16.513 Control Systems -- Final Exam (Spring 2006)

There are 5 problems (including 1 bonus problem, total 100+20 points)

1. (30pts) Given two matrices:

$$A_1=\begin{bmatrix}-1 & -1\\ 1 & -3\end{bmatrix},\qquad A_2=\begin{bmatrix}0 & 1 & 0\\ -1 & -1 & -1\\ 1 & 0 & 0\end{bmatrix}$$

   1) (16) Compute $e^{A_1 t}, e^{A_2 t}$.

   2) (6) For the differential equation

$$\dot{x} = A_2 x + \begin{bmatrix}1\\ -1\\ 1\end{bmatrix}u;\ y = \begin{bmatrix}1 & 0 & 1\end{bmatrix}x,\ \text{with } x(0) = \begin{bmatrix}-1\\ 1\\ 1\end{bmatrix},$$ u(t)=0, what is y(t) for t > 0?

   3) (8) For the same equation as in 2), with x(0)=0, u(t)=1, what is y(t) for t>0?

2. (20pts) Consider the following system

$$\dot{x} = \begin{bmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{bmatrix}x + \begin{bmatrix}a & b\\ 1 & 1\\ b & a\end{bmatrix}u$$

$$y = \begin{bmatrix}a & 1 & b\\ b & 1 & -a\end{bmatrix}x$$

   1) Under what condition on λ_1, λ_2, a and b is the system controllable?
   2) Under what condition on λ_1, λ_2, a and b is the system observable?

3. (25pts) Perform controllable decomposition on the following system

$$\dot{x} = \begin{bmatrix}0 & -1 & 1\\ 2 & 1 & -2\\ 1 & -1 & 0\end{bmatrix}x + \begin{bmatrix}-1 & 1\\ 1 & 0\\ -1 & 1\end{bmatrix}u$$

   Design a controller u=Kx to stabilize the system.

4. (25pts) Given a transfer function

$$G(s) = \frac{2s^3 + s + 1}{s^3 + 2s^2 - 2}$$

   1) Realize G(s) with a controllable canonical form.
   2) Design a state feedback law to assign the poles at -1+j2, -1-j2, and -3.
   3) Use integrators and amplifiers to build a model to realize the system along with the state feedback. (draw a block diagram as in Simulink).

Bonus problem (20pts):

1) Consider $M = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}$, with 0 representing a full block having 0 elements. Assume that Z is square and nonsingular. Show that rank(M)=rank(X)+rank(Z).

2). Given $A \in R^{n\times n}$, $b \in R^{n\times 1}$, $c \in R^{1\times n}$. Suppose that (A,b) is controllable and $\det\begin{bmatrix} A & b \\ c & 0 \end{bmatrix} \neq 0$.

Let $A_L = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix}$, $b_L = \begin{bmatrix} b \\ 0 \end{bmatrix}$. Use result from 1) to show that (A_L, b_L) is controllable.

Solution:

1. (1) For $A_1=\begin{bmatrix}-1 & -1\\ 1 & -3\end{bmatrix}$,

$$\Delta(\lambda)=\left|\lambda I-A_1\right|=\begin{vmatrix}\lambda+1 & 1\\ -1 & \lambda+3\end{vmatrix}=\lambda^2+4\lambda+4=(\lambda+2)^2=0 \quad\Rightarrow\quad \lambda_1=\lambda_2=-2$$

Define $f(\lambda)=e^{\lambda t},\ g(\lambda)=\beta_0+\beta_1\lambda$,

$$f(\lambda_1)=g(\lambda_1) \quad\Rightarrow\quad e^{-2t}=\beta_0-2\beta_1$$

$$f'(\lambda_1)=g'(\lambda_1) \quad\Rightarrow\quad te^{-2t}=\beta_1$$

$$\Rightarrow\quad \beta_0=(1+2t)e^{-2t},\quad \beta_1=te^{-2t}$$

Thus, $g(\lambda)=(1+2t)e^{-2t}+te^{-2t}\lambda$,

$$e^{A_1t}=f(A_1)=g(A_1)=(1+2t)e^{-2t}+te^{-2t}A_1=(1+2t)e^{-2t}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+te^{-2t}\begin{bmatrix}-1 & -1\\ 1 & -3\end{bmatrix}=e^{-2t}\begin{bmatrix}1+t & -t\\ t & 1-t\end{bmatrix}.$$

