16.513 Control Systems

Controllability and Observability
(Chapter 6)

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A General Framework in State-Space Approach

Given an LTI system:

$$\dot{x} = Ax + Bu; \qquad y = Cx \qquad\qquad (*)$$

The system might be unstable or doesn't meet the required performance spec. How can we improve the situation?

The main approach: Let u= v- Kx (state feedback), then

$$\begin{aligned}\dot{x} &= Ax + B(v - Kx); \qquad & y &= Cx + D(v - Kx)\\ &= (A - BK)x + Bv; & &= (C - DK)x - Dv\end{aligned}$$

The performance of the system is changed by matrix K.

Questions:

- Is there a matrix K s.t. A-BK is stable?
- Can eig(A-BK) be moved to desired locations?

These issues are related to the controllability of (*)

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Main Result 1: The eigenvalues of A-BK can be moved to any desired locations iff the system (*) is controllable.

Another situation: the state x is not completely available. Only a linear combination of x, e.g., $y = Cx$, can be measured. How can we realize $u = v - Kx$?

A possible solution: build an observer to estimate x based on measurement of y.

Main result 2: The observer error (difference between the real x and estimated x) can be made arbitrarily small within arbitrarily short time period iff (*) is observable.

We will arrive at these conclusions in Chapter 8. Before that, we need to prepare some tools and go through these fundamental problems: controllability and observability.

Controllability and Observability

Consider an LTI system:

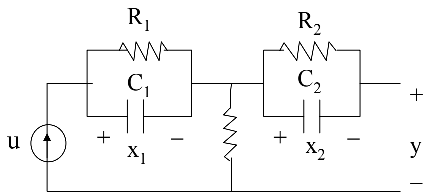
$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

Questions:

a) Can we move the state from one point in the state space to a desired location: $x_0 \rightarrow x_d$, by choosing **u** properly?
b) Can we build a controller to stabilize the system?
c) Suppose that y contains all the quantities that we can measure. Can we evaluate the state x from y?

- The first two questions are related to the controllability of the system
- The last one is related to observability.
- They describe, in some sense, the potential of a system that can be explored through feedback design, e.g., stabilization by output feedback.

An example: A circuit system



Input: u, a current source
State: x_1, x_2, voltages across the two capacitors
Output: y, voltage between two points

Observation:
The input u has no effect on x_2
$\Rightarrow$ x_2 cannot be altered by u $\Rightarrow$ x_2 is not controllable
The measurement y has no relation to x_1, $\Rightarrow$ any change on x_1 is not reflected on y $\Rightarrow$ x_1 is not observable.

Controllability: Definition

Consider the system

$$\dot{x} = Ax + Bu, \quad x \in R^n; \ u \in R^p.$$

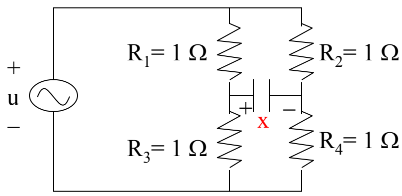
Controllability is a relationship between state and input.

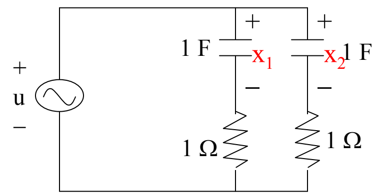
Definition: The system, or the pair (A,B), is said to be controllable if for any initial state $x(0)=x_0$ and any final state x_d, there exist a finite time $T > 0$ and an input u(t), $t \in [0,T]$ such that

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau = x_d \qquad (1)$$

Comment: There may exist different T and u that satisfy (1). As a result, there may be different trajectories starting from x_0 and end at x_d. Controllability does not care about the difference.

Examples: uncontrollable networks.





Observation:

- If x(0)=0, then x(t)=0 for all $t > 0$. The input u can do nothing about it.
- If the resistance is changed so that $R_1/R_3 \neq R_2/R_4$, then you can bring x to any desired value.

Observation:

- If $x_1(0)=x_2(0)=0$, then $x_1(t)=x_2(t)$ for all $t > 0$. You cannot bring x(t) to any point in the plane.
- This situation can be changed by altering the parameters of the components.

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Theorem 1: The pair (A,B) is controllable if and only if the n×n matrix,

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

is nonsingular for every $t > 0$.

