16.513 Control systems (Lecture #11)

Last time,

- Controllability and observability (Chapter 6)
- Two approaches to state feedback design (Chapter 8)
  - Using controllable canonical form
  - By solving matrix equations

Today, we continue to work on feedback design (Chapter 8)

- Other state-feedback problems
  - Regulation and tracking
  - Robust tracking and disturbance rejection
  - Stabilization of uncontrollable systems
- Full dimensional estimator
  - SISO case via observable canonical form
  - MIMO case by solving matrix equation

Project and Final Exam: Due 12pm, Dec 16, 2018

Final exam problems will be sent to
your email box at uml, at 9am, Dec 15 (Saturday).

- The written part of the project should be complete with all results clearly presented. 3 points out of 25 will be given on presentation.
- All the Matlab and Simulink files for the project and the final exam should be contained in a zip file for possible verification.
- The project and final exam should be done independently.

Please send 4 files to me via email between 12 - 12:30pm, 12/16/18
1) Project; 2) Final exam; 3) Homework #12 (pdf file, or MS word)
4) Zip file for all Matlab/Simulink source files

3

Regulation and Tracking (§8.3)

- Generally, regulation is about bringing the output or state to certain desired value asymptotically and keep it there. It can be transformed into a stabilization problem
- Tracking is a relationship between the output and reference signals. It describes the property of how the output y(t) follows a desired reference r(t).
  - The simplest tracking problem is to track a step signal.
  - A more complicated case is to track a sinusoidal signal, a polynomial signal, or periodic signals.
  - We need more advanced tools to address the second case.

4

Tracking a step signal

Recall that we use $u = r - kx$ to stabilize a system and the resulting closed-loop system is

$$\dot{x} = (A - bk)x + br, \quad y = cx$$

By choosing k appropriately, the transfer function from r to y is

$$g(s) = c(sI - A + bk)^{-1}b = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + (\alpha_1 + k_1)s^3 + (\alpha_2 + k_2)s^2 + (\alpha_3 + k_3)s + \alpha_4 + k_4}$$

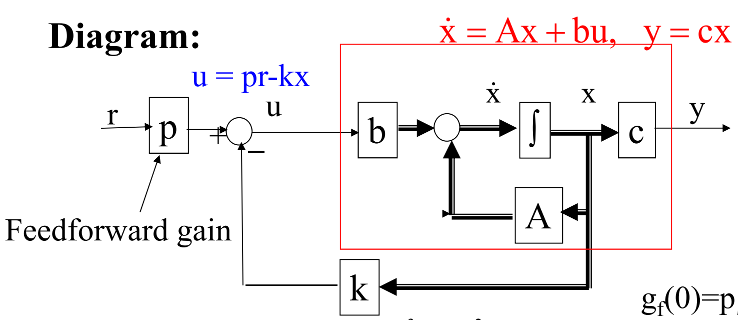
Suppose that $\beta_4 \neq 0$. Then the DC gain from r to y is $g(0)=\beta_4 / (\alpha_4 + k_4)$. If g(0) is 1 and r(t) is a step, then $y(t)-r(t) \rightarrow 0$.

If $g(0)\neq 1$, we need to introduce a feedforward gain p, i.e., let $u = pr-kx$, with $p=(\alpha_4+k_4)/\beta_4$. Then

$$g_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = p\frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + (\alpha_1 + k_1)s^3 + (\alpha_2 + k_2)s^2 + (\alpha_3 + k_3)s + \alpha_4 + k_4}$$

$g_f(0)=1 \Rightarrow$ The output y(t) can track any step signal r(t).

Diagram:



$g_f(0)=p\beta_4/(\alpha_4+k_4)=1$

$$g_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = p\frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + (\alpha_1 + k_1)s^3 + (\alpha_2 + k_2)s^2 + (\alpha_3 + k_3)s + \alpha_4 + k_4}$$

Comments:

- The feedforward strategy should work well when the parameters are accurate and there is no external disturbance.
- However, if the parameters have errors, $p\neq(\alpha_4+k_4)/\beta_4$, then the final DC gain may not be exactly one. Also, some disturbance may cause steady-state tracking errors.
- Robust tracking problem is formulated to deal with these issues.

Robust Tracking and disturbance rejection

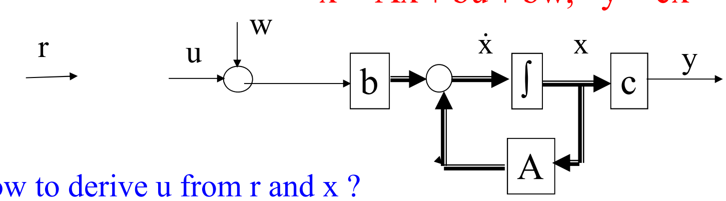
Consider an open-loop system

$$\dot{x} = Ax + bu + bw, \quad y = cx$$

where w is the disturbance. Assume that (A,b) is controllable.
Suppose that there are uncertainties in A, b and c:
$A \to A+\delta A$, $b \to b+\delta b$, $c \to c+\delta c$.

