16.513 Midterm Exam (Spring 2005)

There are 5 problems.

1. (15) For each of the following sets of vectors, determine if it is linearly dependent or independent (LI or LD):

$$S_1 = \{[\sin\theta], [\cos\theta]\}, \quad S_2 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}\right\}, \quad S_3 = \left\{\begin{bmatrix}1+a\\a\end{bmatrix}, \begin{bmatrix}-a\\1-a\end{bmatrix}\right\},$$

$$S_4 = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}, \quad S_5 = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

Solution:

S_1 : LD. Since the number of columns is greater than the number of rows.
For this case, for each θ, there exist $(x_1,x_2) \neq (0,0)$ such that $\sin\theta\, x_1+\cos\theta\, x_2=0$

S_2: LI for $\theta \neq \pi/4 + k\pi$, for any integer k; otherwise LD.

S_3: LI

S_4: LD;

S_5: LI

2. (30) For each of the following matrices A_i,

$$A_1 = [1 \;\; 1 \;\; 1], \quad A_2 = \begin{bmatrix}2 & 1 & 0 & 1\\ 1 & 2 & 1 & 0\end{bmatrix}, \quad A_3 = \begin{bmatrix}1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 0 & 1 & 1 & 1\end{bmatrix},$$

compute

1) the rank of A_i, 2) the nullity of A_i, 3) basis for the range space,
4) basis for the null space

Solution:

For A_1: rank(A_1)=1, nullity(A_1)=2; basis for range space: [1]

Basis for null space: $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

For A_2: rank(A_2)=2, nullity(A_2)=2; basis for range space: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

Basis for null space: $\left\{\begin{bmatrix}1\\0\\-1\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\-2\\-1\end{bmatrix}\right\}$

For A_3: Rank(A_3)=3, nullity(A_3)=1;

basis for range space: any three columns of A_3, or simply $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$,

basis for null space: [1 1 1 -2]'

3. (16) For each of the following matrices

$$A_1=\begin{bmatrix}0 & -2\\1 & -3\end{bmatrix},\qquad A_2=\begin{bmatrix}0 & 1 & 1\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$$

compute 1) the eigenvalues, 2) the (generalized) eigenvectors, 3) The Jordan form

Solution:

For A_1, eigenvalues: λ_1=-1, λ_2=-2; Eigenvectors $v_1=\begin{bmatrix}2\\1\end{bmatrix}$, $v_2=\begin{bmatrix}1\\1\end{bmatrix}$;

Jordan form: $\bar{A}_1=\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}$

For A_2: eigenvalues λ_1=0, λ_2=λ_3=1, Eigenvector of λ_1: $v_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$,

Generalized eigenvetors for λ_2, λ_3: $v_2=\begin{bmatrix}-1\\-1\\0\end{bmatrix}$, $v_3=\begin{bmatrix}1\\-1\\1\end{bmatrix}$

Jordan form: $\bar{A}_2=\begin{bmatrix}0 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$

Caution: v_2 and v_3 must satisfy $v_2=(A-\lambda_2 I)v_3$, and $(A-\lambda_2 I)^2 v_3=0$. If you have $v_2=(\lambda_2 I-A)v_3$, then with $Q=[v_1\ v_2\ v_3]$, you will get $Q^{-1}A_2Q=\begin{bmatrix}0 & 0 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$.

This is not a Jordan form. The off –diagonal element of a Jordan block has to be exactly 1.

4. (20) For the two matrices in Problem 3, compute A_1^k and $e^{A_2 t}$

What are the eigenvalues and eigenvectors of A_1^k and $e^{A_2 t}$? For the equation

$$\dot{x} = A_2 x + \begin{bmatrix}1\\0\\1\end{bmatrix}u;\ y = \begin{bmatrix}1 & 1 & 1\end{bmatrix}x,\ \text{with}\ x(0) = \begin{bmatrix}1\\1\\1\end{bmatrix},\ \text{u(t)=0, what is y(t) for t > 0?}$$

Solution:

For A_1^k: Assume that $f(\lambda)=\lambda^k$, $g(\lambda)=\beta_1\lambda+\beta_0$: Then $f(\lambda_i)=g(\lambda_i)$.

