16.513 Control Systems

Last Time:
- Matrix Operations -- Fundamental to Linear Algebra
  - Determinant
  - Matrix Multiplication
  - Eigenvalue
  - Rank
- Math. Descriptions of Systems ~ Review
  - LTI Systems: State Variable Description
  - Linearization

Today:
- Modeling of Selected Systems
  - Continuous-time systems (§2.5)
    - Electrical circuits
    - Mechanical systems
    - Integrator/Differentiator realization
    - Operational amplifiers
  - Discrete-Time systems (§2.6)
    - Derive state-space equations – difference equations
    - Two simple financial systems
- Linear Algebra, Chapter 3
  - Linear spaces over a field
  - Linear dependence
### 2.5 Modeling of Selected Systems

- We will briefly go over the following systems
  - Electrical Circuits
  - Mechanical Systems
  - Integrator/Differentiator Realization
  - Operational Amplifiers

- For any of the above system, we derive a state space description:
  \[ x(t) = Ax(t) + Bu(t) \]
  \[ y(t) = Cx(t) + Du(t) \]

- Different engineering systems are unified into the same framework, to be addressed by system and control theory.

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### Electrical Circuits

- **State variables?**
  - \( i \) of \( L \) and \( v \) of \( C \)

- How to describe the evolution of the state variables?

  \[
  \frac{di}{dt} = v_L = u - v \\
  \frac{dv}{dt} = \frac{1}{L} v + \frac{1}{L} u \\
  \frac{di_c}{dt} = \frac{v}{R} \\
  \frac{dv_c}{dt} = \frac{1}{C} i_c - \frac{v}{RC} \\
  \]

  In matrix form:

  \[
  \dot{x} = \begin{bmatrix}
  0 & -\frac{1}{L} & \frac{1}{L} \\
  \frac{1}{C} & -\frac{1}{RC} & 0 \\
  \end{bmatrix} x + \begin{bmatrix}
  1 \\
  \frac{1}{L} \\
  \end{bmatrix} u \\
  \]

  State Equation: Two first-order differential equations in terms of state variables and input

- **Output equation:**

  \[
  y = v = \begin{bmatrix}
  0 & 1 \\
  \end{bmatrix} x + 0u \\
  \]

  \[
  \dot{x} = Ax + Bu \\
  y = Cx + Du \\
  \]
Steps to obtain state and output equations:

Step 1: Pick \( \{i_L, v_C\} \) as state variables

Step 2:
\[
\begin{align*}
L \frac{di_L}{dt} &= v_L = f_1(i_L, v_C, u) \\
C \frac{dv_C}{dt} &= i_C = f_2(i_L, v_C, u)
\end{align*}
\]
Linear functions

By using KVL and KCL

Step 3:
\[
\begin{align*}
\frac{di_L}{dt} &= (1/L)f_1(i_C, v_L, u) \\
\frac{dv_C}{dt} &= (1/C)f_2(i_C, v_L, u)
\end{align*}
\]

Step 4: Put the above in matrix form

Step 5: Do the same thing for \( y \) in terms of state variables and input, and put in matrix form

Example

- **State variables?**
  - \( i_1, i_2, \) and \( v, \)

- **State and output equations?**

\[
\begin{align*}
L_1 \frac{di_1}{dt} &= v_{i_1} = u - R_1 i_1 - v \\
L_2 \frac{di_2}{dt} &= v_{i_2} = v - R_2 i_2 \\
C \frac{dv}{dt} &= i_C = i_1 - i_2
\end{align*}
\]

\[
\begin{bmatrix}
\frac{di_1}{dt} \\
\frac{di_2}{dt} \\
\frac{dv}{dt}
\end{bmatrix} =
\begin{bmatrix}
-R_1 & 0 & -\frac{1}{L_1} \\
0 & -R_2 & \frac{1}{L_2} \\
\frac{1}{C} & -\frac{1}{C} & 0
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
v
\end{bmatrix}
\]