Main idea of the proof: Suppose $W_c(t_1)^{-1}$ exists. Recall the solution

$$x(t_1) = e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau$$

If we choose

$$u(t) = -B' e^{A'(t_1-t)} W_c^{-1}(t_1)[e^{At_1} x_0 - x_d]$$

Then,

$$x(t_1) = e^{At_1} x_0 - \int_0^{t_1} e^{A(t_1-\tau)} BB' e^{A'(t_1-\tau)} W_c^{-1}(t_1)[e^{At_1} x_0 - x_d] d\tau$$

$$= e^{At_1} x_0 - \left\{ \int_0^{t_1} e^{A(t_1-\tau)} BB' e^{A'(t_1-\tau)} d\tau \right\} W_c^{-1}(t_1)[e^{At_1} x_0 - x_d]$$

$$= e^{At_1} x_0 - e^{At_1} x_0 + x_d = x_d$$

The necessity of the condition is shown by contradiction. (p.146)

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$$W_c(t) = \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau = \int_0^t e^{A(t-\tau)}BB'e^{A'(t-\tau)}d\tau$$

Equivalent conditions: The following are equivalent conditions for the pair (A,B) to be controllable:

1) $W_c(t)$ is nonsingular for every $t > 0$.
2) $W_c(t)$ is nonsingular for at least one $t > 0$.
3) For every $v \in R^n$, $v \neq 0$, $v' e^{At} B$ is not identically zero.
4) The matrix $G^c = [B \;\; AB \;\; A^2B \; \ldots \; A^{n-1}B]$ has full row rank, i.e., $\rho(G^c) = n$.
5) The matrix $M(\lambda) = [A - \lambda I \;\; B]$ has full row rank at all $\lambda \in C$.
6) $M(\lambda)$ has full row rank at every eigenvalues of A.

Note: $M(\lambda)$ has full row rank if λ is not an eigenvalue of A. We only need to check the rank of $M(\lambda)$ at eigenvalues of A.

Note : Of all the conditions, only 4) and 6) can be practically verified.

Everything starts from the matrix function

$$W_c(t) = \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau$$

We first give some necessary background and tools.

Notice that $W_c(t)$ is a symmetric matrix: $W_c'(t)=W_c(t)$

- Actually it is positive semidefinite. Why?
- All eigenvalues are non-negative.
- $W_c(t_1)$ is nonsingular iff $W_c(t_1) > 0$.
- Controllability is related to the positive definiteness.

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau$$

1) $W_c(t)$ is nonsingular for any $t > 0$.
2) $W_c(t)$ is nonsingular for at least one $t > 0$.

Explanations: 1) $\Leftrightarrow$ 2) Only need to show 2) $\Rightarrow$1)

Assume that $W_c(t_1)$ is nonsingular. Then $W_c(t_1)>0$.

$\Rightarrow$ $v' W_c(t_1) v > 0$ for every $v \in R^n$.

$\Rightarrow$ For a fixed v, $v' W_c(t) v$ is analytic in t and nondecreasing. Also, $v' W_c(0) v=0$, $v' W_c(t_1) v > 0$

$\Rightarrow$ $v' W_c(t) v > 0$, for all $t > 0$.

$\Rightarrow$ For any $t>0$ and $v \in R^n$, $v' W_c(t) v > 0$.

$\Rightarrow$ $W_c(t) > 0$ for all $t > 0$.

$\Rightarrow$ $W_c(t)$ nonsingular for all $t > 0$.

Note: If f(t) is analytic and f(t)=0 for $t \in [a,b]$, then f(t)=0 for all $t \in R$.

The above relation can also be proven with the result from last lecture

Theorem: Let $M(t) \in R^{n \times n}$ be a symmetric matrix function. Suppose that every element of M(t) is an analytic function, i.e., $m_{ij}(t)$ can be differentiated infinitely many times. Then the eigenvalues of M(t) can be arranged as $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$ where each $\lambda_i(t)$ is analytic.

If M(t) is nondecreasing, i.e., $M(t_1) \le M(t_2)$, for any $t_1 < t_2$, then $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$ are nondecreasing.

$$W_c(t) = \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau$$

1) $W_c(t)$ is nonsingular for any $t > 0$.
3) For every $v \in R^n$, $v \neq 0$, $v'e^{At}B$ not identically zero.