Robust tracking requires y(t) to follow a step r(t) in the presence of uncertainties and disturbances.

$$\dot{x} = Ax + bu + bw, \quad y = cx$$

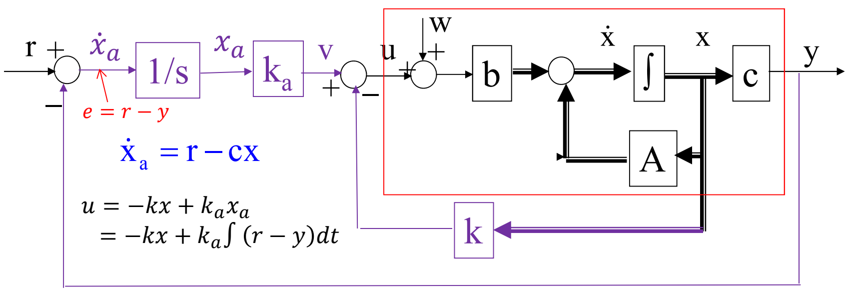


How to derive u from r and x ?

7

Proposed configuration

$$\dot{x} = Ax + bu + bw, \quad y = cx$$



$$\dot{x}_a = r - cx$$

$$u = -kx + k_a x_a = -kx + k_a \int (r-y)dt$$

$$\dot{x} = Ax - bkx + bk_a x_a + bw = (A-bk)x + bkx_a + bw$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A-bk & bk_a \\ -c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} b \\ 0 \end{bmatrix} w, \qquad y = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix}$$

Let $A_L = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix}$, $b_L = \begin{bmatrix} b \\ 0 \end{bmatrix}$. $\Rightarrow \begin{bmatrix} A-bk & bk_a \\ -c & 0 \end{bmatrix} = A_L - b_L \begin{bmatrix} k & -k_a \end{bmatrix}$

- If $c(sI-A)^{-1}b$ has no zero at s=0, then (A_L, b_L) is controllable
- Then the eigenvalues of $A_L - b_L[k \ \ -k_a]$ can be arbitrarily assigned, internal stability can be achieved.

8

- Now we discuss tracking and disturbance rejection:

Let $\Delta_f(s) = \det(sI - A_L + b_L[k \quad -k_a]) = \det\begin{bmatrix} sI - A + bk & -bk_a \\ c & s \end{bmatrix}$

Suppose $c(sI - A + bk)^{-1}b = N(s)/D(s)$ Then $\Delta_f(s) = sD(s) + k_a N(s)$

Let $y(s) = y_r(s) + y_w(s) = \hat{g}_r(s)r(s) + \hat{g}_w(s)w(s)$

$$\hat{g}_r(s) = \frac{y_r(s)}{r(s)} = \frac{k_a N(s)}{\Delta_f(s)} = \frac{k_a N(s)}{sD(s) + k_a N(s)} \quad \Rightarrow \quad \hat{g}_r(0) = 1$$

$$\hat{g}_w(s) = \frac{sN(s)}{\Delta_f(s)}, \quad \hat{g}_w(0) = 0$$ If $w(t)$ is a step signal, $y_w(t) \to 0$

Conclusions:

- The DC gain from r to y is always 1. Tracking step signal asymptotically even if parameters A,b,c change.
- The DC gain from w to y is always 0. Step disturbance can be rejected.

9

Example: $u \to g_o(s) = \dfrac{s^2 + 2s + 3}{s^3 + s^2 + s - 1} \to y$

Design a robust tracking control strategy such that y tracks a step signal r(t) asymptotically.

State space realization of $g_o(s)$:

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$A_L = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -2 & -3 & 0 \end{bmatrix}, \quad b_L = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

The basic requirement is stability. The convergence rate depends on the eigenvalues of $A_L - b_L k_L$ ($k_L = [k \quad -k_a]$)

We first choose k_L that assign the eigenvalues.

10

We use the second approach of pole assignment. Pick

$$F=\begin{bmatrix}-2 & -2 & 0 & 0\\ 2 & -2 & 0 & 0\\ 0 & 0 & -4 & -4\\ 0 & 0 & 4 & -4\end{bmatrix}$$

Pick k0=[1 1 1 1];
T=lyap(AL,-F,-bL*k0)
k_L=k0*inv(T)

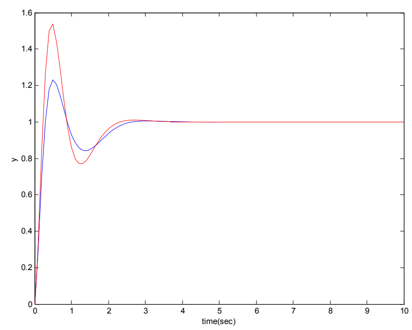
k_L = [11.0000 -14.3333 22.3333 -85.3333]
k=[11.0000 -14.3333 22.3333], ka=85.3333

y(t) is plotted in red curve.

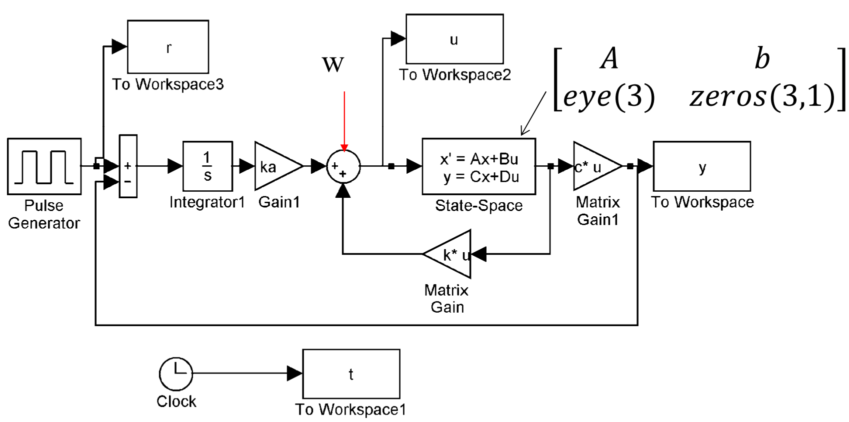
If we pick

$$F=\begin{bmatrix}-2 & -2 & 0 & 0\\ 2 & -2 & 0 & 0\\ 0 & 0 & -4 & 0\\ 0 & 0 & 0 & -8\end{bmatrix}$$

The tracking performance is improved. See y(t) plotted in blue.

