16.513 control systems

期中考试

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1. (a)

$$A_1 = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{vmatrix} \xrightarrow{\text{sw } r_1 \,\&\, r_2} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{vmatrix} \xrightarrow[r_4 = r_4 - 3r_1]{r_3 = r_3 - 2r_1} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$$

$$\xrightarrow{r_4 = r_4 - 2r_3} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix} \xrightarrow{r_3 = r_3 + r_2} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \det\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \neq 0$$

$\therefore \rho(A_1) = 2$

(b)

Corrected

$$A_2 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ \cos\theta & \sin\theta & 0 \end{bmatrix}, \quad \text{since } \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1 \neq 0$$

$\text{rank}(A_2) \geq 2$

$$\det A_2 = \cos\theta - \sin\theta = \begin{cases} 0 & \text{if } \theta = \frac{\pi}{4}, \frac{5\pi}{4} \\ \neq 0 & \text{elsewhere} \end{cases}$$

$$\text{Thus rank}(A_2) = \begin{cases} 2, & \text{if } \theta = \frac{\pi}{4}, \frac{5\pi}{4} \\ 3, & \text{elsewhere.} \end{cases}$$

(c)

$$A_3 = \begin{vmatrix} 1 & a & 1 \\ 1 & -1 & -a \end{vmatrix} \xrightarrow{r_2 = r_1 - r_2} \begin{vmatrix} 1 & a & 1 \\ 0 & a+1 & 1+a \end{vmatrix} \qquad \because a \in R$$

if $a + 1 = 0 \Rightarrow a = -1$, then $\rho(A_3) = 1$

if $a \neq -1$, then $\rho(A_3) = 2$

1 - 4

2 - 5

3 - 7

4 - 7

5 - 6 Bonus — 6

2. Let the old bias be $e_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$, $e_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$, $e_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

$\because \bar{e}_1 = e_2 + e_3$, $\bar{e}_2 = e_1 + e_3$, $\bar{e}_3 = e_1 + e_2$

$$(\bar{e}_1, \bar{e}_2, \bar{e}_3) = (e_1, e_2, e_3) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \Rightarrow Q = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}, \quad P = Q^{-1} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

(a)

For $x = e_1 - e_2 = (e_1, e_2, e_3) \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} \Rightarrow \beta = \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix}$

$$\bar{\beta} = P\beta = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} = \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix}$$

i.e. $x = (\bar{e}_1, \bar{e}_2, \bar{e}_3) \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix} = -\bar{e}_1 + \bar{e}_2$

Thus $a = -1$, $b = 1$, $c = 0$

(b) For $y = 2\bar{e}_1 + 4\bar{e}_2 = (\bar{e}_1, \bar{e}_2, \bar{e}_3) \begin{vmatrix} 2 \\ 4 \\ 0 \end{vmatrix} \Rightarrow \bar{\beta} = \begin{vmatrix} 2 \\ 4 \\ 0 \end{vmatrix}$

$$\beta = Q\bar{\beta} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} 2 \\ 4 \\ 0 \end{vmatrix} = \begin{vmatrix} 4 \\ 2 \\ 6 \end{vmatrix}$$

i.e. $x = (e_1, e_2, e_3) \begin{vmatrix} 4 \\ 2 \\ 6 \end{vmatrix} = 4e_1 + 2e_2 + 6e_3$

Thus $\alpha = 4$, $\beta = 2$, $\gamma = 6$

$= [a_1\ a_2\ a_3\ a_4]$

3. (1) $m=2$, $n=4$, $A = \begin{vmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 1 \end{vmatrix}$, $y = \begin{vmatrix} 2 \\ 3 \end{vmatrix}$

$\because \rho(A) = \rho(A:y) = 2 < n$, $\exists$ infinite # of soln

$\nu(A) = n - \rho(A) = 4-2 = 2.$

$y = a_1 + 2a_2 \Rightarrow [a_1\ a_2\ a_3\ a_4] \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = y \Rightarrow A \begin{vmatrix} 1 \\ 2 \\ 0 \\ 0 \end{vmatrix} = y$

one particular soln is $x_p = \begin{vmatrix} 1 \\ 2 \\ 0 \\ 0 \end{vmatrix}$

$a_1 - a_2 = a_3 \Rightarrow a_1 - a_2 - a_3 = 0 \Rightarrow \quad h_1 = \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix}$