Explanations: 1) $\Leftrightarrow$ 3)

Suppose $W_c(t_1)$ is singular for certain t_1.
Exist $v \in R^n$ such that $v'W_c(t_1)v = 0$.
Since $v'e^{A\tau}BB'e^{A'\tau}v \geq 0$ for all τ, must have
$\Rightarrow v'e^{A\tau}BB'e^{A'\tau}v=0$, $v'e^{A\tau}B=0$ for all $\tau \in [0,t_1]$
$\Rightarrow v'e^{At}B=0$ for all t by analyticity.
On the other hand if $v'e^{At}B=0$ for all t,
$\Rightarrow v'W_c(t)v = 0$ for all t
$\Rightarrow W_c(t)$ must be singular since it is positive semi-definite.

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3) For every $v \in R^n$, $v \neq 0$, $v'e^{At}B$ not identically zero.
4) The matrix $G^c = [B\ \ AB\ \ A^2B \ldots A^{n-1}B]$ has full row rank i.e., $\rho(G) = n$.

Explanation: 3) $\Leftrightarrow$ 4)

The opposite of 3) and 4):

3a) Exists $v \in R^n$, $v \neq 0$, $v'e^{At}B = 0$ for all t.
4a) $\rho(G) < n$

4a) $\Rightarrow$ 3a) Exists v such that $v'G^c=0 \Rightarrow v'B=v'AB=\ldots=v'A^{n-1}B=0$.
Since e^{At} is a linear combination of $\{I, A, A^2, \ldots A^{n-1}\}$,
$e^{At}B = k_0(t)B + k_1(t)AB + k_2(t)A^2B + \ldots + k_{n-1}(t)A^{n-1}B$
$\Rightarrow v'e^{At}B=0$ for all t

3a) $\Rightarrow$ 4a) $v'e^{At}B = 0$ for all $t \Rightarrow d^j(v'e^{At}B)/dt^j|_{t=0} = 0$ for all $j \geq 0$.
$\Rightarrow v'e^{At}A_jB=0$ for all j. $\Rightarrow v'e^{At}G^c=0 \Rightarrow \rho(G^c) < n$.

For the relationship between 4) and 5), or 4) and 6), see page 147.

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$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

Equivalent conditions: The following are equivalent conditions for the pair (A,B) to be controllable:

1) $W_c(t)$ is nonsingular for any $t > 0$.
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Note: $M(\lambda)$ has full row rank if λ is not an eigenvalue of A. We only need to check the rank of $M(\lambda)$ at eigenvalues of A.

Note : Of all the conditions, only 4) and 6) can be practically verified.

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$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau$$

Some observations: Recall from the proof of Theorem 1(slide 8). To bring the state from $x(0) = x_0$ to $x(t_1) = x_d$, a particular input is

$$u(t) = -B' e^{A'(t_1 - t)} W^{-1}(t_1)[e^{At_1} x_0 - x_d]$$

- This is actually the minimal energy control, i.e., if there is another input w(t) to transfer x_0 to x_d within the same time interval, then

$$\int_0^{t_1} w(\tau)' w(\tau) d\tau \geq \int_0^{t_1} u(\tau)' u(\tau) d\tau$$

- If (A,B) is controllable, $W_c(t)^{-1}$ exists for all $t > 0$.

$\Rightarrow$ The transfer of the state can be accomplished in arbitrarily small time interval

Note that as t_1 decrease, both $\lambda_{min}[W_c(t_1)]$ and $\lambda_{max}[W_c(t_1)]$ decrease. Then $\|W_c(t_1)^{-1}\|$ increases. $\Rightarrow$ larger magnitude of u is required. As $t_1 \rightarrow 0$, $\|W_c(t_1)^{-1}\| \rightarrow \infty$, $u(t) \rightarrow \infty$.

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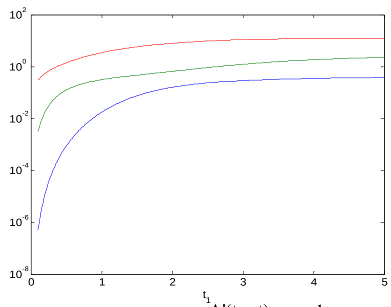
$$W_c(t)=\int_0^t e^{A\tau}BB'e^{A'\tau}d\tau$$

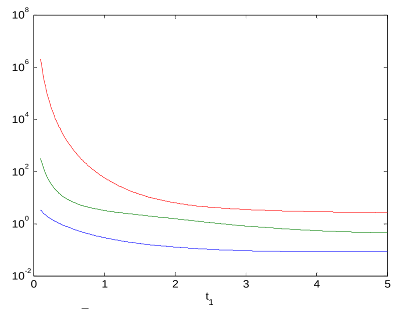
An example:

$$\dot{x}=Ax+Bu,\quad A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix},\quad B=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Eigenvalues of $W_c(t_1)$



Eigenvalues of $W_c(t_1)^{-1}$



$$u(t)=-B'e^{A'(t_1-t)}W^{-1}(t_1)[e^{At_1}x_0-x_d]$$

Magnitude of u increases as t_1 is decreased.