\[
y = R_2 i_2 = \begin{bmatrix} 0 & R_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}
\]

\[
\begin{bmatrix} \mathbf{x} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x} \end{bmatrix} + \mathbf{B} \begin{bmatrix} u \end{bmatrix}
\]

\[
\begin{bmatrix} y \end{bmatrix} = \mathbf{C} \begin{bmatrix} \mathbf{x} \end{bmatrix} + \mathbf{D} \begin{bmatrix} u \end{bmatrix}
\]

\[
\begin{bmatrix} \mathbf{x} \end{bmatrix} = \mathbf{Ax} + \mathbf{Bu}
\]

\[
\begin{bmatrix} y \end{bmatrix} = \mathbf{Cx} + \mathbf{Du}
\]
Mechanical Systems

- Elements: Spring, dashpot, and mass
  - Spring: \( f_s = Ky \), opposite direction, ~ Hooke’s law, \( K \): Stiffness
  - Dashpot: \( f_d = Dy' \), opposite direction, ~ \( D \): Damping coefficient
  - Mass: \( M \), Newton’s law of motion
  - LTI elements, LTI systems \( M\ddot{y} = f_N \) ~ Net force
  - Linear differential equations with constant coefficients

- How to describe the system?
  - Free body diagram:

\[
\begin{align*}
K & \quad y(t) & & u(t) \quad M \\
D & \quad y(t) & & u(t) \quad M \\
Ky & \quad y(t) & & u(t) \quad M \\
Dy' & \quad y(t) & & u(t) \quad M \\
\end{align*}
\]

\[
M\ddot{y} = u - Ky - Dy' \quad \ddot{y} + \frac{D}{M}\dot{y} + \frac{K}{M}y = \frac{1}{M}u \quad \text{~ Input/Output description}
\]

- Number of state variables? Which ones?
  - 2 state variables: \( x_1 \equiv y, x_2 \equiv x_1' \)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \ddot{y} = \frac{u - Ky - Dy'}{M} = \frac{u - Kx_1 - Dx_2}{M} \\
\end{align*}
\]

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
\frac{1}{M} & \frac{1}{M}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

\[
y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u
\]
• Steps to obtain state and output equations:
  Step 1: Determine **ALL** junctions and label the displacement of each one
  Step 2: Draw a free body diagram for each rigid body to obtain the net force on it
  Step 3: Apply Newton's law of motion to each rigid body
  Step 4: Select the displacement and velocity as state variables, and write the state and output equations in matrix form
• For rotational systems: \( \tau = J\alpha \)
  • \( \tau \): Torque = Tangential force \( \times \) arm
  • \( J \): Moment of inertia = \( \int r^2 dm \)
  • \( \alpha \): Angular acceleration
  – There are also angular spring/damper

\[
\begin{align*}
\mathbf{K} & \quad \mathbf{D} \quad \mathbf{M} \\
\mathbf{D}(y_1' - y_2') & \quad \mathbf{y}_1(t) \quad \mathbf{M} \\
\mathbf{M} & \quad \mathbf{y}_2(t) \quad \mathbf{u}(t)
\end{align*}
\]

- How to describe the system?
- How many junctions are there?

• Number of state variables? How to select the state variables?

\[
\begin{align*}
x_1 &= y_1; \quad x_2 = \dot{y}_1; \quad x_3 = y_2 \\
\dot{x}_1 &= \dot{y}_1 = x_2 \\
\dot{x}_2 &= \ddot{y}_1 = \frac{1}{M}(u - Dx_2 + D\dot{x}_3) \\
\dot{x}_3 &= x_2 - \frac{K}{D}x_3
\end{align*}
\]

\[
x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{K}{M} \\ 0 & 1 & -\frac{K}{D} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u
\]
Example: an axial artificial heart pump

