EECE5130 Control systems - Last lecture

Last time, we constructed

- Full dimensional estimator
  - SISO case via observable canonical form
  - MIMO case by solving matrix equation

Today: We conclude the design part

- Reduced order observer
- Connection of state-feedback with state estimation
- LQR optimal control
- Rejection of sinusoidal disturbances

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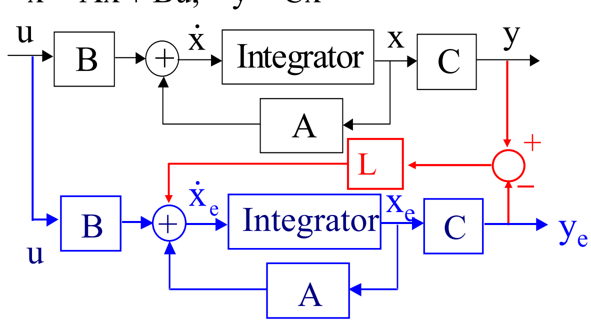
Full-Dimensional State Estimators

- The basic idea: make a copy of the original system

$$\dot{x} = Ax + Bu; \quad y = Cx$$



- Make correction on dx_e/dt based on $(y - y_e)$

$$\dot{x}_e = Ax_e + Bu + L(y - y_e), \quad y_e = Cx_e$$

- The error dynamics for $e = x - x_e$: $\dot{e} = (A - LC)e$
- Main issue: designing L for good convergence of e(t)

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- Main issue: designing L for good convergence of e(t)

$$\dot{e} = (A - LC)e$$

Under what condition can A-LC be stabilized?

- If (A,C) is observable, then the eigenvalue can be arbitrarily assigned.
- If (A,C) is unobservable, the unobservable subsystem must be stable. Suppose

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix}, \quad \bar{C} = CP^{-1} = [\bar{C}_o \quad 0],$$

$$\text{Let } \bar{L} = \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \end{bmatrix}, \quad \text{then } \bar{A} - \bar{L}\bar{C} = \begin{bmatrix} \bar{A}_o - \bar{L}_1\bar{C}_0 & 0 \\ \bar{A}_{21} - \bar{L}_2\bar{C}_o & \bar{A}_{\bar{o}} \end{bmatrix}$$

The eigenvalues of $\bar{A}_o - \bar{L}_1\bar{C}_o$ can be arbitrarily assigned.
The eigenvalues of $\bar{A}_{\bar{o}}$ cannot be changed

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For an observable pair (A,C), we studied two approaches to assign the eigenvalues of A-LC

- through observable canonical form
- by solving matrix equation

Next we study:

- Reduced-order estimator
- Combining state estimator with state-feedback
- Summary of feedback design
- Feedback design for discrete-time systems
- LQR optimal control
- Rejection of sinusoidal disturbances

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Reduced Dimensional State Estimator

- So far, the dimension of the estimator = n
- Is this really needed especially when q is not small?
  - Assume that y = Cx with C: q×n, q > 1, C full row rank.
- What is the minimum estimator dimension needed?
  - The dimension needed is (n - q)
- There are two methods
  - By transforming the state equation into a special form: the structure is clear but the procedure is complicated. An earlier method. Will not be covered.
  - By solving matrix equations: simpler procedure.

  For MIMO systems, this method offers infinite many solutions. Will be discussed next.

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Reduced Dimensional Estimator

- The full-dimensional method via matrix equality can be extended for reduced-dimensional estimator
  - Recall a full dimensional estimator:

$$\dot{x}_e = Ax_e + Bu + L(y - y_e), \quad y_e = Cx_e$$
$$\dot{x}_e = (A - LC)x_e + Ly + Bu$$

- A reduced-order equation modified from above:

$$\dot{z} = Fz + Gy + Hu \,; \qquad F \in R^{(n-q)\times(n-q)},\ G \in R^{(n-q)\times q},\ H \in R^{(n-q)\times p}$$

If $z(t) \to Tx$ for some $T \in R^{(n-q)\times n}$.

then $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \to \begin{bmatrix} Cx(t) \\ Tx(t) \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} x(t)$. If $P := \begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular,

$$\Rightarrow \quad P^{-1}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \to x(t) \qquad \boxed{\text{State recovered from y and z.}}$$

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The crucial points:

1) Ensure that $P = \begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular and
2) $z(t) \to Tx$

We first discuss how to ensure 2). Recall

$$\dot{x} = Ax + Bu \;\Rightarrow\; T\dot{x} = TAx + TBu$$
$$\dot{z} = Fz + Gy + Hu;$$

Define e : =Tx-z

Then $\dot{e} = TAx + TBu - Fz - GCx - Hu.$

If we choose T such that TA=FT+GC and H=TB,

Then $\dot{e} = (FT + GC)x + TBu - Fz - GCx - Hu = F(Tx - z)$
$= Fe$

As long as F is stable, $e(t) \to 0$ and $z(t) \to Tx(t).$ 7

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The algorithm:

- Select $F \in R^{(n-q)\times(n-q)}$ having desired estimator eigenvalues which are disjoint from those of A
- Choose $G \in R^{(n-q)\times q}$ such that $\{F, G\}$ is controllable
- Solve $TA - FT = GC$ to obtain $T \in R^{(n-q)\times n}$
- If the resulting $P = \begin{bmatrix} C \\ T \end{bmatrix}$ is non-singular, $H = TB$ and state estimator can be obtained as

$$\dot{z} = Fz + Gy + Hu$$
$$x_e = P^{-1}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix};$$

Otherwise, choose a different F or G and try again

Note: For randomly chosen G, the probability that P is nonsingular is 1. 8

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Example (Continued)

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x$$

Obtain a reduced-order estimator with pole at -10

- In this case, n = 3, q = 2, n-q = 1
- Select F ((n-q)×(n-q)): F = -10
- Select G ((n-q)×q): G = [1, 0] ~ {F, G} controllable
- Solve TA - FT = GC to obtain T ((n-q)×n)

$$T = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}$$

$$TA = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} t_1 + 3t_2 & 2t_1 - t_2 + 2t_3 & t_2 \end{bmatrix}$$