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Example: Determine the controllability for

$$\dot{x}=Ax+Bu,\quad A=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\quad B=\begin{bmatrix}a\\b\end{bmatrix}$$

Approach 1: $G^c=[B\ \ AB]=\begin{bmatrix}a & -a\\ b & -b\end{bmatrix}$

$\rho(G)<2=n$ for all possible a and b

The system not controllable whatever a and b are.

Approach 2: Check $M(\lambda)=[A-\lambda I\ \ B]$ at $\lambda=-1$

$$M(-1)=\begin{bmatrix}0 & 0 & a\\ 0 & 0 & b\end{bmatrix}$$

$\rho(M(-1))<2$ for all possible a and b

Same conclusion on controllability

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Example:

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Approach 1: $G^c = [B \ \ AB] = \begin{bmatrix} a & -a \\ b & -2b \end{bmatrix}$

$$\det G^c = -ab, \quad \begin{cases} \rho(G^c) = 2, & \text{if } a \neq 0 \text{ and } b \neq 0 \\ \rho(G^c) < 2, & \text{if either } a = 0 \text{ or } b = 0 \end{cases}$$

The system is controllable if $a \neq 0$ and $b \neq 0$.

Approach 2: Check $M(\lambda)=[A-\lambda I \ \ B]$ at $\lambda=-1$

$$M(-1) = \begin{bmatrix} 0 & 0 & a \\ 0 & -1 & b \end{bmatrix}, \quad M(-2) = \begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \end{bmatrix},$$

$\rho(M(-1))= \rho(M(-2))=2$ iff $a \neq 0$ and $b \neq 0$

Same conclusion on controllability

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A general SI system (diagonalizable)

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The above system is controllable if and only if the eigenvalues are distinct and none of the b_i's is zero

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Example:

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$G^c = [B \ \ AB] = \begin{bmatrix} b_1 & \alpha b_1 - \beta b_2 \\ b_2 & \beta b_1 + \alpha b_2 \end{bmatrix}$$

$$\det G^c = \beta(b_1^2 + b_2^2), \quad \begin{cases} \rho(G^c) = 2, & \text{if } \beta \neq 0 \text{ and } b_1^2 + b_2^2 \neq 0 \\ \rho(G^c) < 2, & \text{if either } \beta = 0 \text{ or } b_1^2 + b_2^2 = 0 \end{cases}$$

The system is controllable if $\beta \neq 0$ and $(b_1, b_2) \neq (0,0)$

Example: $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

Suppose that the eigenvalues of A_1 and those of A_2 are disjoint.
$\Rightarrow$ (A,B) is controllable iff (A_1,B_1) is controllable and (A_2,B_2) is controllable.

Proof: Consider $M(\lambda) = [\lambda I - A \quad B] = \begin{bmatrix} \lambda I - A_1 & 0 & B_1 \\ 0 & \lambda I - A_2 & B_2 \end{bmatrix}$

Problem: Show that $\text{rank}\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix} \geq \text{rank}(X) + \text{rank}(Y)$

If Y has full row rank, then $\text{rank}\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix} = \text{rank}(X) + \text{rank}(Y)$

Give an example such that, $\text{rank}\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix} \neq \text{rank}(X) + \text{rank}(Y)$

Theorem: Consider the pair

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}$$

Suppose that the eigenvalues of A_i and those of A_j are disjoint for $i \neq j$. Then (A,B) is controllable iff (A_i,B_i) is controllable for all i.

Theorem: Let $\rho(B) = p$. The pair (A,B) is controllable iff

$$G^c{}_{n-p+1} := [B \;\; AB \;\; A^2B \ldots A^{n-p}B]$$

has full row rank. This is equivalent to $G^c{}_{n-p+1}G^{c\prime}{}_{n-p+1}$ being nonsingular, and to $G^c{}_{n-p+1}G^{c\prime}{}_{n-p+1} > 0$ (positive definite.)

Note: The following are equivalent

- X has full row rank;
- XX' is nonsingular;
- XX' is positive-definite.
- How to prove these?