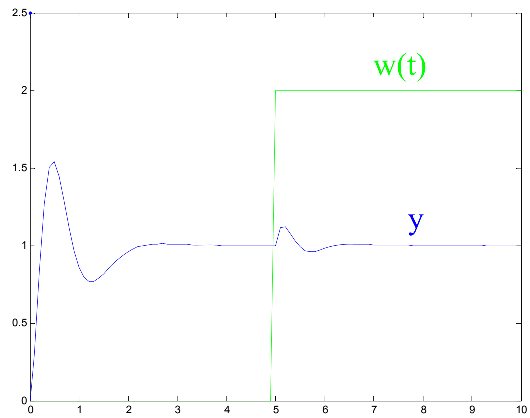


The simulink model



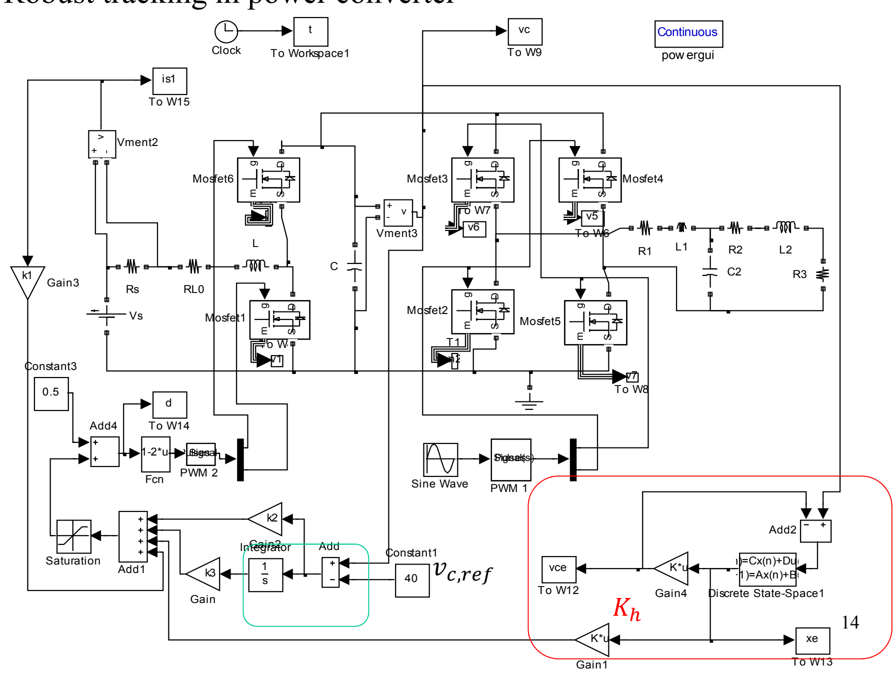
You can try to change the parameters A, or b and the tracking property is maintained.

A response in the presence of step disturbance

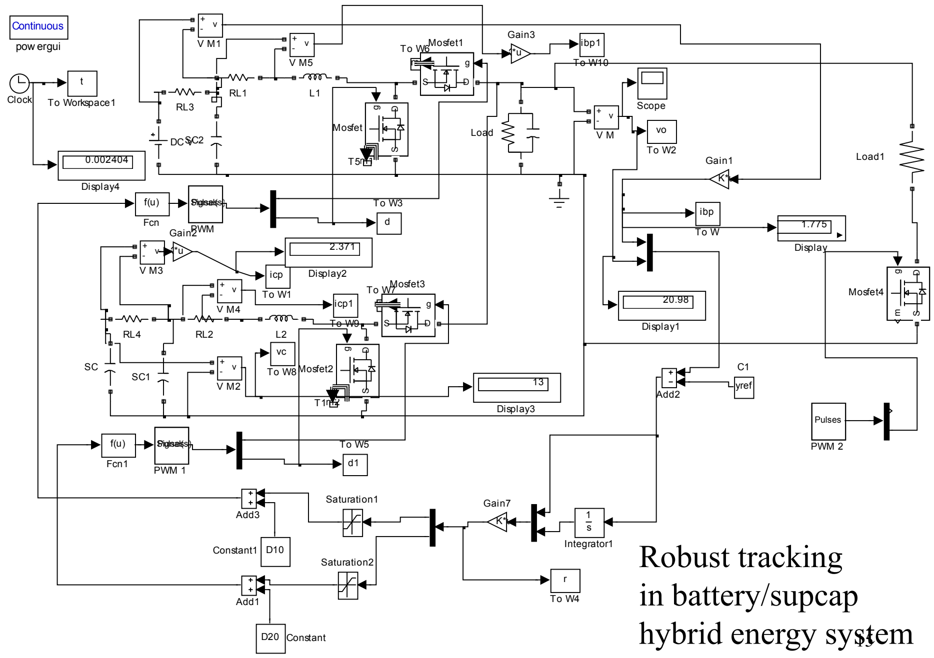


13

Robust tracking in power converter



14



Robust tracking
in battery/supcap
hybrid energy system

Problem set #11

1. Design a robust tracking strategy for the system

$$\xrightarrow{\;u\;} \boxed{g_o(s) = \frac{s^2+s+1}{s^3-s^2+2s-1}} \xrightarrow{\;y\;}$$

so that the output y follows a step signal asymptotically. Choose design parameters so that the closed-loop poles are at -2+j2, -2-j2, -4 and -8. Simulate the system from t=0s to t=20s. (print the simulink model)