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$$FT = (-10) \cdot \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = \begin{bmatrix} -10t_1 & -10t_2 & -10t_3 \end{bmatrix}$$

$$GC = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$TA - FT - GC = \begin{bmatrix} 11t_1 + 3t_2 & 2t_1 + 9t_2 + 2t_3 & t_2 + 10t_3 - 1 \end{bmatrix} = 0$$

T is obtained as $T = \begin{bmatrix} \frac{3}{454} & \frac{-11}{454} & \frac{93}{908} \end{bmatrix}$

$$P = \begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ \frac{3}{454} & \frac{-11}{454} & \frac{93}{908} \end{bmatrix}$$

$$H = TB = \begin{bmatrix} \frac{3}{454} & \frac{-11}{454} & \frac{93}{908} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \frac{83}{908}$$

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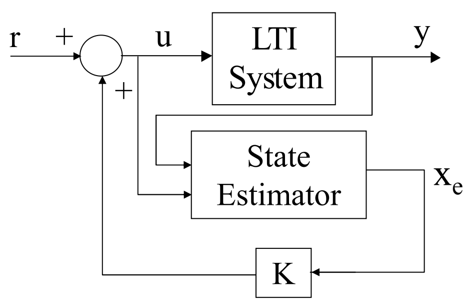
- Putting things together:

$$\dot{z} = Fz + Gy + Hu$$
$$= -10z + [1 \quad 0]y + \frac{83}{908}u$$
$$x_e = P^{-1}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}\begin{bmatrix} y \\ z \end{bmatrix} = [Q_1 \quad Q_2]\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ \frac{3}{454} & \frac{-11}{454} & \frac{93}{908} \end{bmatrix}^{-1}\begin{bmatrix} y \\ z \end{bmatrix}$$

Connection of State Estimation and Feedback

- We will assume controllability and observability
  - How can we use the estimated state?
  - Can we use $x_e(t)$ in state feedback?
  - What are the complications?



- How to analyze the combined system?
  - Put all the equations together, and then analyze them

$$\dot{x} = Ax + Bu; \quad y = Cx$$

$$\dot{z} = Fz + Gy + Hu; \; TA - FT = GC; \; x_e = [Q_1 \quad Q_2]\begin{bmatrix} y \\ z \end{bmatrix}$$

~ Assuming a reduced dimensional estimator

$$u = r - Kx_e$$

  - There are two equations involving x and z:

$$\dot{x} = Ax + B(r - Kx_e) = Ax + B(r - KQ_1 y - KQ_2 z)$$

$$= (A - BKQ_1C)x - BKQ_2 z + Br$$

$$\dot{z} = Fz + Gy + Hu \; = Fz + GCx + H(r - KQ_1Cx - KQ_2 z)$$

$$= (GC - HKQ_1C)x + (F - HKQ_2)z + Hr$$

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$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BKQ_1C & -BKQ_2 \\ GC - HKQ_1C & F - HKQ_2 \end{bmatrix}}_{A_L}\begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ H \end{bmatrix}}_{B_L} r; \quad y = \underbrace{[C \quad 0]}_{C_L}\begin{bmatrix} x \\ z \end{bmatrix}$$

- Introducing the following equivalent transformation:

$$\begin{bmatrix} x \\ e \end{bmatrix} \equiv \begin{bmatrix} x \\ z - Tx \end{bmatrix} = \begin{bmatrix} I & 0 \\ -T & I \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}$$

S, with $S^{-1} = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix}$

Transformed estimation error

$$\left(\overline{A}_L = SA_LS^{-1}; \overline{B}_L = SB_L; \overline{C}_L = C_LS^{-1}\right)$$

Recall $[Q_1 \quad Q_2]\begin{bmatrix} C \\ T \end{bmatrix} = Q_1C + Q_2T = I, \;\; TA - FT = GC; H = TB$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BKQ_2 \\ 0 & F \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r; \quad y = [C \quad 0]\begin{bmatrix} x \\ e \end{bmatrix}$$

- What can be said about poles of the combined system?
- They are the union of eig(A-BK) and eig(F)

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$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BKQ_2 \\ \boxed{0} & F \end{bmatrix} \begin{bmatrix} x \\ \boxed{e} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r; \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

- Poles of the combined system: eig(A-BK) and eig(F).
- Eigenvalues of state feedback are not affected by the eigenvalues of state estimator F, and vice versa
- Design of state feedback and state estimator can be carried out independently ~ the Separation Property and the Certainty Equivalence Property (not true in general)
- What is the transfer function from r to y?
- e is uncontrollable, as it cannot be controlled directly from r or indirectly from x $\Rightarrow$ Will not show up in $\hat{G}(s)$

$$\boxed{\hat{G}(s) = C(sI - A - Bk)^{-1} B}$$

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$$\hat{G}(s) = C(sI - A - Bk)^{-1} B$$

- In deriving the transfer function, initial conditions are assumed to be 0, i.e., $x(0) - x_e(0) = 0$, or $x(0) = x_e(0)$. The dynamics of state estimator therefore will not show up
- If $x(0) \neq x_e(0)$, the estimation error will show up in y. The error will vanish quickly if the eigenvalues of F are further to the left as compared to the eigenvalues of (A - BK)

Rule of thumb: The poles of state estimator should be 2 to 3 times faster than the poles of state feedback

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Example. A DC motor driving a load

$$u \longrightarrow \boxed{\frac{1}{s(s+1)}} \longrightarrow y$$

Design state feedback with poles at $-1 \pm j$, and a reduced state estimator with pole at -2

The two designs can be done separately

State Feedback:

Step 1: $\hat{G}(s) = \dfrac{1}{s(s+1)} = \dfrac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} \sim \alpha_1 = 1,\ \alpha_2 = 0,\ \beta_1 = 0,\ \beta_2 = 1$