Illustration

The forces acting on the rotor:

\[ F_1: \text{the active force that can be generated as desired}\]
\[ (F = k \frac{I_1^2}{(c+y_1)^2} - k \frac{I_2^2}{(c-y_1)^2}) \]

\[ F_2, F_3: \text{passive forces, } F_2 = -k_1y_2, F_3 = -k_3y_3, \text{ similar to springs} \]
The motion of the rotor:

\[ M\ddot{y}_c = F_1 + F_2 + F_3, \quad y_c = (l_1y_2 + l_2y_1)/l \]

\[ J\ddot{\alpha} = -l_1F_1 + l_2F_2 - l_3F_3, \quad \alpha = (y_2 - y_1)/l \]

F_2 and F_3 depend on y_1 and y_2. Equation can be expressed in terms of y_1 and y_2

\[ \ddot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1F_1 \]

\[ \ddot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2F_1 \]

Let \( x_1 = y_1, \quad x_2 = \dot{y}_1, \quad x_3 = y_2, \quad x_4 = \dot{y}_2 \)

In matrix form:

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
a_{11} & 0 & a_{12} & 0 \\
0 & 0 & 0 & 1 \\
a_{21} & 0 & a_{22} & 0 \\
\end{bmatrix} x + \begin{bmatrix}
0 \\
b_1 \\
0 \\
b_2 \\
\end{bmatrix} F_1 \\
= Ax + Bu
\]
Integrator/Differentiator Realization

• **Elements**: Amplifiers, differentiators, and integrators

  Amplifier: \( y(t) = af(t) \)
  Differentiator: \( y(t) = \frac{df(t)}{dt} \)
  Integrator: \( y(t) = \int_{t_0}^{t} f(\tau) d\tau + y(t_0) \)

• Are they LTI elements? Yes
• Which one has memory? What are their dimensions?
  – Integrator has memory. Dimensions: 0, 0, and 1, respectively
• They can be connected in various ways to form LTI systems
  – Number of state variables = number of integrators
  – Linear differential equations with constant coefficients

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u - 2x_1 \\
y &= x_2
\end{align*}
\]

- What are the state variables?
- Select output of integrators as SVs
- What are the state and output equations?

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 u
\]

- Linear differential equations with constant coefficients
• Steps to obtain state and output equations:
  Step 1: Select outputs of integrators as state variables
  Step 2: Express inputs of integrators in terms of state variables and input based on the interconnection of the block diagram
  Step 3: Put in matrix form
  Step 4: Do the same thing for \( y \) in terms of state variables and input, and put in matrix form

**Exercise:** derive state equations for the following sys.
Operational Amplifiers (Op Amps)

- Usually, \( A > 10^4 \)
  - Ideal Op Amp:
    - \( A \rightarrow \infty \) \( \sim \) Implying that \( (v_a - v_b) \rightarrow 0 \), or \( v_a \rightarrow v_b \)
    - \( i_a \rightarrow 0 \) and \( i_b \rightarrow 0 \)
  - Problem: How to analyze a circuit with ideal Op Amps

\[ v_0 = A (v_a - v_b), \text{ with } -V_{CC} \leq v_0 \leq V_{CC} \]

- On-inverting terminal, \( v_a \)
- Inverting terminal, \( v_b \)
- Output, \( v_o \)

- Key ideas:
  - Make effective use of \( i_a = i_b = 0 \) and \( v_a = v_b \)
  - Do not apply the node equation to output terminals of op amps and ground nodes, since the output current and power supply current are generally unknown
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  - Discrete-Time systems (§2.6):
    - Derive state-space equations – difference equations
    - Two simple financial systems
- Linear Algebra, Chapter 3
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2.6 Discrete-Time Systems

- Thus far, we have considered continuous-time systems and signals

\[ y(t) \]

- In many cases signals are defined only at discrete instants of time
  - T: Sampling period
  - No derivative and no differential equations
  - The corresponding signal or system is described by a set of difference equations

\[ y[k] = u[k-1] \]