1) with the given $g_o(s)$

2) Keep all the design parameters but replace $g_o(s)$ with

$$g_1(s) = \frac{s^2+(1+\delta)s+1}{s^3-s^2+(2-\delta)s-1} \qquad \text{for } \delta = 0,\ 0.5,\ 1,\ 2$$

Plot y(t) for each of the cases with 0 initial condition for the state. You may plot all responses in the same figure and identify them with the value of δ.

What is the minimal δ to make the closed-loop system unstable?

(Note: use the same state-feedback for all cases.)

Stabilization of uncontrollable systems

Recall: If (A,B) is controllable, then the eigenvalues of (A+BK) can be arbitrarily assigned.

- What if (A,B) is not controllable? Can the system be stabilized?

The original system: $\dot{x} = Ax + Bu$

Suppose that the system is transformed (by $z = Px$) into the following:

$$\dot{z} = \bar{A}z + \bar{B}u, \quad \bar{A} = PAP^{-1}, \quad \bar{B} = PB, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u, \qquad (\bar{A}_c, \bar{B}_c) \text{ controllable}$$

Note: $\text{eig}(A) = \text{eig}(\bar{A}) = \text{eig}(\bar{A}_c) \cup \text{eig}(\bar{A}_{\bar{c}})$

17

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u, \qquad (\bar{A}_c, \bar{B}_c) \text{ controllable}$$

Note: $\text{eig}(A) = \text{eig}(\bar{A}) = \text{eig}(\bar{A}_c) \cup \text{eig}(\bar{A}_{\bar{c}})$

Consider a state feedback: $u = r - Kx = r - \bar{K}z = r - \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad K = \bar{K}P$

$$\Rightarrow \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_c - \bar{B}_c\bar{K}_1 & \bar{A}_{12} - \bar{B}_c\bar{K}_2 \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} r,$$

Then $\text{eig}(A - BK) = \text{eig}(\bar{A} - \bar{B}\bar{K}) = \text{eig}(\bar{A}_c - \bar{B}_c\bar{K}_1) \cup \text{eig}(\bar{A}_{\bar{c}})$

Conclusion:

- Since $(\bar{A}_c, \bar{B}_c)$ is controllable, $\text{eig}(\bar{A}_c - \bar{B}_c\bar{K}_1)$ can be arbitrarily assigned
- $\text{eig}(\bar{A}_{\bar{c}})$ cannot be changed by any state feedback. They are called uncontrollable modes.
- For the system to be stabilizable, the uncontrollable modes have to be stable, i.e., $\text{Re}(\lambda_i(\bar{A}_{\bar{c}})) < 0$ for each i.

18

Today's topics:

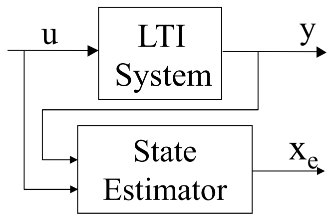
- Other state-feedback problems
  - Regulation and tracking
  - Robust tracking and disturbance rejection
  - Stabilization of uncontrollable systems
- Full dimensional estimator
  - SISO case via observable canonical form
  - MIMO case by solving matrix equation

State Estimators

- Previously, we assumed that x is available. However, generally x is not available

Q. What to do?

- Construct a system to estimate x ~ State estimator (or observer)



- Want $x_e(t)$ to be a "good" estimate of $x(t)$, i.e.,
  - $x(t) - x_e(t)$ should go to zero asymptotically with fast convergence rate.

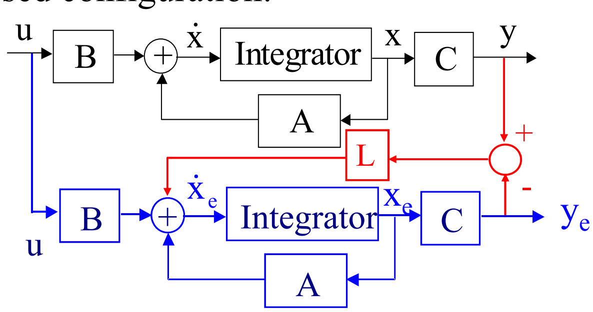
- Main result: If the system is observable, then by properly designing the state estimator, the poles of the dynamics associated with $x(t) - x_e(t)$ can be arbitrarily assigned
  - ➢ Error goes to zero as quickly as possible
- We will discuss several types of estimators
  - Full-dimensional state estimator
    - Single output case
    - Multivariable output case
  - Reduced-dimensional state estimator
- Finally, connecting state estimator and state feedback

21

Full-Dimensional State Estimators

- How to estimate x based on u and y?