For $A_2=\begin{bmatrix}0 & 1 & 0\\ -1 & -1 & -1\\ 1 & 0 & 0\end{bmatrix}$

$$\Delta(\lambda)=\left|\lambda I-A_2\right|=\begin{vmatrix}\lambda & -1 & 0\\ 1 & \lambda+1 & 1\\ -1 & 0 & \lambda\end{vmatrix}=\lambda^2(\lambda+1)+\lambda+1=(\lambda+1)(\lambda^2+1)=0 \quad\Rightarrow\quad \lambda_1=-1,\ \lambda_2=j,\ \lambda_3=-j$$

Define $f(\lambda)=e^{\lambda t},\ g(\lambda)=\beta_0+\beta_1\lambda+\beta_2\lambda^2$,

$$f(\lambda_1)=g(\lambda_1) \quad\Rightarrow\quad e^{-t}=\beta_0-\beta_1+\beta_2$$

$$f(\lambda_2)=g(\lambda_2) \quad\Rightarrow\quad e^{jt}=\beta_0+j\beta_1-\beta_2$$

$$f(\lambda_3)=g(\lambda_3) \quad\Rightarrow\quad e^{-jt}=\beta_0-j\beta_1-\beta_2$$

$$\Rightarrow\quad \beta_0=\frac{e^{-t}+\cos t+\sin t}{2},\quad \beta_1=\sin t,\quad \beta_2=\frac{e^{-t}-\cos t+\sin t}{2}$$

Thus, $g(\lambda)=\dfrac{e^{-t}+\cos t+\sin t}{2}+\sin t\lambda+\dfrac{e^{-t}-\cos t+\sin t}{2}\lambda^2$,

$$e^{A_2t}=f(A_2)=g(A_2)=\frac{e^{-t}+\cos t+\sin t}{2}+\sin tA_2+\frac{e^{-t}-\cos t+\sin t}{2}A_2^2$$

$$=\frac{e^{-t}+\cos t+\sin t}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+\sin t\begin{bmatrix}0 & 1 & 0\\ -1 & -1 & -1\\ 1 & 0 & 0\end{bmatrix}+\frac{e^{-t}-\cos t+\sin t}{2}\begin{bmatrix}-1 & -1 & -1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}\cos t & -\dfrac{e^{-t}-\cos t-\sin t}{2} & -\dfrac{e^{-t}-\cos t+\sin t}{2}\\ -\sin t & \dfrac{e^{-t}+\cos t-\sin t}{2} & \dfrac{e^{-t}-\cos t-\sin t}{2}\\ \sin t & \dfrac{e^{-t}-\cos t+\sin t}{2} & \dfrac{e^{-t}+\cos t+\sin t}{2}\end{bmatrix}.$$

(2) Given the initial condition $x(0)=\begin{bmatrix}-1\\1\\1\end{bmatrix}$, input $u(t)=0$, the zero-input response is,

$$y(t)=Ce^{A_2t}x(0)=[1\quad 0\quad 1]\begin{bmatrix}\cos t & -\frac{e^{-t}-\cos t-\sin t}{2} & -\frac{e^{-t}-\cos t+\sin t}{2}\\ -\sin t & \frac{e^{-t}+\cos t-\sin t}{2} & \frac{e^{-t}-\cos t-\sin t}{2}\\ \sin t & \frac{e^{-t}-\cos t+\sin t}{2} & \frac{e^{-t}+\cos t+\sin t}{2}\end{bmatrix}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$=[\cos t+\sin t\quad \sin t\quad \cos t]\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$=0\ (t>0).$$