Elements: Amplifiers, delay elements, sources (inputs)

Amplifiers:

\[ u[k] \xrightarrow{a} y[k] = a u[k] \]

~ LTI and memoryless

Delay Element:

\[ u[k] \xrightarrow{z^{-1}} y[k] = u[k-1] \]

~ LTI with memory (1 initial condition)

They can be interconnected to form an LTI system
Example

- How to describe the above mathematically?
  - I/O description:
    \[ y[k+1] = -y[k] + 3u[k] + (u[k-1] - Ky[k-1]), \text{ or} \]
    \[ y[k+1] = -y[k] - Ky[k-1] + 3u[k] + u[k-1] \]
  - A linear difference equation with constant coefficient

- State space description: Select output of delay elements as state variables

\[ \begin{align*}
  x_1[k+1] &= -x_1[k] + x_2[k] + 3u[k] \\
  x_2[k+1] &= -K x_1[k] + u[k] \\
  y[k] &= x_1[k]
\end{align*} \]
Exercise:

- Two state variables, $x_1[k]$, $x_2[k]$
- $x_1[k+1] = u[k] + x_2[k]$
- $x_2[k+1] = -bx_1[k] + ax_2[k]$
- $y[k] = x_1[k]$
- $x[k+1] = \begin{bmatrix} 0 & 1 \\ -b & a \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k]$
- $y[k] = [1 \ 0] x[k]$

Example 1: Balance in your bank account

- A bank offers interest $r$ compounded every day at 12am
  - $u[k]$: The amount of deposit during day $k$
    ($u[k] < 0$ for withdrawal)
  - $y[k]$: The amount in the account at the beginning of day $k$
- What is $y[k+1]$?
  - $y[k+1] = (1 + r) y[k] + u[k]$
Example 2: Amortization

- How to describe paying back a car loan over four years with initial debt $D$, interest $r$, and monthly payment $p$?
  - Let $x[k]$ be the amount you owe at the beginning of the $k$th month. Then
    \[ x[k+1] = (1 + r) x[k] - p \]
  - Initial and terminal conditions: $x[0] = D$ and final condition $x[48] = 0$
  - How to find $p$?

The system:
\[ x[k+1] = \begin{bmatrix} 1 + r \\ -1 \end{bmatrix} x[k] + \begin{bmatrix} -1 \\ p \end{bmatrix} \]

Solution:
\[
x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-m-1} B u[m]
\]
\[
= (1 + r)^k x[0] + \sum_{m=0}^{k-1} (1 + r)^{k-m-1} (-1)p
\]
\[
= (1 + r)^k D - \left( \sum_{m=0}^{k-1} (1 + r)^{k-m-1} \right) p = (1 + r)^k D - \frac{(1 + r)^k - 1}{r} p
\]

Given $D=20000; \ r=0.004; \ x[48]=0$;

\[
0 = (1 + 0.004)^{48} 20000 - \frac{(1 + 0.004)^{48} - 1}{0.004} p
\]

Your monthly payment $p=458.7761$
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Linear Algebra:

Tools for System Analysis and Design

- Our modeling efforts lead to a state-space description of LTI system
  \[
  \begin{align*}
  \dot{x}(t) &= Ax(t) + Bu(t) & x[k + 1] &= Ax[k] + Bu[k] \\
  y(t) &= Cx(t) + Du(t) & y[k] &= Cx[k] + Du[k]
  \end{align*}
  \]

- Analysis problems: stability; transient performances; potential for improvement by feedback control, …
• Consider an LTI continuous-time system
\[
\begin{align*}
    x(t) &= Ax(t) + Bu(t) \\
    y(t) &= Cx(t) + Du(t)
\end{align*}
\]

• For a practical system, usually there is a natural way to choose the state variables, e.g.,
\[
    x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} i_L \\ v_c \end{bmatrix}
\]