$a_4 - a_1 = a_3 \Rightarrow a_1 + a_3 - a_4 = 0 \Rightarrow \quad h_2 = \begin{vmatrix} 1 \\ 0 \\ 1 \\ -1 \end{vmatrix}$

general soln:

$$x = x_p + k_1 h_1 + k_2 h_2 = \begin{vmatrix} 1 \\ 2 \\ 0 \\ 0 \end{vmatrix} + k_1 \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix} + k_2 \begin{vmatrix} 1 \\ 0 \\ 1 \\ -1 \end{vmatrix} = \begin{vmatrix} 1+k_1+k_2 \\ 2-k_1 \\ -k_1+k_2 \\ -k_2 \end{vmatrix}, \quad k_1, k_2 \in R$$

(2) $m=n=2$, $A = [a_1\ a_2] = \begin{vmatrix} -3 & 1 \\ 6 & -2 \end{vmatrix}$, $y = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

$\because \det \begin{vmatrix} -3 & 1 \\ 6 & -2 \end{vmatrix} = 0$, $\rho(A) \neq \rho(A:y)$, No Solution!

(3) $A = [1\ 0\ -1\ -2] = [a_1\ a_2\ a_3\ a_4]$, $y = [-2]$, $m=1$, $n=4$

$\because \rho(A) = \rho([A:y]) = 1 < n$, $\exists$ infinite solution

$\nu(A) = n - \rho(A) = 4-1 = 3$

$y = -2a_1 \Rightarrow x_p = [-2\ 0\ 0\ 0]^T$ is one particular soln.

$a_2 = a_1 + a_3 \Rightarrow h_1 = [1\ -1\ 1\ 0]^T$

$a_3 = -a_1 \Rightarrow h_2 = [1\ 0\ 1\ 0]^T$

$a_4 = 2a_1 \Rightarrow h_3 = [2\ 0\ 0\ -1]^T$

general solution:

$$x = x_p + k_1 h_1 + k_2 h_2 + k_3 h_3 = \begin{vmatrix} -2 \\ 0 \\ 0 \\ 0 \end{vmatrix} + k_1 \begin{vmatrix} 1 \\ -1 \\ 1 \\ 0 \end{vmatrix} + k_2 \begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix} + k_3 \begin{vmatrix} 2 \\ 0 \\ 0 \\ -1 \end{vmatrix} = \begin{vmatrix} -2+k_1+k_2+2k_3 \\ -k_1 \\ k_1+k_2 \\ -k_3 \end{vmatrix}, \quad k_1, k_2, k_3, k_4 \in R$$

4. For A_1

$$\Delta(\lambda) = |\lambda I - A_1| = \begin{vmatrix} \lambda-5 & -6 \\ 3 & \lambda+4 \end{vmatrix} = (\lambda-5)(\lambda+4) + 18 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = -1$$

$$(\lambda_1 I - A_1) v_1 = 0 \Rightarrow \begin{vmatrix} -3 & -6 \\ 3 & 6 \end{vmatrix} v_1 = 0 \Rightarrow v_1 = \begin{vmatrix} -2 \\ 1 \end{vmatrix}$$

$$(\lambda_2 I - A_1) v_2 = 0 \Rightarrow \begin{vmatrix} -6 & -6 \\ 3 & 3 \end{vmatrix} v_2 = 0 \Rightarrow v_2 = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

$$Q_1 = [v_1\ v_2] = \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} \quad \text{thus } Q_1^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\bar{A}_1 = Q_1^{-1} A_1 Q_1 = \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 5 & 6 \\ -3 & -4 \end{vmatrix} \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix}$$

For A_2

$$\Delta(\lambda) = \begin{vmatrix} \lambda-2 & 0 & 0 \\ -3 & \lambda+2 & 1 \\ 3 & -1 & \lambda \end{vmatrix} = \lambda(\lambda-2)(\lambda+2) + \lambda - 2 = (\lambda+1)^2(\lambda-2) = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = -1,\quad \lambda_3 = 2$$

$$(\lambda_1 I - A_2) v_1 = 0 \Rightarrow \begin{vmatrix} -3 & 0 & 0 \\ -3 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} v_1 = 0 \Rightarrow v_1 = \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix}$$

$\because \lambda_1 = \lambda_2$ consider $(A_2 - \lambda_1 I) v_2 = v_1$

$$\begin{vmatrix} 3 & 0 & 0 \\ 3 & -1 & -1 \\ -3 & 1 & 1 \end{vmatrix} v_2 = \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix} \Rightarrow v_2 = \begin{vmatrix} 0 \\ 0 \\ -1 \end{vmatrix}$$

$$(\lambda_3 I - A_2) v_3 = 0 \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ -3 & 4 & 1 \\ 3 & -1 & 2 \end{vmatrix} v_3 = 0 \Rightarrow v_3 = \begin{vmatrix} 1 \\ 1 \\ -1 \end{vmatrix}$$

$$Q_2 = [v_1\ v_2\ v_3] = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{vmatrix} \quad \text{thus } Q_2^{-1} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$\bar{A}_2 = Q_2^{-1} A_2 Q_2 = \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 2 & 0 & 0 \\ 3 & -2 & -1 \\ -3 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