  $\dot{x} = Ax + Bu; \quad y = Cx$ ~ Assuming that D = 0
- A proposed configuration:



  - Duplicate the system dynamics
  - Make correction on dx_e/dt when y and y_e are different
  - Pick the correction as a linear function of $(y - y_e)$

22

Will the proposed configuration work?

The original system

$$\dot{x} = Ax + Bu; \quad y = Cx$$

The duplicated system (estimator)

$$\dot{x}_e = Ax_e + Bu + L(y - y_e), \quad y_e = Cx_e$$

- The correction term $L(y-y_e)$ plays the essential role.
- Can we choose L appropriately to make x_e approach x?

23

- We now analyze the system
  - First, consider the estimator dynamics

$$\begin{aligned}\dot{x}_e &= Ax_e + Bu + L(y - y_e), \qquad y_e = Cx_e \\ &= Ax_e + Bu + L[Cx - Cx_e] \\ &= (A - LC)x_e + LCx + Bu\end{aligned}$$

  - Next , consider the dynamics of $e := x - x_e$

$$\begin{aligned}\dot{e} = \dot{x} - \dot{x}_e &= (Ax + Bu) - [(A - LC)x_e + LCx + Bu] \\ &= (A - LC)(x - x_e) = (A - LC)e\end{aligned}$$

$$\dot{e} = (A - LC)e$$

  - If the eigenvalues of (A - LC) have negative real parts, any error will converge to 0, $x - x_e \to 0$.
  - Can the eigenvalues of A-LC be arbitrarily assigned?

24

Theorem. If the system (A,C) is observable, then all the eigenvalues of (A - LC) can be arbitrarily placed, provided that complex eigenvalues appear in pairs

Proof : The result follows from duality

- The eigenvalues of (A-LC) and (A'-C'L') are the same.
- By Theorem 6.5: (A, C) is observable iff (A', C') is controllable (page 155)
- Now, since (A, C) is observable, (A', C') is controllable
- Eigenvalues of A' – C'L' can be arbitrarily placed

- Note: All the state feedback design procedure can be used to design state estimator, and some of them will be highlighted next

Full Dimensional State Estimator, SISO Case

Theorem. If a SISO system is observable, then it can be transformed, by an equivalent transformation, to an observable canonical form

- Observable Canonical Form:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & .. & -\alpha_n \\ 1 & 0 & 0 & .. & -\alpha_{n-1} \\ 0 & 1 & 0 & .. & -\alpha_{n-2} \\ : & : & : & : & : \\ 0 & 0 & .. & 1 & -\alpha_1 \end{bmatrix} x + \begin{bmatrix} \beta_n \\ \beta_{n-1} \\ \beta_{n-2} \\ : \\ \beta_1 \end{bmatrix} u; \quad y = [0 \quad 0 \quad 0 \quad .. \quad 1] x + du$$

$$c(sI - A)^{-1} b + d = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + .. + \beta_n}{s^n + \alpha_1 s^{n-1} + .. + \alpha_{n-1} s + \alpha_n} + d$$

The transformation matrix Q is formed as follows:

$$q_n \equiv c'$$

$$q_{n-1} \equiv A'q_n + \alpha_1 c' = A'c' + \alpha_1 c'$$

$$q_{n-i} \equiv A'q_{n-i+1} + \alpha_i c' = (A')^{i-1} + \alpha_1 (A')^{i-2} + \cdots + \alpha_{i-1} c'$$

$$q_1 \equiv A'q_2 + \alpha_{n-1} c' = (A')^{n-1} + \alpha_1 (A')^{n-2} + \cdots + \alpha_{n-1} c'$$

For example, with n=4

$$Q = \left[(A')^3 c' \quad (A')^2 c' \quad A'c' \quad c'\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 1 \end{bmatrix}$$

– With $P \equiv Q'$, then (PAP^{-1}, Pb, cP^{-1}) is in the observable canonical form.

27

Example $\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$

– Is the system observable?

$$G^o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 6 & -2 & 2 \end{bmatrix}$$

$$\left|G^o\right| = -12 \neq 0 \quad \sim \text{Observable}$$

$$\Delta(\lambda) = \begin{vmatrix} \lambda - 1 & -2 & 0 \\ -3 & \lambda + 1 & -1 \\ 0 & -2 & \lambda \end{vmatrix} = \lambda\left(\lambda^2 - 1\right) - 2(\lambda - 1) - 6\lambda$$

$$= \lambda^3 + 0\lambda^2 - 9\lambda + 2 \qquad \sim \alpha_1 = 0,\ \alpha_2 = -9,\ \alpha_3 = 2$$

28

$$q_3 = c' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \qquad q_2 = A'q_3 + \alpha_1 c' = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$q_1 = A'q_2 + \alpha_2 c' = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \\ -7 \end{bmatrix}$$

$$Q = \begin{bmatrix} 6 & 0 & 0 \\ -2 & 2 & 0 \\ -7 & 0 & 1 \end{bmatrix}; \quad P \equiv Q' = \begin{bmatrix} 6 & -2 & -7 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \overline{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 9 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\overline{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}; \qquad \overline{b} = Pb = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