(3) Given the initial condition $x(0)=0$, input $u(t)=1$, the zero-state response is,

$$y(t)=\int_0^t Ce^{A_2(t-\tau)}Bu(\tau)d\tau+Du(t)$$

$$=\int_0^t[1\quad 0\quad 1]\begin{bmatrix}\cos(t-\tau) & -\frac{e^{-(t-\tau)}-\cos(t-\tau)-\sin(t-\tau)}{2} & -\frac{e^{-(t-\tau)}-\cos(t-\tau)+\sin(t-\tau)}{2}\\ -\sin(t-\tau) & \frac{e^{-(t-\tau)}+\cos(t-\tau)-\sin(t-\tau)}{2} & \frac{e^{-(t-\tau)}-\cos(t-\tau)-\sin(t-\tau)}{2}\\ \sin(t-\tau) & \frac{e^{-(t-\tau)}-\cos(t-\tau)+\sin(t-\tau)}{2} & \frac{e^{-(t-\tau)}+\cos(t-\tau)+\sin(t-\tau)}{2}\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix}d\tau$$

$$=\int_0^t[\cos(t-\tau)+\sin(t-\tau)\quad \sin(t-\tau)\quad \cos(t-\tau)]\begin{bmatrix}1\\-1\\1\end{bmatrix}d\tau$$

$$=2\int_0^t\cos(t-\tau)d\tau$$

$$=2\sin t\ (t>0).$$

2. (1) Since this is a Jordan form equation, with the eigenvalues $\lambda_1,\ \lambda_2$,

$$A=\begin{bmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{bmatrix},\quad B=\begin{bmatrix} & * & \\ 1 & & 1\\ b & & a\end{bmatrix}\quad \begin{matrix}\\ b_{11}\\ b_{21}\end{matrix}$$

If $\lambda_1\neq\lambda_2$, Consider $b_{11},\ b_{21}$ seperately, $b_{11}=[1\quad 1]$ is LI itself, $b_{21}=[b\quad a]$is LI only if $a\neq 0$ or $b\neq 0$.

If $\lambda_1=\lambda_2$, $\{b_{11}, b_{21}\}$ are LI only if $\det\begin{bmatrix}b_{11}\\b_{21}\end{bmatrix}\neq 0$, i.e., $\det\begin{bmatrix}1 & 1\\ b & a\end{bmatrix}=a-b\neq 0\ \Rightarrow\ a\neq b.$

In summary, the condition for the system to be controllable is,

$$\lambda_1 \neq \lambda_2,\ |a|+|b| \neq 0 \quad \text{or} \quad \lambda_1 = \lambda_2,\ a \neq b$$

(2) Similar as (1), with the eigenvalues $\lambda_1,\ \lambda_2$,

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$C = \begin{bmatrix} a & * & b \\ b & & -a \end{bmatrix}$$

$$c_{11} \quad * \quad c_{21}$$

If $\lambda_1 \neq \lambda_2$, Consider $c_{11},\ c_{21}$ seperately, $c_{11} = \begin{bmatrix} a \\ b \end{bmatrix}$ is LI only if $a \neq 0$ or $b \neq 0$, same conclusion for $c_{21} = \begin{bmatrix} b \\ -a \end{bmatrix}$.

If $\lambda_1 = \lambda_2$, $\{c_{11}, c_{21}\}$ are LI only if $\det\begin{bmatrix} c_{11} & c_{21} \end{bmatrix} \neq 0$, i.e., $\det\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = -a^2 - b^2 \neq 0 \ \Rightarrow\ a \neq 0$ or $b \neq 0$.

In summary, the condition for the system to be controllable is $|a|+|b| \neq 0$.

3. Given $A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix},\quad n = 3,\ p = 2,$

$$G^c_{n-p+1} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix},\quad \rho(G^c_{n-p+1}) = 2 < n,\quad (A,\ B) \text{ is uncontrollable.}$$

Controllable decomposition: Let $Q = \begin{bmatrix} b_1 & b_2 & q \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, then $P = Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$,

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}, \qquad \text{where } \bar{A}_c = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},\ \bar{A}_{12} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ \bar{A}_{\bar{c}} = \begin{bmatrix} -1 \end{bmatrix}.$$

$$\bar{B} = PB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}, \qquad \text{where } \bar{B}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The pair $(\bar{A}_c,\ \bar{B}_c)$ is controllable, and the equivalent system is,

$$\begin{cases}\dot{z}_1 = \bar{A}_c z_1 + \bar{A}_{12} z_2 + \bar{B}_c u \\ \dot{z}_2 = \bar{A}_{\bar{c}} z_2\end{cases}, \qquad \text{where } z = Px.$$

Controller design: Since

$$\det(\lambda I - \bar{A}_c) = \begin{vmatrix}\lambda - 1 & 0 \\ 1 & \lambda - 1\end{vmatrix} = (\lambda-1)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 1$$

the controllable subsystem is unstable, but fortunately, the uncontrollable subsystem is stable, $\bar{A}_{\bar{c}} = [-1]$, so we can design a feedback to make the system to be stable.