Step 2: CCF $\quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

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Step 3: $\Delta_d(s) = (s + 1 - j)\,(s + 1 + j) = s^2 + 2s + 2$

$\sim\ \bar{\alpha}_1 = 2,\ \ \bar{\alpha}_2 = 2$

Step 4: $\bar{k}_1 = \bar{\alpha}_2 - \alpha_2 = 2 - 0 = 2$

$\bar{k}_2 = \bar{\alpha}_1 - \alpha_1 = 2 - 1 = 1$

$\bar{k} = [\ \bar{k}_1\ \ \bar{k}_2] = [2\ 1]$

$k = P\ \bar{k} = \bar{k}$

State Estimator :

Step 1: F= -2

Step 2: Choose G = -2 $\Rightarrow$ {F, G} is controllable

Step 3: Solve TA - FT = GC to obtain T

$$\begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - (-2)\begin{bmatrix} t_1 & t_2 \end{bmatrix} = -2\begin{bmatrix} 1 & 0 \end{bmatrix};$$

$$\begin{bmatrix} 2t_1 & t_1 + t_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \end{bmatrix} \quad \Rightarrow \quad T = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

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Step 4: $P = \begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ nonsingular. $\Rightarrow$ $P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

$$H = TB = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

The estimator:

$$\dot{z} = Fz + Gy + Hu; \quad x_e = P^{-1} \begin{bmatrix} y \\ z \end{bmatrix}$$

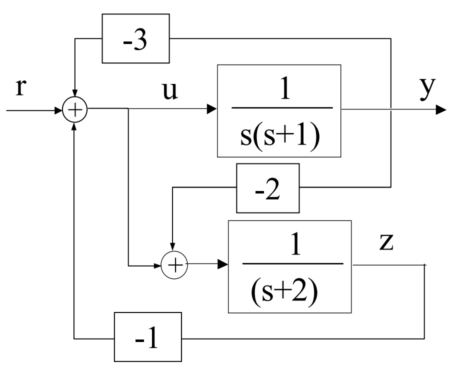
$$\dot{z} = -2z - 2y + u$$

$$x_e = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y+z \end{bmatrix};$$

Combining state feedback and state estimator:

$$u = r - kx_e = r - \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = r + \begin{bmatrix} -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix}$$

– Schematically (with $\dot{z}$ = -2z - 2y + u):



Example (same as in last lecture)

$$\dot{x} = Ax + bu = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = cx = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x$$

A full dimensional observer was built as

$$\dot{x}_e = Ax_e + Bu + L(y - y_e), \quad L = \begin{bmatrix} -3.5 & 24 \\ -3.75 & 15 \\ 3 & 2 \end{bmatrix}$$

So that the estimator poles are at -5, -5, -10

Since A is unstable, the output diverges to infinity.

We need to design a state feedback law. Let the desired eigenvalues of A-bk be -1+j1, -1-j1, -2. The feedback gain is

k=[0.3750 1.7500 1.5000]

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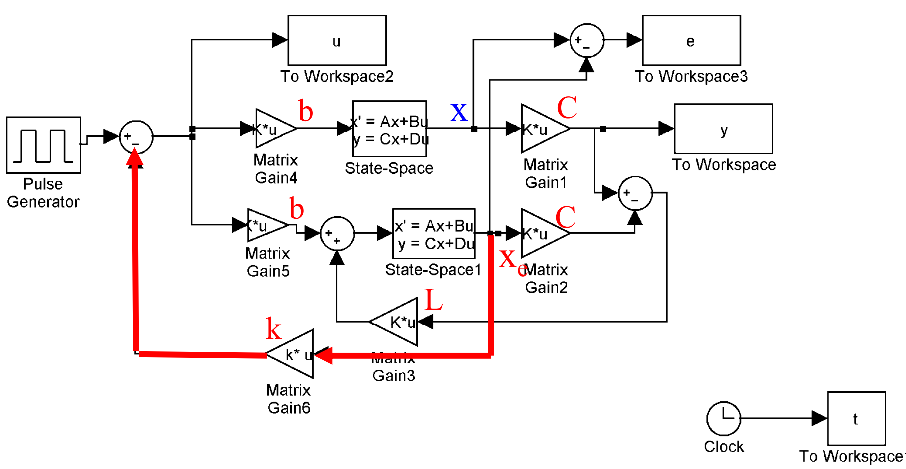

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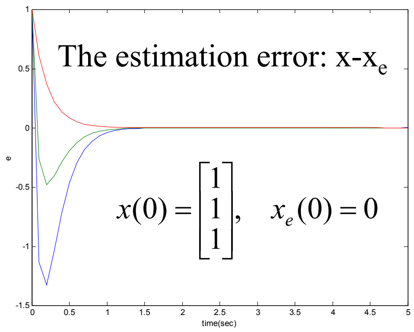

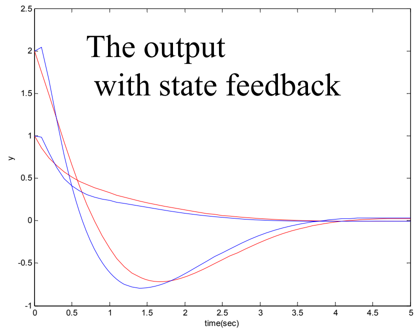

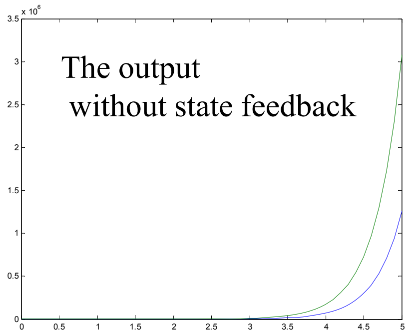

blue: from the original state x

$u = kx$

red: from estimated state x_e

$u = kx_e$

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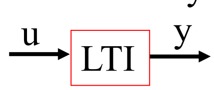
Summary of feedback design

A linear time invariant system



The system can be described by

- a proper rational transfer function (matrix) G(s)
- state space equation

$$\dot{x} = Ax + Bu; \quad y = Cx$$

If by a transfer function, we need to obtain a state-space realization (controllable and observable)

Suppose that (A,B) is controllable and (A,C) is observable.