29

- Now, how to select L?
  - Consider the equivalent system in Observable Canonical Form:

$$\dot{\overline{x}} = \begin{bmatrix} 0 & 0 & 0 & .. & -\alpha_n \\ 1 & 0 & 0 & .. & -\alpha_{n-1} \\ 0 & 1 & 0 & .. & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .. & 1 & -\alpha_1 \end{bmatrix} \overline{x} + \begin{bmatrix} \beta_n \\ \beta_{n-1} \\ \beta_{n-2} \\ \vdots \\ \beta_1 \end{bmatrix} u; \quad y = \begin{bmatrix} 0 & 0 & 0 & .. & 1 \end{bmatrix} \overline{x}$$

$$\dot{\overline{e}} = \left(\overline{A} - \overline{L}\overline{c} \right) \overline{e}$$

$$\overline{A} - \overline{L}\overline{c} = \begin{bmatrix} 0 & 0 & 0 & .. & -\alpha_n \\ 1 & 0 & 0 & .. & -\alpha_{n-1} \\ 0 & 1 & 0 & .. & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .. & 1 & -\alpha_1 \end{bmatrix} - \begin{bmatrix} \bar{l}_1 \\ \bar{l}_2 \\ \bar{l}_3 \\ \vdots \\ \bar{l}_n \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & .. & 1 \end{bmatrix}$$

30

$$\bar{A} - \bar{L}\bar{c} = \begin{bmatrix} 0 & 0 & 0 & .. & -(\alpha_n + \bar{l}_1) \\ 1 & 0 & 0 & .. & -(\alpha_{n-1} + \bar{l}_2) \\ 0 & 1 & 0 & .. & -(\alpha_{n-2} + \bar{l}_3) \\ : & : & : & : & : \\ 0 & 0 & .. & 1 & -(\alpha_1 + \bar{l}_n) \end{bmatrix}$$

– The characteristic polynomial:

$$\Delta_d(s) = s^n + (\alpha_1 + \bar{l}_n)s^{n-1} + .. + (\alpha_{n-1} + \bar{l}_2)s + (\alpha_n + \bar{l}_1)$$

– Suppose that the desired estimator poles are $\{\hat{\lambda}_i\}_{i=1}^n$ and the desired characteristic function is

$$\Delta_d(s) = s^n + \hat{\alpha}_1 s^{n-1} + .. + \hat{\alpha}_{n-1}s + \hat{\alpha}_n$$

Then it is clear that

$$\bar{l}_1 = \hat{\alpha}_n - \alpha_n; \bar{l}_2 = \hat{\alpha}_{n-1} - \alpha_{n-1}; .. \bar{l}_{n-1} = \hat{\alpha}_2 - \alpha_2; \bar{l}_n = \hat{\alpha}_1 - \alpha_1$$

31

$$\bar{L} = \begin{bmatrix} \hat{\alpha}_n - \alpha_n \\ \hat{\alpha}_{n-1} - \alpha_{n-1} \\ : \\ \hat{\alpha}_1 - \alpha_1 \end{bmatrix}$$

– Finally, Let $L = P^{-1}\bar{L}$, then

$$A - LC = P^{-1}\bar{A}P - P^{-1}\bar{L}\bar{C}P = P^{-1}(\bar{A} - \bar{L}\bar{C})P$$

A-LC also has the desired eigenvalues.

– L is uniquely determined by the desired eigenvalues of the estimator.

32

Procedure for designing the estimator gain

Step 1. Choose the desired eigenvalue set $\{\lambda_i, i=1,2,\ldots n\}$ of A-Lc and obtain the coefficients of

$$\Delta_d(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda^n) = s^n + \bar{\alpha}_1 s^{n-1} + \cdots + \bar{\alpha}_{n-1} s + \bar{\alpha}_n$$

Step 2. Compute the characteristic polynomial of A

$$\Delta(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

and the transformation matrix, e.g., for n = 4

$$Q = \begin{bmatrix} (A')^3 c' & (A')^2 c' & A'c' & c' \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 1 \end{bmatrix}, \quad P = Q'$$

33

Then $\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 0 & 0 & -\alpha_4 \\ 1 & 0 & 0 & -\alpha_3 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \end{bmatrix}, \quad \bar{c} = CP^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$

$$\bar{A}\text{-}\bar{b}\bar{k} = \begin{bmatrix} 0 & 0 & 0 & -\alpha_4 - \bar{l}_1 \\ 1 & 0 & 0 & -\alpha_3 - \bar{l}_2 \\ 0 & 1 & 0 & -\alpha_2 - \bar{l}_3 \\ 0 & 0 & 1 & -\alpha_1 - \bar{l}_4 \end{bmatrix}$$

Step 3: Choose $\bar{l}_i = \bar{\alpha}_i - \alpha_i$

$$\text{Then } \bar{A}\text{-}\bar{L}\bar{c} = \begin{bmatrix} 0 & 0 & 0 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & -\bar{\alpha}_3 \\ 0 & 1 & 0 & -\bar{\alpha}_2 \\ 0 & 0 & 1 & -\bar{\alpha}_1 \end{bmatrix}$$

Step 4: Compute $L = P^{-1}\bar{L}$.

then $A - Lc = P^{-1}(\bar{A} - \bar{L}\bar{c})P$ has the desired eigenvalues $\{\lambda_i, i = 1,2,\ldots,n\}$

34

Example (Continued)

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

– Find L s.t. the estimator poles are all at -3, -3, -10
– As discussed earlier, the system is observable, and

$$\Delta(\lambda) = \lambda^3 + 0\lambda^2 - 9\lambda + 2 \quad \sim \alpha_1 = 0,\ \alpha_2 = -9,\ \alpha_3 = 2$$