• However, the natural state selection may not be the best for analysis. There may exist other selection to make the structure of A, B, C, D simple for analysis

• If T is a nonsingular matrix, then \( z = Tx \) is also the state and satisfies
\[
\begin{align*}
    \dot{z}(t) &= TAT^{-1}z(t) + TBu(t) \\
    y(t) &= CT^{-1}z(t) + Du(t)
\end{align*}
\]

• Two descriptions
\[
\begin{align*}
    \dot{x}(t) &= Ax(t) + Bu(t) \\
    y(t) &= Cx(t) + Du(t) \\
    \dot{z}(t) &= \tilde{A}z(t) + \tilde{B}u(t) \\
    y(t) &= \tilde{C}z(t) + \tilde{D}u(t)
\end{align*}
\]
are equivalent when I/O relation is concerned.

• For a particular analysis problem, a special form of \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) may be the most convenient, e.g.,
\[
\begin{align*}
    \tilde{A} &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \\
    \tilde{A} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}, \\
    \tilde{B} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
    \tilde{C} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{align*}
\]

• We need to use tools from Linear Algebra to get a desirable description.
The operation \( x \rightarrow z = Tx \) is called a linear transformation.
- It plays the essential role in obtaining a desired state-space description
  \[
  \begin{align*}
  \dot{z}(t) &= \tilde{A}z(t) + \tilde{B}u(t) \\
  y(t) &= \tilde{C}z(t) + \tilde{D}u(t)
  \end{align*}
  \]

Linear algebra will be needed for the transformation and analysis of the system
- Linear spaces over a vector field
- Relationship among a set of vectors: LD and LI
- Representations of a vector in terms of a basis
- The concept of perpendicularity: Orthogonality
- Linear Operators and Representations

### 3.1 Linear Vector Spaces and Linear Operators

**Notation:**
- \( \mathbb{R}^n \): n-dimensional real linear vector space
- \( \mathbb{C}^n \): n-dimensional complex linear vector space
- \( \mathbb{R}^{n \times m} \): the set of \( n \times m \) real matrices (also a vector space)
- \( \mathbb{C}^{n \times m} \): the set of \( n \times m \) complex matrices (a vector space)

- A matrix \( T \in \mathbb{R}^{n \times m} \) represents a linear operation from \( \mathbb{R}^m \) to \( \mathbb{R}^n \): \( x \in \mathbb{R}^m \rightarrow Tx \in \mathbb{R}^n \).
- All the matrices \( A, B, C, D \) in the state space equation are real
**Linear Vector Spaces R^n and C^n**

R: The set of real numbers;  C: The set of complex numbers

If x is a real number, we say x ∈ R;
If x is a complex number, we say x ∈ C

R^n: n-dimensional real vector space
C^n: n-dimensional complex vector space

\[
\begin{align*}
R^n &= \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}, \\
C^n &= \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, x_2, \ldots, x_n \in \mathbb{C} \right\}
\end{align*}
\]

If x,y ∈ R^n, a,b ∈ R, then ax + by ∈ R^n  \rightarrow  R^n is a linear space.
If x,y ∈ C^n, a,b ∈ C, then ax + by ∈ C^n  \rightarrow  C^n is a linear space.

---

**Subspace**

- Consider Y ⊂ R^n. Y is a **subspace** of R^n iff Y itself is a linear space
  - Y is a subspace iff \( \alpha_1 y_1 + \alpha_2 y_2 \in Y \) for all \( y_1, y_2 \in Y \) and \( \alpha_1, \alpha_2 \in \mathbb{R} \) (linearity condition)
  - Subspace of C^n can be defined similarly

Example: Consider R^2. The set of \((x_1, x_2)\) satisfying

\[ x_1 - 2x_2 + 1 = 0 \]

can be written as

\[
Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 - 2x_2 + 1 = 0, x_1, x_2 \in \mathbb{R} \right\}
\]

Is the linearity condition satisfied?
Then how about the set of \((x_1, x_2)\) satisfying \(x_1 - 2x_2 = 0\)?