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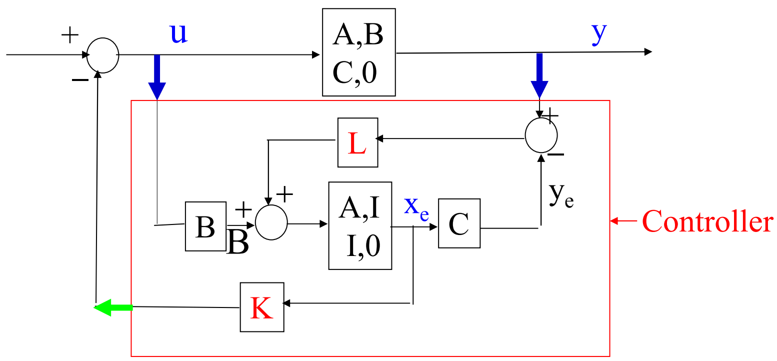
State feedback gain: find K such that A-BK has the desired eigenvalues;

Observer gain: find L such that A-LC has the desired eigenvalues. Usually assign eig(A-LC) to be further away from the imaginary axis than eig(A-BK)



The inputs to the controller are u and y, the output is Kx_e

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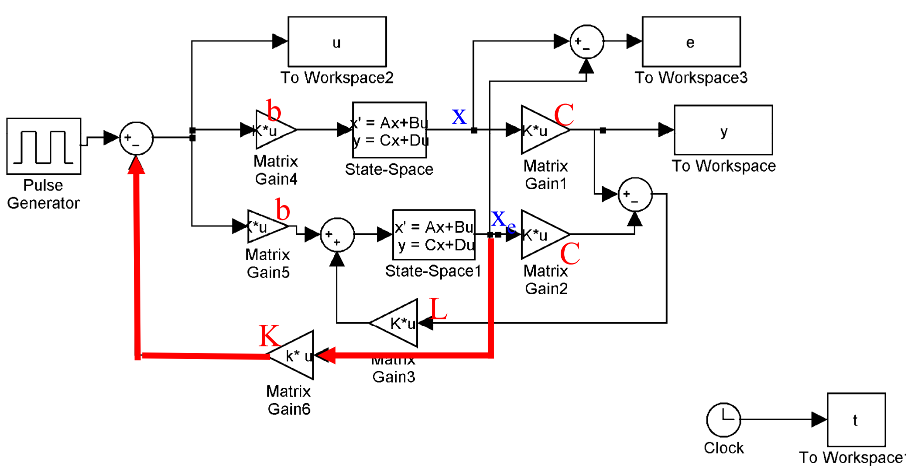
When performing simulation, we can break (A,B,C,0) into three components in serial, B, (A,I,I,0), C so that we can examine the state x



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Today's topics:

- Reduced-order estimator
- Combining state estimator with state-feedback
- Summary of feedback design
- Feedback design for discrete-time systems
- LQR optimal control
- Rejection of sinusoidal disturbances

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Feedback design for discrete-time systems

The system:

$x[k+1]=Ax[k]+Bu[k], \quad y = Cx[k]$

- The procedure for designing state feedback and observer is the same as that for continuous-time systems except for the desired eigenvalues for A-BK and A-LC: eig(A-BK) and eig(A-LC) are required to be all inside the unit circle.
- The convergence rate for $x[k+1]=(A-BK)x[k]$ is faster if the eigenvalues of (A-BK) have smaller absolute values. $(A-BK)^k$ goes to 0 faster.
- What happens if the eigenvalues of A-BK are all 0?

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- What happens if the eigenvalues of A-BK are all 0?

➢ $(A\text{-}BK)^n = 0$, $(A\text{-}BK)^{n+i}=0$, ….

How to see this? In this case, there exist a similar transformation such that

$$P(A-BK)P^{-1} = J\,,\quad J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_m \end{bmatrix}$$

Each J_i of the form

$$J_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ldots\ldots$$

$\Rightarrow$ $J_i^n = 0$ $\Rightarrow$ $J^n = 0$ $\Rightarrow$ $(A\text{-}BK)^n = P^{-1}J^n P = 0$

$\Rightarrow$ $x[k]=(A\text{-}BK)^k x[0]=0$ for all $k \geq n$.

$u = -Kx$ is called dead-beat control.

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Same thing happens for the observer

If A-LC has all zero eigenvalues, we have

$e[k]=(A\text{-}LC)^k e[0]=0$ for all $k \geq n$.

$\Rightarrow$ $x[k]=x_e[k]$ for all $k \geq n$.
$u=Kx_e=Kx$, same as direct state feedback.
$x[n+k]= (A\text{-}BK)^k x[n] = 0$ for all $k \geq n$.
$x[k]=0$ for all $k \geq 2n$.

Dead-beat control still achieved.

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Today's topics:

- Reduced-order estimator
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LQR optimal control: Motivation

An open-loop system:

$$\dot{x} = Ax + Bu; \quad y = Cx, \quad x \in R^n, \ u \in R^p, \ y \in R^q$$

With state feedback $u = r - K u$, we have

$$\dot{x} = (A - BK)x + Br; \quad y = Cx,$$

The closed-loop performance is closely related to the eigenvalues of A-BK, but the relationship can be complicated. Generally, large real parts yield fast convergence rate.

Then why not simply assign eigenvalues with large real parts?

Note that to assign the eigenvalues to the far left of the imaginary axis, the elements of K have to be large. $u = Kx$ is large, requiring large control capacity, magnitude, or energy.

- There is a conflict between good response and limited control capacity.