– The desired estimator characteristic function

$$\Delta_d(s) = (s+3)^2(s+10) = s^3 + 16s^2 + 69s + 90$$

$$\Rightarrow \bar{\alpha}_1 = 16,\ \bar{\alpha}_2 = 69,\ \bar{\alpha}_3 = 90; \qquad \bar{L} = \begin{bmatrix} \bar{\alpha}_3 - \alpha_3 \\ \bar{\alpha}_2 - \alpha_2 \\ \bar{\alpha}_1 - \alpha_1 \end{bmatrix} = \begin{bmatrix} 88 \\ 78 \\ 16 \end{bmatrix}$$

35

– From the previous example,

$$P = Q^T = \begin{bmatrix} 6 & -2 & -7 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad P^{-1} = \begin{bmatrix} 1/6 & 1/6 & 7/6 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = P^{-1}\bar{L} = \begin{bmatrix} 1/6 & 1/6 & 7/6 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 88 \\ 78 \\ 16 \end{bmatrix} = \begin{bmatrix} 139/3 \\ 39 \\ 16 \end{bmatrix}$$

– Verification:

$$A - Lc = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 139/3 \\ 39 \\ 16 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -139/3 \\ 3 & -1 & -38 \\ 0 & 2 & -16 \end{bmatrix}$$

36

$$|\lambda-(A-Lc)| = \begin{vmatrix} \lambda-1 & -2 & 139/3 \\ -3 & \lambda+1 & 38 \\ 0 & -2 & \lambda+16 \end{vmatrix}$$

$$= (\lambda^2-1)(\lambda+16)+278+76(\lambda-1)-6(\lambda+16)$$

$$= (\lambda^3+16\lambda^2-\lambda-16)+278+(76\lambda-76)-(6\lambda+96)$$

$$= \lambda^3+16\lambda^2+69\lambda+90 \quad \sim \text{As desired}$$

37

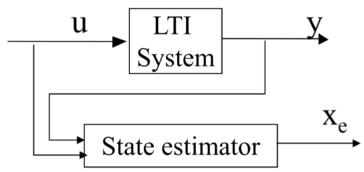
Today's topics:

- Other state-feedback problems
  - Regulation and tracking
  - Robust tracking and disturbance rejection
  - Stabilization of uncontrollable systems
- Full dimensional estimator
  - SISO case via observable canonical form
  - MIMO case by solving matrix equation

38

General MIMO State Estimator

- Dual to MIMO state feedback ~ Methods discussed earlier has a counterpart here
  - We will discuss the method based on matrix equation, assuming that the system is observable



  - A full-dimensional state estimator with $D = 0$:

$$\dot{x}_e = Ax_e + Bu + L(y - y_e), \quad y_e = Cx_e$$

Let $e = x - x_e$, the error dynamics:

$$\dot{e} = (A - LC)e$$

39

Similar to the state feedback design, the objective is to pick an observer gain L such that A-LC is equivalent to a certain F which has the desired eigenvalues, i.e.,

$A - LC = T^{-1}FT$ for a nonsingular T

$\Leftrightarrow$ $TA - FT = TLC, \quad \text{Let } L_0 = TL$

$\Leftrightarrow$ $TA - FT = L_0C,$

The procedure: Given A,C and F. Pick L_0, solve $TA - FT = L_0C$ for T.
If T is nonsingular, let $L = T^{-1}L_0$. Then $A - LC$ has the desired eigenvalues.

40

About the solution to $TA - FT = L_0C$, we have:

- The matrix equation has a unique solution iff A and F have no common eigenvalues;
- The solution T is nonsingular only if (A,C) is observable and (F,L_0) is controllable;
- In case that C has one row, T is nonsingular if and only if (A,C) is observable and (F,L_0) is controllable.

The above result can be derived from the results for state feedback design from duality: taking transpose of the equation, we obtain

$$A'T' - T'F' = C'L_0' \quad \text{as compared with}$$

$$A\,T - T\,F = B\,K_0 \quad \text{for the state feedback design}$$

Example

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x$$

Find T s.t. the estimator poles are at -5, -5, -10

– Check observability:

$$G^o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 3 & 1 & 1 \\ x & x & x \\ x & x & x \end{bmatrix}$$

~ Observable

– Select F: $F = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -10 \end{bmatrix}$

– Select L_0: $L_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \{F, L_0\}$ controllable

– Solve TA-FT=L_0C with matlab: T=lyap(-F,A,-L0*C)

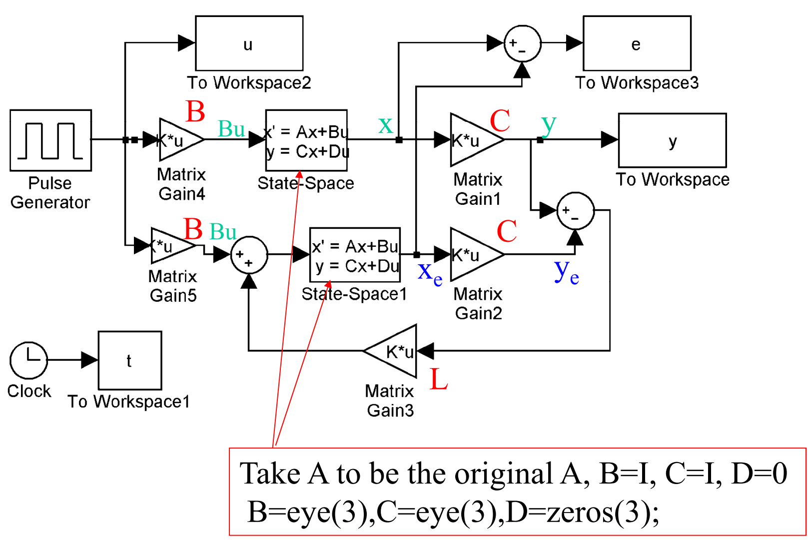
$$T = \begin{bmatrix} 0.1250 & -0.2500 & 0.2500 \\ -0.1875 & 0.3750 & 0.1250 \\ -0.0602 & 0.2206 & 0.1779 \end{bmatrix}, \quad L = T^{-1}L_0 = \begin{bmatrix} -3.5 & 24 \\ -3.75 & 15 \\ 3 & 2 \end{bmatrix}$$

