that $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

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