16.513 Midterm Exam (Spring 2005)

There are 5 problems and one bonus problem.

1. (15) For each of the following sets of vectors, determine if it is linearly dependent or independent (LI or LD):

$$S_1 = \{[\sin\theta], [\cos\theta]\}, \qquad S_2 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}\right\}, \qquad S_3 = \left\{\begin{bmatrix}1+a\\ a\end{bmatrix}, \begin{bmatrix}-a\\ 1-a\end{bmatrix}\right\},$$

$$S_4 = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}, \quad S_5 = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

2. (30) For each of the following matrices A_i,

$$A_1 = [1 \;\; 1 \;\; 1], \quad A_2 = \begin{bmatrix}2 & 1 & 0 & 1\\ 1 & 2 & 1 & 0\end{bmatrix}, \qquad A_3 = \begin{bmatrix}1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 0 & 1 & 1 & 1\end{bmatrix},$$

compute

1) the rank of A_i, 2) the nullity of A_i, 3) basis for the range space,
4) basis for the null space.

3. (16) For each of the following matrices

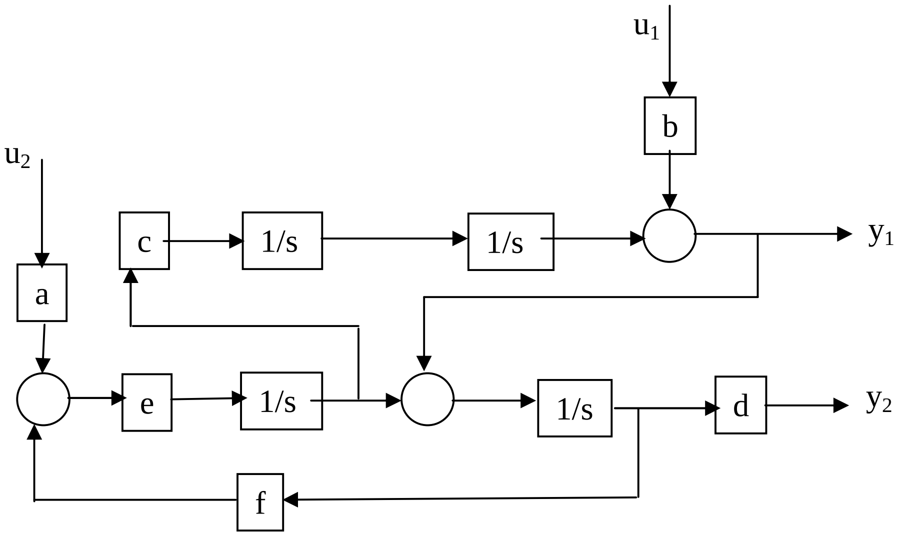
$$A_1 = \begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}, \qquad A_2 = \begin{bmatrix}0 & 1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

compute 1) the eigenvalues, 2) the (generalized) eigenvectors, 3) The Jordan form

4. (20) For the two matrices in Problem 3, compute A_1^k and $e^{A_2 t}$.
What are the eigenvalues and eigenvectors of A_1^k and $e^{A_2 t}$? For the equation

$$\dot{x} = A_2 x + \begin{bmatrix}1\\0\\1\end{bmatrix} u; \; y = [1 \;\; 1 \;\; 1]x, \text{ with } x(0) = \begin{bmatrix}1\\1\\1\end{bmatrix}, \text{ u(t)=0, what is y(t) for t > 0?}$$

5. (19) Find a state-space description (with the matrices (A,B,C,D))for the following system



6. Bonus problem (20) : Given a state equation

$$\dot{x} = Ax + Bu \quad \text{with} \quad A = \begin{bmatrix} -1 & 22 & 33 \\ 0 & -2 & 22 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let P= $\begin{bmatrix} A^2B & AB & B \end{bmatrix}^{-1}$. Introduce new state z = Px.

a) (5) Show that

$$\dot{z} = \overline{A}z + \overline{B}u$$

where $\overline{A} = PAP^{-1}$, $\overline{B} = PB$.

b) (15) Compute $\overline{A}, \overline{B}$. (Hint: don't use brute force)