Procedure to determine controllability:

Step 1: form the $n\times(p\times(n\text{-}p+1))$ matrix

$G^c_{n-p+1}=[b_1\ b_2\ \ldots\ b_p\ Ab_1\ Ab_2\ \ldots\ Ab_p\ \ldots\ldots\ A^{n-p}b_1\ A^{n-p}b_2\ \ldots A^{n-p}b_p]$

Step 2: From left to right, throw away columns that are dependent on its LHS columns (or LI columns) until n LI columns are detected or until the last column.

– Recall: y is dependent on the columns of Z iff $\rho(Z) = \rho([Z\ y])$

Step 3:

- If the maximal number of LI columns is n, controllable.

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Example:

$$A=\begin{bmatrix}0&1&0&0\\3&0&0&2\\0&0&0&1\\0&-2&0&0\end{bmatrix},\quad B=\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}$$

n=4, p=2. $\rho(B)=2=p$.

$$G^c_{n-p+1}=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=\begin{bmatrix}0&0&1&0&0&2\\1&0&0&2&-1&0\\0&0&0&1&-2&0\\0&1&-2&0&0&-4\end{bmatrix}$$

$$\begin{matrix}b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2\end{matrix}$$

The first 4 columns are LI. $\Rightarrow \rho(G^c_{n-p+1})=4=n$

$\Rightarrow$ (A,B) controllable

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Example:

$$A=\begin{bmatrix}1&0&0&0\\0&2&1&0\\0&0&2&0\\0&0&0&3\end{bmatrix},\quad B=\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}\qquad n=4,\ p=2.\ \ \rho(B)=2.$$

$$G^c_{n-p+1}=\left[B\ \ AB\ \ A^2B\right]=\begin{bmatrix}1&0&1&0&1&0\\1&0&3&0&8&0\\1&0&2&0&4&0\\1&1&3&3&9&9\end{bmatrix}$$

$$\quad b_1\quad b_2\quad Ab_1\quad Ab_2\quad A^2b_1\quad A^2b_2$$

The first 3 columns are LI.
The 4th is dependent on the first 3.

$$\left[b_1\ b_2\ Ab_1\ A^2b_1\right]=\begin{bmatrix}1&0&1&1\\1&0&3&8\\1&0&2&4\\1&1&3&9\end{bmatrix}\ \text{has full row rank}$$

Hence (A,B) is controllable.

Effect of equivalence transformation

Recall that equivalence transformation can make the structure cleaner and simplify analysis.

Question:
Does similarity transformation retain the controllability property?

Theorem: The controllability property is invariant under any equivalence transformation

Proof: Consider (A,B) with $G^c=[B\ AB\ A^2B\ \ldots.\ A^{n-1}B]$.
Let the transformation matrix be P. Then $(A,B) \Leftrightarrow (PAP^{-1}, PB)$

$$\begin{aligned}\overline{G}^c &= [\overline{B}\ \ \overline{AB}\cdots\overline{A}^{n-1}\overline{B}]\\ &=[PB\ \ PAP^{-1}PB\ \cdots\ PA^{n-1}P^{-1}PB]\\ &=[PB\ \ PAB\ \cdots\ PA^{n-1}B]\\ &=P[B\ \ AB\ \cdots\ A^{n-1}B]\\ &=PG^c\end{aligned}$$

Since P is nonsingular,

$$\rho(\overline{G}^c)=\rho(G^c)$$

Review: Definition of Controllability

Consider the system

$$\dot{x} = Ax + Bu, \quad x \in R^n; \; u \in R^p.$$

Controllability is a relationship between state and input.

Definition: The system, or the pair (A,B), is said to be controllable if for any initial state $x(0)=x_0$ and any final state x_d, there exist a finite time $T > 0$ and an input $u(t)$, $t \in [0,T]$ such that

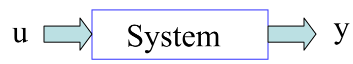
$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau = x_d \qquad (1)$$

Next Problem: Observability

Observability: A dual concept

Consider an n-dimensional, p-input, q-output system:

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$



Assume that we know the input and can measure the output, but has no access to the state.

Definition: The system, is said to be observable if for any unknown initial state x(0), there exists a finite $t_1 > 0$ such that x(0) can be exactly evaluated over $[0,t_1]$ from the input u and the output y. Otherwise the system is said to be unobservable.