For the subsystem $(\bar{A}_c,\ \bar{B}_c)$, choose the poles as $-2,\ -3$, thus, $F = \begin{bmatrix}-2 & 0 \\ 0 & -3\end{bmatrix}$, choose $\bar{K}_0 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$,

$$\bar{A}_c T - T\ F = \bar{B}_c \bar{K}_0 \quad \Rightarrow \quad \begin{bmatrix}1 & 0 \\ -1 & 1\end{bmatrix}\begin{bmatrix}t_{11} & t_{12} \\ t_{21} & t_{22}\end{bmatrix} - \begin{bmatrix}t_{11} & t_{12} \\ t_{21} & t_{22}\end{bmatrix}\begin{bmatrix}-2 & 0 \\ 0 & -3\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}3t_{11} & 4t_{12} \\ 3t_{21} - t_{11} & 4t_{22} - t_{12}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad T = \begin{bmatrix}1/3 & 0 \\ 1/9 & 1/4\end{bmatrix}, \qquad T^{-1} = \begin{bmatrix}3 & 0 \\ -4/3 & 4\end{bmatrix}$$

Thus, $\bar{K} = \bar{K}_0 T^{-1} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0 \\ -4/3 & 4\end{bmatrix} = \begin{bmatrix}3 & 0 \\ -4/3 & 4\end{bmatrix}$,

In the original system, let $\bar{K} = \begin{bmatrix}3 & 0 & a \\ -4/3 & 4 & b\end{bmatrix}$, where $a,\ b$ can be any real values, just choose $a = b = 0$ since they do not affect the eigenvalues, then,

$$K = \bar{K}P = \begin{bmatrix}3 & 0 & 0 \\ -4/3 & 4 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 3 & 0 \\ 0 & 8/3 & 4\end{bmatrix}$$

Verify:

$$A - BK = \begin{bmatrix}0 & -1 & 1 \\ 2 & 1 & -2 \\ 1 & -1 & 0\end{bmatrix} - \begin{bmatrix}-1 & 1 \\ 1 & 0 \\ -1 & 1\end{bmatrix}\begin{bmatrix}0 & 3 & 0 \\ 0 & 8/3 & 4\end{bmatrix} = \begin{bmatrix}0 & -1 & 1 \\ 2 & 1 & -2 \\ 1 & -1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1/3 & 4 \\ 0 & 3 & 0 \\ 0 & -1/3 & 4\end{bmatrix}$$

$$= \begin{bmatrix}0 & -2/3 & -3 \\ 2 & -2 & -2 \\ 1 & -2/3 & -4\end{bmatrix}$$

$$\det(\lambda I - A + BK) = \begin{vmatrix}\lambda & 2/3 & 3 \\ -2 & \lambda + 2 & 2 \\ -1 & 2/3 & \lambda + 4\end{vmatrix} = (\lambda+1)(\lambda+2)(\lambda+3) = 0 \quad \Rightarrow \quad \lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_2 = -3.$$

4. (1) Given the transfer function

$$G(s)=\frac{2s^3+s+1}{s^3+2s^2-2}=2+\frac{-4s^2+s+5}{s^3+2s^2-2},$$

the state space realization of $G(s)$ is,

$$A=\begin{bmatrix}-2 & 0 & 2\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\quad B=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad C=\begin{bmatrix}-4 & 1 & 5\end{bmatrix},\quad D=2.$$

Verify:

$$sI-A=\begin{bmatrix}s+2 & 0 & -2\\ -1 & s & 0\\ 0 & -1 & s\end{bmatrix},\quad (sI-A)^{-1}=\frac{1}{s^3+2s^2-2}\begin{bmatrix}s^2 & * & *\\ s & * & *\\ 1 & * & *\end{bmatrix}$$

$$C(sI-A)^{-1}B+D=\frac{1}{s^3+2s^2-2}\begin{bmatrix}-4 & 1 & 5\end{bmatrix}\begin{bmatrix}s^2 & * & *\\ s & * & *\\ 1 & * & *\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}+2=\frac{2s^3+s+1}{s^3+2s^2-2}$$