$\Rightarrow$ $(-1)^k=-\beta_1+\beta_0$; $(-2)^k=-2\beta_1+\beta_0$,

$\Rightarrow$ $\beta_1=(-1)^k-(-2)^k$, $\beta_0= 2(-1)^k-(-2)^k$;

$$A_1^k = \beta_1 A_1 + \beta_0 I = \begin{bmatrix} 2(-1)^k-(-2)^k & -2(-1)^k+2(-2)^k \\ (-1)^k-(-2)^k & -(-1)^k+2(-2)^k \end{bmatrix}$$

For e^{A2}: Assume $f(\lambda)=e^{\lambda t}$, $g(\lambda)= \beta_2\lambda^2 + \beta_1\lambda + \beta_0$. $f'(\lambda)= te^{\lambda t}$, $g'(\lambda)=2\beta_2\lambda+\beta_1\lambda$

$\lambda_1=0$; $\lambda_2=\lambda_3=1$;

$f(\lambda_1)=g(\lambda_1)$: $1=\beta_0$;

$f(\lambda_2)=g(\lambda_2)$: $e^t=\beta_2+\beta_1+\beta_0$;

$f'(\lambda_2)=g'(\lambda_2)$: $te^t =2\beta_2+\beta_1$;

$\beta_2=te^t-e^t+1$; $\beta_1=-te^t+2e^t-2$; $\beta_0 = 1$.

$$A_2{}^2 = \begin{bmatrix}0 & 1 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix};\quad e^{A_2 t} = \beta_2 A^2 + \beta_1 A + \beta_0 I = \begin{bmatrix}1 & e^t-1 & -te^t+2e^t-2\\ 0 & e^t & -te^t\\ 0 & 0 & e^t\end{bmatrix}$$

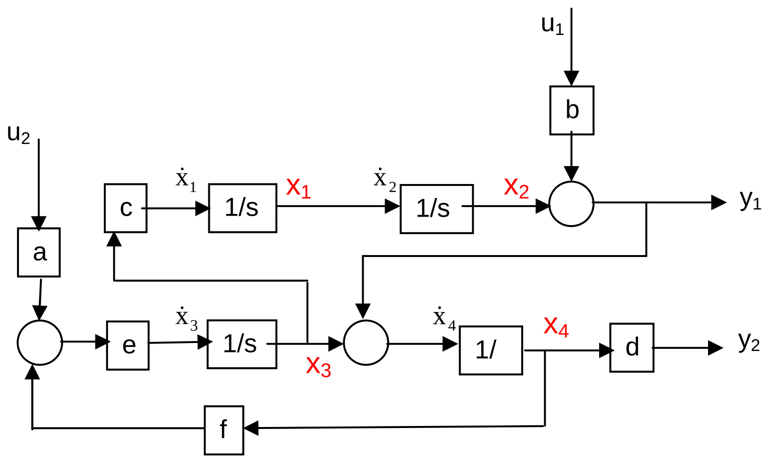
The eigenvalues of A_1^k: $(-1)^k$, $(-2)^k$; eigenvectors are the same as those of A_1.

The eigenvalues of e^{A2t} : 1, e^t, e^t ; eigenvectors are the same as those of A_2.

The zero-input response:

$$y(t) = Ce^{A_2 t}x(0) = \begin{bmatrix}1 & 1 & 1\end{bmatrix} e^{A_2 t}\begin{bmatrix}1\\1\\1\end{bmatrix} = -2e^t + 5e^t - 3.$$

5. (19) Find a state-space description for the following system



Solution:

$\dot{x}_1 = cx_3;$

$\dot{x}_2 = x_1;$ $\quad y_1 = x_2 + bu_1$

$\dot{x}_3 = e(fx_4 + u_2);$ $\quad y_2 = dx_4$

$\dot{x}_4 = x_3 + (bu_1 + x_2)$

$$\Rightarrow \quad \dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & c & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & ef \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & e \\ b & 0 \end{bmatrix}}_{B} u\,; \quad y = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}}_{D} u$$

6…Bonus problem (20) : Given a state equation

$$\dot{x} = Ax + Bu \quad \text{with} \quad A = \begin{bmatrix} -1 & 22 & 33 \\ 0 & -2 & 22 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let P= $\begin{bmatrix} A^2B & AB & B \end{bmatrix}^{-1}$. Introduce new state z = Px.

a) (5) Show that

$$\dot{z} = \bar{A}z + \bar{B}u$$

where $\bar{A} = PAP^{-1}, \bar{B} = PB.$

b). (15) Compute $\bar{A}, \bar{B}$. (Hint: don't use brute force)

To be explained in class.