Yes. In fact, any straight line passing through 0 form a subspace.

What would be a subspace for \(\mathbb{R}^3\)?

– Any plane or straight line passing through 0

\(\{(x_1, x_2, x_3) : ax_1 + bx_2 + cx_3 = 0\}\) for constants \(a, b, c\) denote a plane. How to represent a line in the space?

The set of solutions to a system of homogeneous equation is a subspace: \(\{x \in \mathbb{R}^n : Ax = 0\}\).

How about \(\{x \in \mathbb{R}^n : Ax = c\}\) ?

Consider \(\mathbb{R}^n\),

– Given any set of vectors \(\{x_i\}_{i=1}^n\), \(x_i \in \mathbb{R}^n\).

– Form the set of linear combinations

\[
Y \equiv \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in \mathbb{R} \right\}
\]

– Then \(Y\) is a linear space, and is a subspace of \(\mathbb{R}^n\).

– It is the space spanned by \(\{x_i\}_{i=1}^n\).
Linear Independence

- Relationship among a set of vectors.
  - A set of vectors \( \{x_1, x_2, \ldots, x_m\} \) in \( \mathbb{R}^n \) is linearly dependent (LD) iff \( \exists \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \) in \( \mathbb{R} \), not all zero, s.t.
    \[
    \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m = 0 \tag{\*}
    \]
    - If (\*) holds and assume for example that \( \alpha_1 \neq 0 \), then
      \[ x_1 = -\frac{\alpha_2 x_2 + \ldots + \alpha_m x_m}{\alpha_1} \]
      i.e., \( x_1 \) is a linear combination of \( \{\alpha_i\}_{i=2}^{m} \)
  - If the only set of \( \{\alpha_i\}_{i=1}^{m} \) s.t. the above holds is
    \[ \alpha_1 = \alpha_2 = \ldots = \alpha_m = 0 \]
    then \( \{x_i\}_{i=1}^{m} \) is said to be linearly independent (LI)
    - None of \( x_i \) can be expressed as a linear combination of the rest

- A linearly dependent set ~ Some redundancy in the set

**Example.** Consider the following vectors:

- For the following sets, are they linearly dependent (LD) or independent (LI)?
  - \( \{x_1, x_2\} \)
  - \( \{x_1, x_3\} \)
  - \( \{x_1, x_3, x_4\} \)
  - \( \{x_1, x_2, x_3, x_4\} \)

If you have a LD set, \( \{x_1, x_2, \ldots, x_m\} \), then \( \{x_1, x_2, \ldots, x_m, y\} \) is LD for any \( y \).
• Given a set of vectors, \( \{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^n \), how to find out if there are LD or LI?

• A general way to detect LD or LI:
  - \( \{x_1, x_2, \ldots, x_m\} \) are LD iff \( \exists \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \), not all zero, s.t.
    \[
    \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m = 0
    \]

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m
\end{bmatrix} = \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m
\end{bmatrix}
\]

\[A\alpha = 0\]

- \( \{x_1, x_2, \ldots, x_m\} \) are LD iff \( A\alpha = 0 \) has a nonzero solution

Need to understand the solution to a homogeneous equation. There is always a solution \( \alpha = 0 \).

Question: under what condition is the solution unique?

Detecting LD and LI through solutions to linear equations

Given \( \{x_1, x_2, \ldots, x_m\} \), form \[
A = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix}
\]

Consider the equation \( A\alpha = 0 \)

If the equation has a unique solution, LI;
If the equation has nonunique solution, LD.

This is related to the rank of \( A \).

If rank(\( A \))=m, (\( A \) has full column rank), the solution is unique;
If rank(\( A \))<m, the solution is not unique.