Question:

- how can we balance the conflict between good transient response and small control effort?

Problem formulation:

- Use energy of u, denoted as $J_1(u)$, to measure control effort:
  - small energy implies small control effort.
- Use the energy of y, denoted $J_2(y)$, to measure the quality of the transient response
  - small energy related to fast convergence and small oscillation.
- Construct a performance index as the total sum of energy of the input and the output.
- Add flexibility by using weights, e.g., $J=c_1J_1(u)+c_2J_2(y)$
  - large c_1 implies that the control is expensive and we intend to keep it small
  - small c_1 indicates that the control is cheap and we don't care if we need to use large control magnitude or energy.

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Linear Quadratic Regulator Problem

- Problem posed and solved by R. E. Kalman (1960)

An LTI system

$$\dot{x} = Ax + Bu; \quad y = Cx, \quad x \in R^n, \ u \in R^p, \ y \in R^q$$

Assume that (A,B) is controllable and (A,C) observable.

- Objective: Given $Q \in R^{q\times q}$, $R \in R^{p\times p}$, $Q \geq 0$, $R>0$.
  For $x(0)=x_0$, find a control u(t), t>0, to minimize

$$J = \int_0^\infty (y'(t)Qy(t) + u'(t)Ru(t))dt$$

J is called the cost function. It measures the total weighted energy of the output and the control.

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$$J=\int_0^\infty (y'(t)Qy(t) + u'(t)Ru(t))d\ t$$

J contains two parts:

$$J_1=\int_0^\infty y'(t)Qy(t)dt \quad \text{and} \quad J_2=\int_0^\infty u'(t)Ru(t)dt$$

J_1 is a measure of energy for the output;
J_2 is a measure of energy for the input.

Since $Q \geq 0$, $R > 0$, we know $J_1 \geq 0$, $J_2>0$.

Usually, Q and R are chosen to be diagonal matrices. Each diagonal element represents a penalty on the corresponding output or input, e.g., suppose p=3, q=2;

$$y'Qy=\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=y_1^2+5y_2^2$$

$$u'Ry=\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=10u_1^2+u_2^2+0.1u_3^2$$

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Problem:

$$\min_{u(t)} \int_0^\infty (y'(t)Qy(t) + u'(t)Ru(t))d\ t$$

$$\text{s.t. } \dot{x}=Ax+Bu; \quad y=Cx, \quad x(0)=x_0$$

Solution:

$$\boxed{\begin{array}{l} u(t) = -R^{-1}B'Px(t), \text{ where } P>0 \text{ satisfies} \\ PA + A'P - PBR^{-1}B'P + C'QC = 0 \qquad (***) \end{array}}$$

Comments:

- (***) is called an Algebraic Riccati Equation (ARE)
- Same formula for all initial condition x_0;
- A simple linear state feedback
- The closed-loop system is

$$\dot{x}=(A-BR^{-1}BP)x, \quad x(0)=x_0$$

It is stable.

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$$J = \int_0^\infty (y'Qy + u'Ru)dt \qquad (1)$$

Proof: From standard calculus we have

$$\frac{d}{dt}(x'Px) = \dot{x}'Px + x'P\dot{x}$$

Also, $$\int_0^\infty \frac{d}{dt}(x'Px) = (x'Px)(\infty) - (x'Px)(0) \qquad (2)$$

Combining (1) and (2) to obtain

$$J - (x'Px)(0) = -(x'Px)(\infty) + \int_0^\infty (y'Qy + u'Ru + \frac{d}{dt}(x'Px))dt$$
$$= -(x'Px)(\infty) + \int_0^\infty (y'Qy + u'Ru + \dot{x}'Px + x'P\dot{x})dt$$
$$= -(x'Px)(\infty) + \int_0^\infty (x'C'QCx + u'Ru + (x'A' + u'B')Px + x'P(Ax + Bu))dt$$
$$= -(x'Px)(\infty) + \int_0^\infty (x'(C'QC + A'P + PA)x + u'Ru + u'B'Px + x'PBu)dt$$

Recall that P satisfies $PA + A'P - PBR^{-1}B'P + C'QC = 0$ ⇒

$$J - (x'Px)(0) = (x'Px)(\infty) + \int_0^\infty (x'PBR^{-1}B'Px + u'Ru + u'B'Px + x'PBu)dt$$
$$= (x'Px)(\infty) + \int_0^\infty (u + x'PBR^{-1})R(u + R^{-1}BPx)dt$$

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From last slide:

$$J - (x'Px)(0) = (x'Px)(\infty) + \int_0^\infty (u + x'PBR^{-1})R(u + R^{-1}BPx)dt$$

Now, suppose that J is finite, we should have $x(t) \to 0$ as t goes to infinity. Thus $(x'Px)(\infty) = 0$.

$$J = (x'Px)(0) + \int_0^\infty (u + x'PBR^{-1})R(u + R^{-1}BPx)dt$$

Since $R > 0$, the integrand is nonnegative. To minimize J, we have to choose $u = -R^{-1}BPx$. By doing this, we also have

$$\boxed{\min J = (x'Px)(0) = x(0)'Px(0)} \qquad \text{Q.E.D.}$$

Comments:

- The optimal cost depends only on P and x(0)
- The ARE: $PA + A'P - PBR^{-1}B'P + C'QC = 0$ has many solutions. But there is only one $P > 0$.
- The optimal control is a linear state feedback.
- The closed-loop matrix $A - BR^{-1}B'P$ is stable.