(2) Given the poles $-1+j2,\ -1-j2,$ and $-3,$

$$\Delta_d(s)=(s+1-j2)(s+1+j2)(s+3)=s^3+5s^2+11s+15 \quad\Rightarrow\quad \bar{\alpha}_1=5,\ \ \bar{\alpha}_2=11,\ \ \bar{\alpha}_3=15.$$

The characteristic polynomial of A is,

$$\det(sI-A)=\det\begin{bmatrix}s+2 & 0 & -2\\ -1 & s & 0\\ 0 & -1 & s\end{bmatrix}=s^3+2s^2-2 \quad\Rightarrow\quad \alpha_1=2,\ \ \alpha_2=0,\ \ \alpha_3=-2.$$

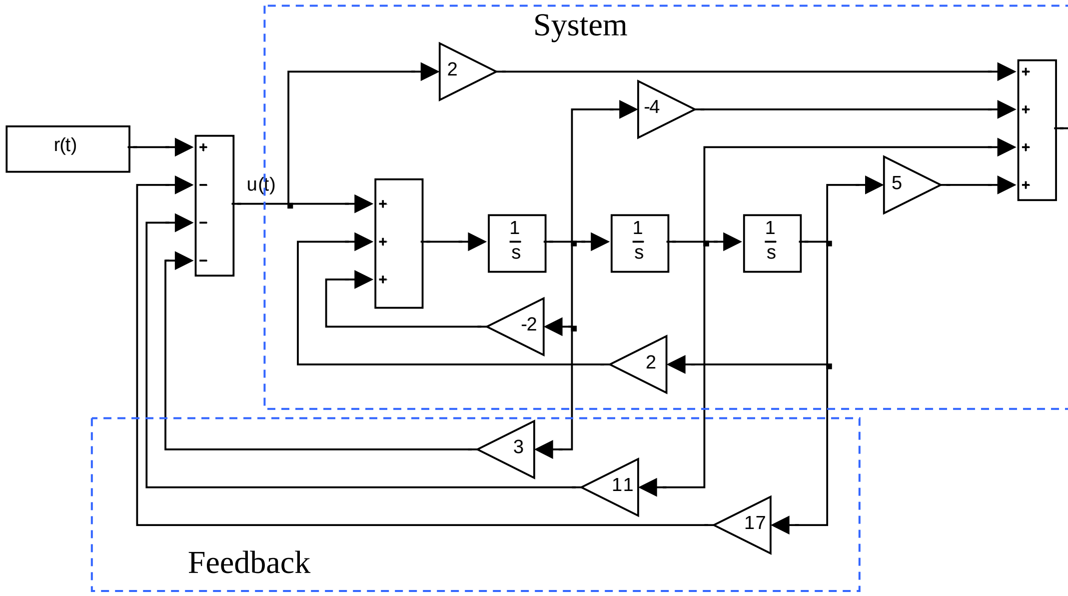
Thus, $\bar{k}_1=\bar{\alpha}_1-\alpha_1=3,\ \ \bar{k}_2=\bar{\alpha}_2-\alpha_2=11,\ \ \bar{k}_3=\bar{\alpha}_3-\alpha_3=17 \quad\Rightarrow\quad \bar{k}=\begin{bmatrix}3 & 11 & 17\end{bmatrix}$

Verify:

$$A-B\bar{k}=\begin{bmatrix}-2 & 0 & 2\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}3 & 11 & 17\end{bmatrix}=\begin{bmatrix}-5 & -11 & -15\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$\det(sI-A+B\bar{k})=\det\begin{bmatrix}s+5 & 11 & 15\\ -1 & s & 0\\ 0 & -1 & s\end{bmatrix}=s^3+5s^2+11s+15 \quad\Rightarrow\quad s_1=-1+j2,\ s_2=-1-j2,\ s_3=-3$$

(3) The simulink model is,



5. (1) Given $M = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}$,

Denote the dimension as, $Z \in R^{q\times q}$, $X \in R^{m\times n}$, $Y \in R^{q\times n}$. Since Z is nonsingular,

$$rank(Z) = q$$

Obviously, $rank([Y \quad Z]) \geq rank(Z)$ always holds, but Also,

$$[Y \quad Z] \in R^{q\times(n+q)},\ rank([Y \quad Z]) \leq \min(q, n+q) = q$$

So, $rank([Y \quad Z]) = q$, *i.e.*, for each row vector $y_i \in R^{1\times n}$, $z_i \in R^{1\times q}$, $i = 1, 2, ..., q$,

$\{[y_i \quad z_i]\}$ are LI.