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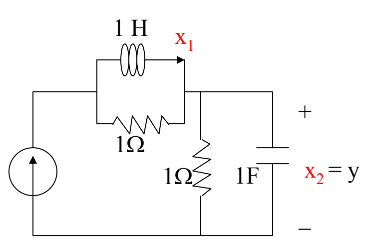
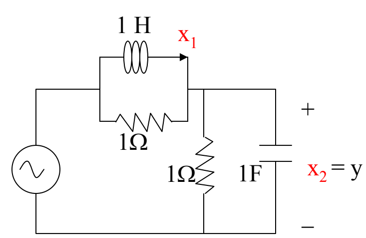
Examples: Two circuits



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u,$$

$$y = x_2$$

The output is independent of x_1; Or any change on x_1 will not be reflected from the output.
$\Rightarrow$ Unobservable.



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u,$$

$$y = x_2$$

x_1 has some effect on $x_2 = y$. We will see later that the circuit is observable.

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Some observations:

Recall the state-space solution,

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Since both u and y are known, define

$$\bar{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

Then $\bar{y}(t)$ is known. Also we have

$$\bar{y}(t) = Ce^{At}x_0$$

The problem is whether x_0 can be detected given $\bar{y}(t)$ over an interval $[0, t_1]$

Clearly, the observability depends only on matrices A and C. We can assume that B=0 and D=0.

Theorem: The system (A,B,C,D) is observable if and only if the n×n matrix

$$W_o(t) = \int_0^t e^{A'\tau}C'Ce^{A\tau}d\tau$$

is nonsingular for any $t > 0$.

Proof. Consider the case where B = 0 and D = 0. Then

$$y(t) = Ce^{At}x_0 \Rightarrow e^{A't}C'y(t) = e^{A't}C'Ce^{At}x_0$$

$$\Rightarrow \int_0^{t_1} e^{A't}C'y(t)dt = \int_0^{t_1} e^{A't}C'Ce^{At}x_0dt$$

$$= \int_0^{t_1} e^{A't}C'Ce^{At}dt\ x_0 = W_o(t_1)x_0$$

If $W_o(t_1)$ is nonsingular, then

$$x_0 = W_o(t_1)^{-1}\int_0^{t_1} e^{A't}C'y(t)dt$$

and observability is ensured. (Sufficiency proved)

Proof continued. (necessity)

Take a comparison between W_o and W_c

$$W_o(t)=\int_0^t e^{A'\tau}C'Ce^{A\tau}d\tau \qquad W_c(t)=\int_0^t e^{A\tau}BB'e^{A'\tau}d\tau$$

Mathematically they have the same structure. So we have the following equivalent conditions:

1) $W_o(t)$ is nonsingular for any $t>0$.
2) $W_o(t)$ is nonsingular for at least one $t>0$.
3) For every $v\in R^n$, $v\neq 0$, $Ce^{At}v$ not identically zero.

If $W_0(t)$ is singular for any $t>0$, then there exists $v\in R^n$, $v\neq 0$ such that $Ce^{At}v=0$ for all t.
If we have $y(t)=Ce^{At}x_0$ for some x_0, then

$$Ce^{At}(x_0+kv)=Ce^{At}x_0+kCe^{At}v=y(t)$$

This means that different initial conditions will yield the same output. And the difference can never to told. $\Rightarrow$ unobservable.

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Duality between controllability and observability

By comparing

$$W_o(t)=\int_0^t e^{A'\tau}C'Ce^{A\tau}d\tau \qquad W_c(t)=\int_0^t e^{A\tau}BB'e^{A'\tau}d\tau$$

It is straightforward to conclude the following

Theorem of duality: The pair (A,B) is controllable if and only if $(A_1, C_1) = (A', B')$ is observable.

$$\dot{x}=Ax+Bu \quad \xleftrightarrow{\text{Dual systems}} \quad \begin{aligned}\dot{z}&=A_1z=A'z\\ y&=C_1z=B'z\end{aligned}$$

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Equivalent conditions for observability:

1) The pair (A,C) is observable.
2) $W_o(t)$ is nonsingular for some $t > 0$.
3) The observability matrix

$$G^o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank, i.e., $\rho(G^o) = n$.

4) The matrix

$$M^o(\lambda) = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

has full column rank at every eigenvalue of A.

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Theorem: The pair (A,C) is observable if and only if

$$G^o_{n-q+1} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-q} \end{bmatrix}$$

has full column rank, where $q=\rho(C)$.