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Example (same as in last lecture)

$$\dot{x} = Ax + bu = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = Cx = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x$$

$$\text{Case 1: Pick } Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1, \ (Q = [1\ 0;0\ \ 1])$$

k=lqr(A,b,C'*Q*C,R),   k = [1.3260   2.1651   2.3134]
eig(A-b*k)={-0.3588 , -3.3858 + 0.1672i, -3.3858 - 0.1672i}

Case 2: Q=[10 0;0 10]; R=1,
k=[1.0197   4.0322   5.2158],
eig(A-b*k)={-0.4702, -3.2555, -7.5618}

Case 3: Q=[1000 0 ;0 1000]; R=1,
k=[-1.8875   31.6460   46.6354]
eig(A-b*K)={-0.4961   -3.2487   -70.7616}

- 0.4963 as Q goes to infinity

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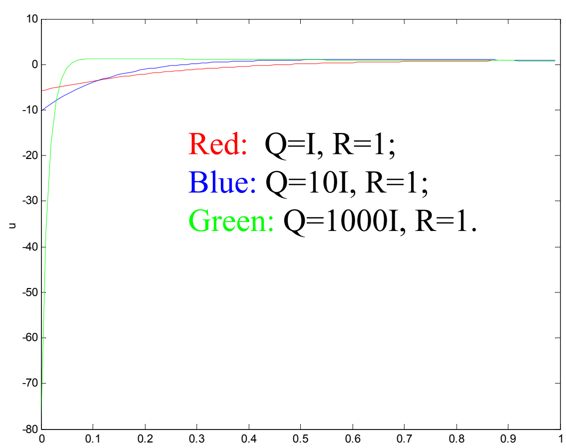
Control u: larger Q results in larger magnitude of u. Larger Q, heavier weight on y, control is relatively cheaper.



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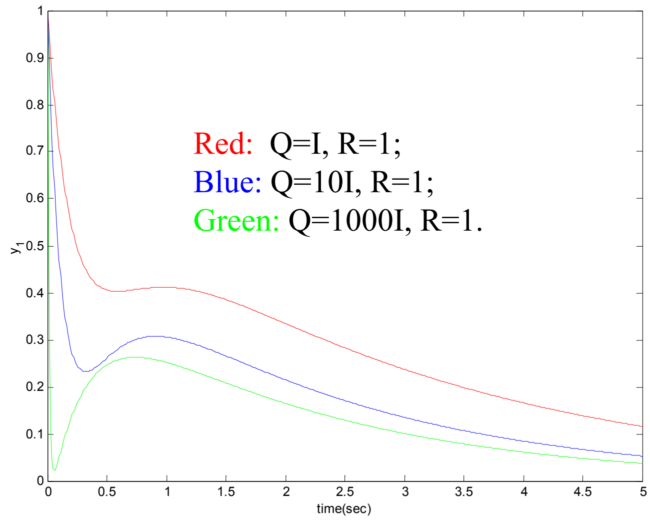
Output y_1: Larger Q results in faster convergence rate of y and smaller magnitude



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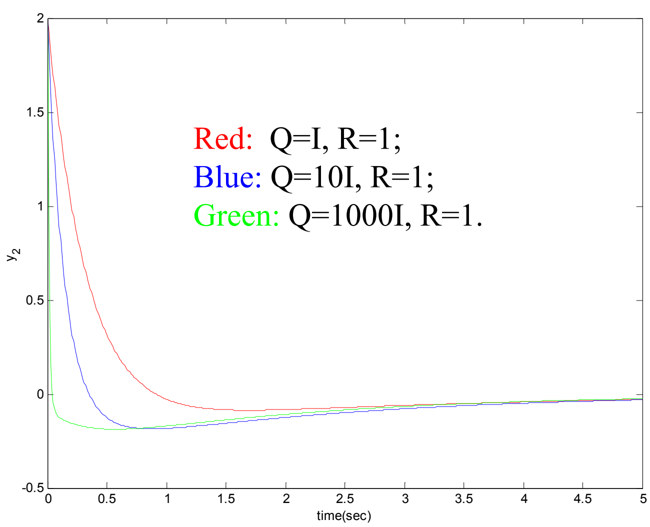
Output y_2: Larger Q results in faster convergence rate of y and smaller magnitude



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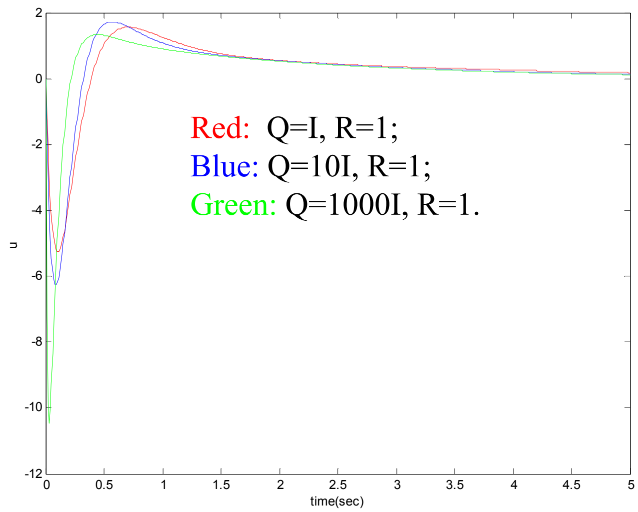
Simulation with feedback from estimated state

Control u: larger Q results in larger magnitude of u



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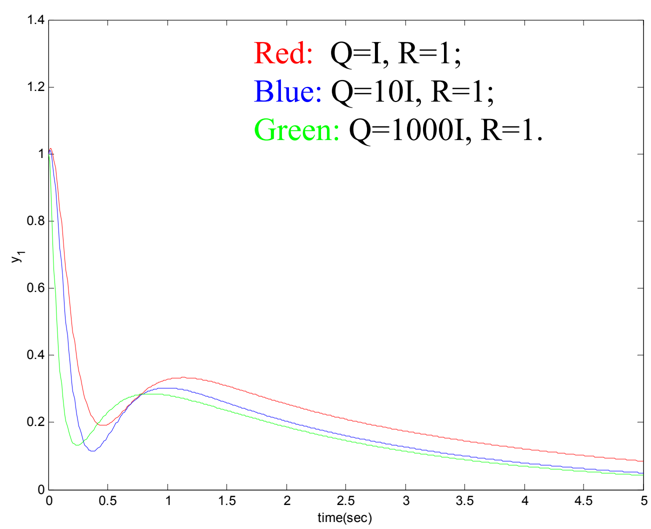
Output y_1: Larger Q results in faster convergence rate of y and smaller overshoot