• If \( n=m \) and \( A \) is nonsingular, \( \det(A) \neq 0 \), rank(\( A \))=m
  only \( \alpha=0 \) satisfies. \( A\alpha=0 \), hence LI
• If \( n=m \) and \( A \) is singular, \( \det(A) = 0 \), rank(\( A \)) < m
  \( \exists \alpha \neq 0 \) s.t. \( A\alpha=0 \), hence LD
Are the following vectors LD or LI?

\[ x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, x_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \]

\[ \det(A) = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{vmatrix} = 3 \times 6 \times 1 + 2 \times 5 \times 3 + 2 \times 4 \times 4 - 3 \times 3 \times 4 - 2 \times 2 \times 6 - 5 \times 4 \times 1 \]
\[ = 18 + 30 + 32 - 36 - 24 - 20 \]
\[ = 0 \]

\( \Rightarrow \text{LD} \)

How about

\[ x_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \]

\[ \det(A) = ? \]

\[ \det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 4 \\ 4 & 5 & 7 \end{vmatrix} = -10 \neq 0 \]

\( \Rightarrow \text{LI} \)

All depends on the uniqueness of solution for \( A\alpha = 0 \)

- If \( m > n \), \( A \) is a wide matrix, \( \text{rank}(A) < m \), always has a nonzero solution, e.g.,

\[ x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} \]

- If \( m < n \) and \( \text{rank}(A) = m \), LI;
- If \( m < n \) and \( \text{rank}(A) < m \), LD;

\[ A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \]

\[ \text{rank}(A_1) = 2 = m, \text{ LI} \]
\[ \text{rank}(A_2) = 1 < m, \text{ LD} \]
• **Examples:** determine the LD/LI for the following group of vectors

\[
\begin{bmatrix}
\sin \theta \\
\cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \theta \\
\sin \theta
\end{bmatrix},
\begin{bmatrix}
\sin \theta \\
\cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \theta \\
\sin \theta
\end{bmatrix},
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
3
\end{bmatrix},
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\begin{bmatrix}
1 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\begin{bmatrix}
2 \\
3
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\begin{bmatrix}
2 \\
3 \\
5
\end{bmatrix}
\]

---

**Dimension**

• For a linear vector space, the maximum number of LI vectors is called the **dimension** of the space, denoted as $D$

• Consider $\{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^n$
  
  — If $m>n$, they are always dependent, $D \leq n$
  
  — For $m=n$, there exist $x_1, x_2, \ldots, x_n$ such that with $A=[x_1, x_2, \ldots, x_n], |A| \neq 0$, $x_i$’s LI, $D \geq n$
  
  — Hence $D = n$
Today:
- Modeling of Selected Systems
  - Continuous-time systems (§2.5)
    - Electrical circuits, Mechanical systems
    - Integrator/Differentiator realization
    - Operational amplifiers
  - Discrete-Time systems (§2.6):
    - Derive state-space equations – difference equations
    - Two simple financial systems
- Linear Algebra, Chapter 3
  - Linear spaces over a field
  - Linear dependence
- Next time: Linear algebra continued.

Homework Set #3
1. Derive state-space description for the circuit:

2. Derive state-space description for the diagram:
4. Are the following sets subspace of $\mathbb{R}^2$?

\[
Y_1 = \left\{ a \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\},
\]

\[
Y_2 = \left\{ a \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\},
\]

\[
Y_3 = \left\{ a \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \end{bmatrix} : a \geq 0, b \in \mathbb{R} \right\}.
\]

5. Are the following groups of vectors LD or LI?

1) \[
\begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ -1 & 3 & 4 \end{bmatrix}
\]

2) \[
\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}
\]

3) \[
\begin{bmatrix} a & 1 \\ 1 & -a \end{bmatrix}
\]

4) \[
\begin{bmatrix} \cos \theta & 0 \\ 2 \sin \theta & 1 \end{bmatrix}
\]

5) \[
\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}
\]

6) \[
\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -1 \end{bmatrix}
\]