Also we have,

$$rank([X \quad 0]) = rank(X),$$

denoted as r, *i.e.*, $\exists x_i \in R^{1\times n}$, $i = 1, 2, ..., r$ s.t. $\{[x_i \quad 0]\}$ are LI.

Now, Consider the linear combination of $\{[y_i \quad z_i]\}$ and $\{[x_i \quad 0]\}$,

$$\sum_{i=1}^{r} a_i [x_i \quad 0] + \sum_{i=1}^{q} b_i [y_i \quad z_i] = \left[\sum_{i=1}^{r} a_i x_i + \sum_{i=1}^{q} b_i y_i \qquad \sum_{i=1}^{q} b_i z_i \right] = 0$$

$$\Rightarrow \quad \sum_{i=1}^{q} b_i z_i = 0 \quad \Rightarrow \quad b_i = 0 \text{ (Z is nonsingular)},$$

$$\Rightarrow \quad \sum_{i=1}^{r} a_i x_i + \sum_{i=1}^{q} b_i y_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{r} a_i x_i = 0 \quad \Rightarrow \quad a_i = 0 \ (x_i \text{ are LI})$$

Thus, the linear combination of $\{[y_i \quad z_i]\}$ and $\{[x_i \quad 0]\}$ equals 0 only if $a_i = b_i = 0$, $\{[y_i \quad z_i]\}$ and $\{[x_i \quad 0]\}$ are LI,

$$rank(M) = \#\text{of LI rows} = r + q = rank(X) + rank(Z).$$

(2) Given $A \in R^{n\times n},\ b \in R^{n\times 1},\ c \in R^{1\times n}, A_L = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix},\ b_L = \begin{bmatrix} b \\ 0 \end{bmatrix}$,

$$\det(\lambda I - A_L) = \det\begin{bmatrix} \lambda I - A & 0 \\ c & \lambda \end{bmatrix} = \lambda \det(\lambda I - A) \quad \Rightarrow \quad \lambda = 0,\ \lambda \in \lambda(A)$$

i.e., the eigenvalues' set of A_L is that of A and 0. Since,

$$M(\lambda) = \begin{bmatrix} A_L - \lambda I & b_L \end{bmatrix} = \begin{bmatrix} A - \lambda I & 0 & b \\ -c & -\lambda & 0 \end{bmatrix}$$

For $\lambda = 0$,

$$rank(M(0)) = rank\left(\begin{bmatrix} A & 0 & b \\ -c & 0 & 0 \end{bmatrix}\right) = rank\left(\begin{bmatrix} A & b \\ -c & 0 \end{bmatrix}\right) = rank\left(\begin{bmatrix} A & b \\ c & 0 \end{bmatrix}\right) = n+1 \text{ due to } \det\begin{bmatrix} A & b \\ c & 0 \end{bmatrix} \neq 0$$

For $\lambda \in \lambda(A)$ and $\lambda \neq 0$,

$$rank(M(\lambda)) = rank\left(\begin{bmatrix} A - \lambda I & 0 & b \\ -c & -\lambda & 0 \end{bmatrix}\right) = rank\left(\begin{bmatrix} A - \lambda I & b & 0 \\ -c & 0 & -\lambda \end{bmatrix}\right)$$

$$= rank\left(\begin{bmatrix} A - \lambda I & b \end{bmatrix}\right) + rank(-\lambda)$$

$$= n+1 \qquad \left(\begin{bmatrix} A - \lambda I & b \end{bmatrix} \text{ has full row rank since that (A,b) is controllable}\right)$$

Thus, for every eigenvalues of A_L, $M(\lambda)$ has full row rank, $(A_L,\ b_L)$ is controllable.