5. For A_1

$$\Delta(\lambda) = \begin{vmatrix} \lambda-3 & -2 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = 2$$

$$f^{(0)}(\lambda_1) = \lambda_1^{k+2} = 1^{k+2}, \quad f^{(0)}(\lambda_2) = \lambda_2^{k+2} = 2^{k+2}$$

$$g^{(0)}(\lambda_1) = \beta_0 + \beta_1\lambda_1 = \beta_0 + \beta_1 = 1^{k+2} = 1^k = 1$$

$$g^{(0)}(\lambda_2) = \beta_0 + \beta_1\lambda_2 = \beta_0 + 2\beta_1 = 2^{k+2} = 4\cdot 2^k$$

$$\beta_1 = 4\cdot 2^k - 1, \quad \beta_0 = 2 - 4\cdot 2^k$$

$$g(A_1) = (2 - 4\cdot 2^k)I + (4\cdot 2^k - 1)A_1 = A_1^{k+2}$$

For A_2

$$\Delta(\lambda) = \begin{vmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda & 1 \\ 1 & -1 & \lambda-2 \end{vmatrix} = (\lambda-2)(\lambda-1)^2 = 0 \Rightarrow \lambda_1 = 2, \ \lambda_2 = \lambda_3 = 1$$

$k > 3$

define $f(\lambda) = \lambda^k$, $g(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$

$$f(2) = g(2) \Rightarrow 2^k = \beta_0 + 2\beta_1 + 4\beta_2$$

$$f(1) = g(1) \Rightarrow 1^k = \beta_0 + \beta_1 + \beta_2 \Rightarrow 1 = \beta_0 + \beta_1 + \beta_2$$

$$f'(1) = g'(1) \Rightarrow k\cdot 1^{k-1} = \beta_1 + 2\beta_2 \Rightarrow k = \beta_1 + 2\beta_2$$

Thus

$$\beta_0 = 2^k - 2k$$

$$\beta_1 = 2 + 3k - 2\cdot 2^k$$

$$\beta_2 = 2^k - 1 - k$$

$$\therefore A_2^k = f(A_2) = g(A_2) = (2^k - 2k)I + (2 + 3k - 2\cdot 2^k)A_2 + (2^k - 1 - k)A_2^2$$

6. $i_1 = \frac{1}{4}\frac{dV_1}{dt}$, $i_2 = \frac{1}{8}\frac{dV_2}{dt}$, $V_L = L\frac{di}{dt} = 2\frac{di}{dt}$

$u_1 = V_1 + V_2 + 2i_1 = V_1 + V_2 + \frac{1}{2}\frac{dV_1}{dt} \Rightarrow \frac{dV_1}{dt} = -2V_1 - 2V_2 + 2u_1$ ✓

KCL: $i_1 = i_2 + \frac{V_2}{4} + i$

$\Rightarrow \frac{1}{4}\frac{dV_1}{dt} = \frac{1}{8}\frac{dV_2}{dt} + \frac{V_2}{4} + i \Rightarrow -\frac{V_1}{2} - \frac{V_2}{2} + \frac{u_1}{2} = \frac{1}{8}\frac{dV_2}{dt} + \frac{V_2}{4} + i$

$\downarrow$

$-4V_1 - 4V_2 + 4u_1 = \frac{dV_2}{dt} + 2V_2 + 8i$

$\downarrow$

$\frac{dV_2}{dt} = -4V_1 - 6V_2 - 8i + 4u_1$ ✓

$V_2 = V_L + u_2 + 8i = 2\frac{di}{dt} + u_2 + 8i$

$\Rightarrow \frac{di}{dt} = \frac{V_2}{2} - 4i - \frac{u_2}{2}$ ✓

$$\begin{vmatrix} \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \\ \frac{di}{dt} \end{vmatrix} = \begin{vmatrix} -2 & -2 & 0 \\ -4 & -6 & -8 \\ 0 & \frac{1}{2} & -4 \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \\ i \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 4 & 0 \\ 0 & -\frac{1}{2} \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}$$ ✓

$$\therefore A = \begin{vmatrix} -2 & -2 & 0 \\ -4 & -6 & -8 \\ 0 & \frac{1}{2} & -4 \end{vmatrix} \qquad B = \begin{vmatrix} 2 & 0 \\ 4 & 0 \\ 0 & -\frac{1}{2} \end{vmatrix}$$