Theorem: The obserbability property is invariant under any equivalence transformation;

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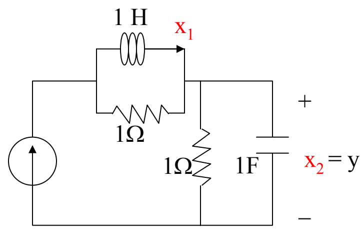
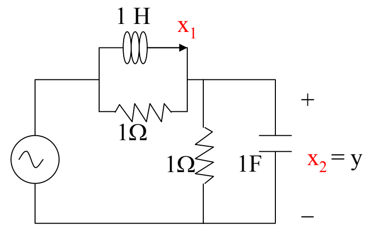
Examples: Two circuits





$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$$G^o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$\rho(G^o) = 1 < 2$

⇨ Unobservable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$$G^o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$\rho(G^o) = 2$

⇨ Observable

Theorem: Consider the pair

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix}$$

Suppose that the eigenvalues of A_i and those of A_j are disjoint for $.i \neq j$. Then (A,C) is observable iff (A_i,C_i) is observable for all i.

Comments on controllability and observability

Controllability: Recall from the proof of the theorem on slide 8. To bring the state from $x(0) = x_0$ to $x(t_1) = x_d$, a particular input is

$$u(t) = -B'e^{A'(t_1-t)}W_c^{-1}(t_1)[e^{At_1}x_0 - x_d]$$

- Usually we don't apply such an input to move the state between points in the state space. (open-loop control, very sensitive).
- A more practical problem is to use a simple linear feedback control, such as $u = Fx + u_0$, to bring the state asymptotically toward a certain point.
  - Such a problem is related to stabilizability.

A future result:

- If a system is controllable, then it is stabilizable.

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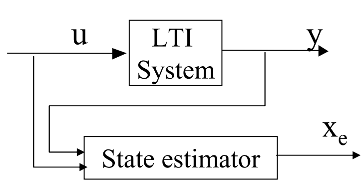
Observability: Theoretically, the initial state x(0) can be computed from the output y(t) by the following formula

$$x_0 = W_o(t_1)^{-1}\int_0^{t_1} e^{A't}C'y(t)dt$$

And the time interval $[0,t_1]$ can be arbitrarily small.

- Again, this approach is complicated and sensitive.
- Actually, we are not only interested in computing x(0) with other available information. Most often, we would like to detect the state for all time.
- What we will be doing later is to build a LTI system with u and y as input and z as state so that $z(t) \rightarrow x(t)$ as $t \rightarrow 0$.



If the system is observable, then the estimated state can be made to approach the real state asymptotically.

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So far, we have learned

- Controllability
- Observability

Next, we will study

- Canonical decomposition: to divide the state space into controllable/uncontrollable, observable/unobservable subspaces

Canonical Decomposition

Consider an LTI system,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

Let z = Px, where P is nonsingular, then

$$\dot{z} = \overline{A}x + \overline{B}u, \quad y = \overline{C}z + \overline{D}u$$

where $\overline{A} = PAP^{-1}, \ \overline{B} = PB, \quad \overline{C} = CP^{-1}, \quad \overline{D} = D$

Recall that under an equivalence transformation, all properties, such as stability, controllability and observability are preserved.

We also have $\overline{G^c} = PG^c, \quad \overline{G^o} = G^o P^{-1}$

Next we are going to use equivalence transformation to obtain certain specific structures which reflect controllability and observability.

Controllability decomposition

Recall $G^c=[B\ AB\ \ldots\ A^{n-1}B]$. Suppose that $\rho(G^c) = n_1 < n$.
Then G^c has at most n_1 LI columns.
They form a basis for the range space of G^c.

Theorem: Suppose that $\rho(G^c) = n_1 < n$. Let Q be a nonsingular matrix whose first n_1 columns are LI columns of G^c. Let $P=Q^{-1}$. Then

$$\overline{A} = PAP^{-1} = \begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ 0 & \overline{A}_{\bar{c}} \end{bmatrix}, \quad \overline{B} = PB = \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix}, \quad \overline{A}_c \in R^{n_1 \times n_1}, \overline{B}_c \in R^{n_1 \times p}$$

$$\overline{C} = \begin{bmatrix} \overline{C}_c & \overline{C}_{\bar{c}} \end{bmatrix}$$

Moreover, the pair $(\overline{A}_c, \overline{B}_c)$ is controllable and

$$\overline{C}_c(sI - \overline{A}_c)^{-1}\overline{B}_c + D = C(sI - A)^{-1}B + D$$

See page 159 for the proof.