– Verify: $\mathrm{eig}(A - LC) = \{-5, -5, -10\}$

– If we pick: $L_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} -117 & 24 \\ -60.5 & 15 \\ 3 & 2 \end{bmatrix}$

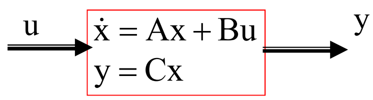
43

$$\dot{x} = Ax + Bu; \quad y = Cx; \qquad \dot{x}_e = Ax_e + Bu + L(y - y_e), \quad y_e = Cx_e$$

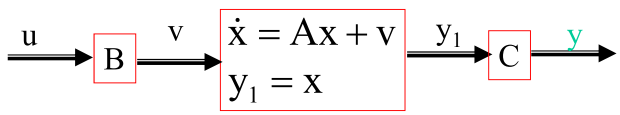


44

Explanation:

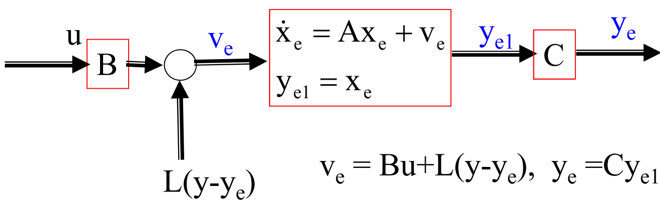


Can be equivalently realized with



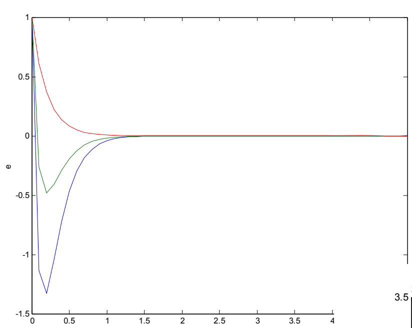
$v = Bu$, $y=Cy_1=Cx$

The purpose of doing this is to get x and x_e



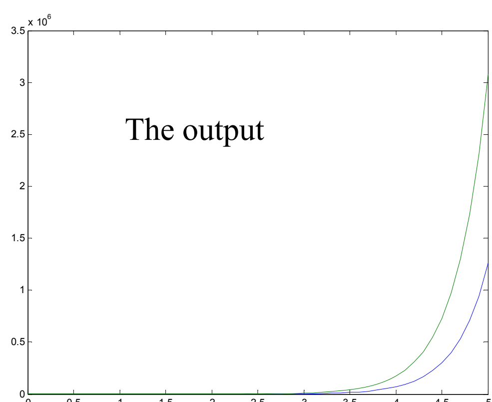
$v_e = Bu+L(y-y_e)$, $y_e =Cy_{e1}=Cx_e$

45



The estimation error: $x-x_e$

$$x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_e(0) = 0$$



Observer for discrete-time system

The original system

$$x(k+1) = Ax(k) + Bu(k);\ y(k) = Cx(k)$$

The duplicated system (estimator)

$$x_e(k+1) = Ax_e(k) + Bu(k) + L\left(y(k) - y_e(k)\right), \quad y_e(k) = Cx_e(k)$$

Define the observer error, $e(k) := x(k) - x_e(k)$

$$e(k+1) = \left(Ax(k) + Bu(k)\right) - \left[\left(A - LC\right)x_e(k) + LCx(k) + Bu(k)\right]$$

$$= \left(A - LC\right)e(k)$$

$$\boxed{e(k+1) = \left(A - LC\right)e(k)}$$

- If the eigenvalues of (A - LC) are inside the unit disk, $|\lambda_i(A)| < 1$, any error will converge to 0, $x(k) - x_e(k) \to 0$.

Use same algorithms to assign the eigenvalues of A-LC.

47

Problem set #11

2. For the system

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \\ 0 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u; \quad y = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x$$

Design an observer to estimate the state x. The poles of the observer are -4+j4, -4-j4 and -8. Simulate the system from t=0 to t=10 with x(0)=[1 1 1]', $x_e(0)$=[0 0 0]' and u=0. Plot the state x(t) in one figure and $e = x(t) - x_e(t)$ in another figure. Also print the simulink model.

48

Today's topics:

- Other state-feedback problems
  - Regulation and tracking
  - Robust tracking and disturbance rejection
  - Stabilization of uncontrollable systems
- Full dimensional estimator
  - SISO case via observable canonical form
  - MIMO case by solving matrix equation

Next Time:

- Reduced-order estimator
- Connection of state-feedback with state estimation
- LQR optimal control

Problem set #11: see slides 16, 48

49