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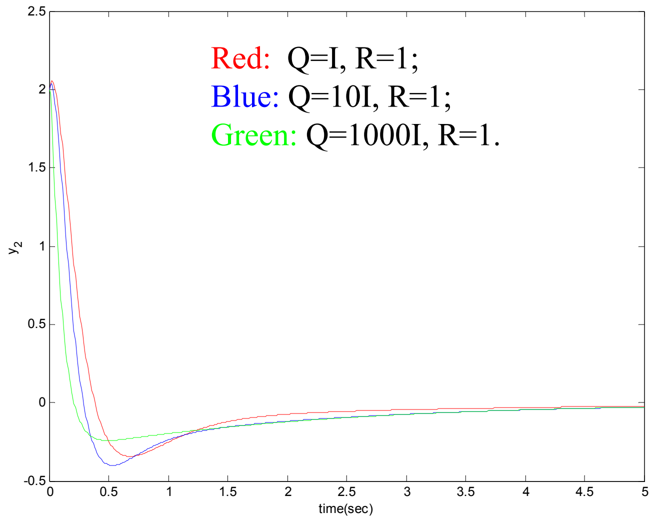
Output y_2: Larger Q results in faster convergence rate of y and small overshoot.



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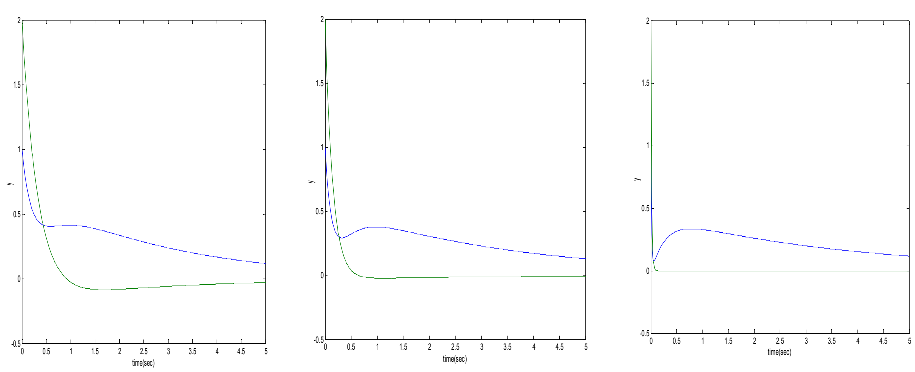
Responses under different weight on y_2 (same weight on y_1)

$$Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, R=1 \qquad Q=\begin{bmatrix}1 & 0\\ 0 & 10\end{bmatrix}, R=1 \qquad Q=\begin{bmatrix}1 & 0\\ 0 & 1000\end{bmatrix}, R=1$$



Blue: y_1
Green: y_2

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Today's topics:

- Reduced-order estimator
- Combining state estimator with state-feedback
- Summary of feedback design
- Feedback design for discrete-time systems
- LQR optimal control
- Rejection of sinusoidal disturbances

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Rejection of sinusoidal disturbances

The system

$$\dot{x} = Ax + Bu + BG\,d,\quad y = Cx$$

The disturbance d is a sinusoidal signal with frequency ω, $d(t) = d_m \sin(\omega t + \theta)$. If we know exactly the magnitude and the phase of d(t), then we can let u=-Kx-Gd, then

$$\dot{x} = Ax - BKx - BGd + BGd = (A - BK)x$$

If (A-BK) is stable, then $x(t) \to 0$.

Question: What can we do if the magnitude and phase of d(t) is unknown?

Solution: build an observer.

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The key point: we can represent $d(t) = d_m \sin(\omega t + \theta)$ as the output of a linear system:

$$\dot{v} = Sv = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} v, \quad d = c_v v = [1 \quad 0] v$$

Recall that $e^{St} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$

$$d(t) = c_v e^{St} v(0) = [1 \quad 0] \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix},$$

Since $(v_{10}, v_{20}) = (\rho \sin\theta,\ \rho\cos\theta)$, for $\rho = \| v_0 \|$, some $\theta \in [0, 2\pi)$,

$$d(t) = \rho(\sin\theta \cos\omega t + \cos\theta \sin\omega t)$$
$$= \rho \sin(\omega t + \theta)$$

Hence $d_m = \|v_0\|$ and θ are uniquely determined by the initial condition of v and there is a one to one corresp.

On the other hand, given d_m and θ, $v(0) = [d_m \sin(\theta) \quad d_m \cos(\theta)]'$

The magnitude and the phase are the polar coordinate of v(0) 49

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Now we have $\qquad d = c_v v$

$$\dot{x} = Ax + Bu + BGc_v v, \quad y = Cx$$
$$\dot{v} = Sv$$

Let $z = \begin{bmatrix} x \\ v \end{bmatrix}$

$$\dot{z} = \begin{bmatrix} A & BGc_v \\ 0 & S \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u, \qquad y = [C \quad 0] z$$

$$\Rightarrow \quad \dot{z} = A_z z + B_z u, \qquad y = C_z z$$

If (A_z, C_z) is observable, then an observer (with state z_e) can be constructed to estimate the state z. Partition z_e as

$$z_e = \begin{bmatrix} x_e \\ v_e \end{bmatrix} \Rightarrow \quad x_e \to x, \quad v_e \to v,$$

$$\Rightarrow \quad c_v v_e(t) \to c_v v(t) = d(t) = d_m \sin(\omega t + \theta)$$

The disturbance d(t) is reconstructed as $d_e(t) = c_v v_e(t)$ 50

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Now we have x and v estimated with x_e and v_e, let

$$u(t) = - Kx_e(t) - Gc_v v_e(t)$$

The closed loop system is:

$$\begin{aligned}\dot{x} &= Ax - BKx_e - BGc_v v_e + BGd \\ &= (A - BK)x + BK(x - x_e) + BG(c_v v - c_v v_e)\end{aligned}$$

Since $x\text{-}x_e \to 0$, $c_v v - c_v v_e \to 0$ and A-BK is stable, we have $x \to 0$.