Discussion:

After state transformation, the equivalent system is

$$\begin{aligned} \dot{z}_1 &= \overline{A}_c z_1 + \overline{A}_{12} z_2 + \overline{B}_c u \\ \dot{z}_2 &= \qquad\quad \overline{A}_{\bar{c}} z_2 \end{aligned}$$

The input u has no effect on z_2. This part of state is uncontrollable.
The first sub-system is controllable if $z_2=0$. If $z_2 \neq 0$, then

$$z_1(t_1) = e^{\overline{A}_c t_1} z_{10} + \int_0^{t_1} e^{\overline{A}_c (t_1-\tau)} \overline{B}_c u(\tau) d\tau + \int_0^{t_1} e^{\overline{A}_c (t_1-\tau)} \overline{A}_{12} z_2(\tau) d\tau$$

$$z_2(\tau) = e^{\overline{A}_{\bar{c}}\tau} z_{20}$$

Given a desired value for z_1, say z_{1d}. If we let

$$v(t_1) = \int_0^{t_1} e^{\overline{A}_c (t_1-\tau)} \overline{A}_{12} e^{\overline{A}_{\bar{c}}\tau} z_{20} d\tau, \quad \overline{W}_c(t_1) = \int_0^{t_1} e^{\overline{A}_c\tau} \overline{B}_c \overline{B}_c' e^{\overline{A}_c'\tau} d\tau$$

and $u(t) = -\overline{B}_c' e^{\overline{A}_c'(t_1-t)} \overline{W}_c^{-1}(t_1)[e^{\overline{A}_c t_1} z_{10} + v(t_1) - z_{1d}]$

Then you can verify that $z_1(t_1)=z_{1d}$.

Example: $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

n=3, p=2, n-p+1=2.
Only need to check G^c_2

$$G_2^c = [B \quad AB] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \rho(G_2^c) = 2 < 3, \quad \text{uncontrollable}$$

$$\text{Let } Q = [b_1 \ b_2 \ q] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ P = Q^{-1}$$

q is picked to make Q nonsingular

$$\bar{A} = PAQ = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix},$$

$$\bar{B} = PB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

Note: the last column of Q is different from the book (page 161).
As a result, $\bar{A}_{12}$ is different from that in the book, which is 0.

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Observability decomposition (follows from duality)

$$\text{Recall} \quad G^o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Theorem: Suppose that $\rho(G^o) = n_1 < n$. Let P be a nonsingular matrix whose first n_1 rows are LI rows of G^o. Then

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix}, \quad \bar{B} = PB = \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix}, \quad \bar{A}_o \in R^{n_1 \times n_1}, \bar{B}_o \in R^{n_1 \times p}$$

$$\bar{C} = [\bar{C}_o \quad 0], \qquad \bar{C}_o \in R^{q \times n_1}$$

Moreover, the pair $(\bar{A}_o, \bar{C}_o)$ is observable and
$\bar{C}_o (sI - \bar{A}_o)^{-1} \bar{B}_o + D = C(sI - A)^{-1} B + D$

Discussion: After state transformation, the equivalent system is

$$\begin{aligned} \dot{z}_1 &= \bar{A}_o z_1 + \bar{B}_o u \\ \dot{z}_2 &= \bar{A}_{21} z_1 + \bar{A}_{\bar{o}} z_2 + \bar{B}_{\bar{o}} u, \\ y &= \bar{C}_o z_1 + Du \end{aligned}$$

z_2 may be affected by z_1
but has no effect on y or z_1

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Summary for today:

- Controllability
- Observability
- Canonical decomposition
  - Controllable/uncontrollable
  - Observable/unobservable

Next Time:

- Controllability and observability continued
  - Controllability/observability decomposition
  - Minimal realization
  - Conditions for Jordan form conditions
  - Parallel results for discrete-time systems
  - Controllability after sampling
- State feedback design (introduction)

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Problem Set #9

1. Is the following state equation controllable? observable?

$$\dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

If not controllable, reduce it to a controllable one;
If not observable, reduce it to an observable one.

2. Is the following state equation controllable? observable?

$$\dot{x} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

If not controllable, reduce it to a controllable one;
If not observable, reduce it to an observable one.

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Bonus Problem: Show that $\text{rank}\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix} \geq \text{rank}(X) + \text{rank}(Y)$

If Y has full row rank, then $\text{rank}\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix} = \text{rank}(X) + \text{rank}(Y)$

Give an example such that, $\text{rank}\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix} \neq \text{rank}(X) + \text{rank}(Y)$