Again, the key point is to consider the disturbance as part of the original system and is estimated with an observer.

The procedure of design is illustrated in the following example.

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Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + d), \quad d(t) = d_m \sin(t + \theta), \ \ y = [1 \ \ 0]x$$

Since $\omega = 1$, take $S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\quad c_v = [1 \quad 0]$

Then $\dot{v} = Sv$, $\quad d(t) = c_v v(t) \quad$ Let $z = \begin{bmatrix} x \\ v \end{bmatrix}$, we have

$$\dot{z} = A_z z + B_z u, \qquad y = C_z z$$

where $A_z = \begin{bmatrix} A & Bc_v \\ 0 & S \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$, $\quad B_z = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; $\quad C_z = [1 \ \ 0 \ \ 0 \ \ 0]$

Construct the observer,

$$\dot{z}_e = A_z z_e + B_z u + L(y - C_z z_e),$$

Need to design L

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The observer:

$$\dot{z}_e = A_z z_e + B_z u + L(y - C_z z_e),$$

The desired eigenvalue for the observer: -4+j4, -4-j4, -6, -10.
The resulting L is

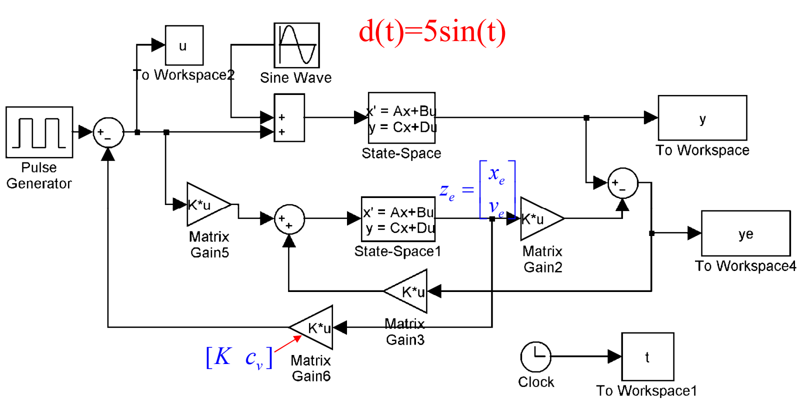
L=[25 245 968 1701]';

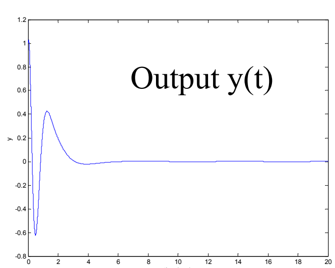
Next design the state feed back gain K.

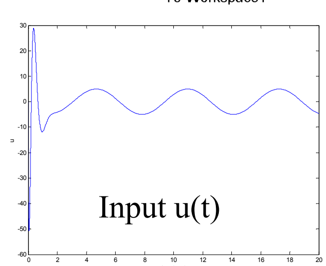
The desired eigenvalue for A-BK: -1+j1, -1-j1
The resulting K is, K=[3 3]

The control law:

$$u = -Kx_e - c_v v_e = -[K\ c_v]z_e = -[3\ 3\ 1\ 0]z_e$$







• The disturbance rejection problem mentioned above is an output regulation problem.

• The method can be extended to deal with the case where d(t) has several frequency components, such as

$$d(t)=d_1\sin(\omega_1 t+\theta_1) + d_2\sin(\omega_2 t+\theta_2) + \ldots.$$

or a periodic signal with a few harmonics

• The method can also be extended for the purpose of tracking a sinusoidal or periodic signals.

• One of my papers studies output regulation with input constraints

T. Hu and Z. Lin,``Output regulation of linear systems with bounded continuous feedback," IEEE Trans. on Automat. Contr., Vol.49, No.11, pp.1941-1953, 2004.

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<u>Project and Final Exam: Due 12pm, Dec 15, 2019</u>

Final exam problems will be sent to
<u>your email box at uml</u>, at 9am, Dec 14 (Saturday).

- The written part of the project should be complete with all results clearly presented. 3 points out of 25 will be given on presentation.
- All the Matlab and Simulink files for the project and the final exam should be contained in a zip file for possible verification.
- <u>The project and final exam should be done independently</u>.

Please send 4 files to me via email between 12 - 12:30pm, 12/15/19
1) Project; 2) Final exam; 3) Homework #12 (pdf file, or MS word)
4) Zip file for all Matlab/Simulink source files

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Problem Set #12

Problem 1: The open-loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

1) Assume that x is available for state feedback. Design an LQR control law by letting R=1 and choosing Q so that all the elements of the feedback gain K have absolute value less than 50. Requirement: $|y_1(t)|,|y_2(t)| \le 0.05$ for all $t > 5$. Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0,15]$.
2) Assume that only the output y is available. Design an observer so that the poles of the observer are -5+j5, -5-j5, -10. Choose the observer gain so that all the elements have absolute value less than 80. Form a closed-loop system along with the LQR controller in part 1). Plot $y_1(t)$ and $y_2(t)$ in the same figure for $t \in [0,15]$.

Problem 2: The open-loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} (u + d), \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The disturbance is $d(t) = d_m \sin(2t + \theta)$.

Construct a feedback law from u and y such that the disturbance is rejected. Given the initial condition of x and d(t)=sin(2t). Adjust controller parameters (K and L) such that $|u(t)| \le 20$ and $|y(t)| \le y_{max}$ for all t and y_{max} is as small as possible. Plot u(t) in one figure, y(t) in another figure.

Follow the steps of the example on slide 52. Be careful with the dimension of the matrices.

Due together with your project and final exam.