INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2001; **11**:555–588 (DOI: 10.1002/rnc.565)

Practical stabilization of exponentially unstable linear systems subject to actuator saturation nonlinearity and disturbance

Tingshu Hu*,† and Zongli Lin

Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22903, U.S.A.

SUMMARY

This paper investigates the problem of practical stabilization for linear systems subject to actuator saturation and input additive disturbance. Attention is restricted to systems with two anti-stable modes. For such a system, a family of linear feedback laws is constructed that achieves semi-global practical stabilization on the asymptotically null controllable region. This is in the sense that, for any set χ_0 in the interior of the asymptotically null controllable region, any (arbitrarily small) set χ_∞ containing the origin in its interior, and any (arbitrarily large) bound on the disturbance, there is a feedback law from the family such that any trajectory of the closed-loop system enters and remains in the set χ_∞ in a finite time as long as it starts from the set χ_0. In proving the main results, the continuity and monotonicity of the domain of attraction for a class of second-order systems are revealed. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: nonlinearities; semi-global stabilization; disturbance rejection; actuator saturation; limit cycle; high gain feedback

1. INTRODUCTION

We consider the problem of controlling an exponentially unstable linear system with saturating actuators. This control problem involves issues ranging from such basic ones as controllability and stabilizability to closed-loop performances beyond stabilization. In regard to controllability, the issue is the characterization of the null controllable region (or the asymptotically null controllable region), the set of all initial states that can be driven to the origin by the bounded

* Correspondence to: Tingshu Hu, Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22903, U.S.A.
† E-mail: th7f@virginia.edu

Contract/grant sponsor: US office of Naval Research Young Investigator Program; contract/grant number: N00014-99-1-0670

Received 14 September 1999
Accepted 4 September 2000

input provided by the saturating actuators in a finite time (or asymptotically). On the other hand, the issue of stabilizability is the determination of the existence of feedback laws that stabilize the system within the asymptotically null controllable region and the actual construction of these feedback laws.

It turns out that these seemingly simple issues are actually quite difficult to address for general linear systems. As a result, they have been systematically studied only for linear systems that are not exponentially unstable (all open-loop poles are in the closed left-hand-plane). In particular, it is now well known [1–3] that if a linear system has all its open-loop poles in the closed left-half-plane and is stabilizable in the usual linear system sense, then, when subject to actuator saturation, its asymptotically null controllable region is the entire state space. For this reason, such a linear system is usually referred to as asymptotically null controllable with bounded controls (ANCBC).

In regard to stabilizability, it is shown in Reference [4] that a linear system subject to actuator saturation can be globally asymptotically stabilized by nonlinear feedback if and only if it is ANCBC. A nested feedback design technique for designing nonlinear globally asymptotically stabilizing feedback laws was proposed in References [5–7]. Alternative solutions to the global stabilization problem consisting of scheduling a parameter in an algebraic Riccati equation according to the size of the state vector were later proposed in References [8–10]. The question of whether or not a general linear ANCBC system subject to actuator saturation can be globally asymptotically stabilized by linear feedback was answered in References [11, 12], where it was shown that a chain of integrators of length greater than 2 cannot be globally asymptotically stabilized by saturated linear feedback.

The notion of semi-global asymptotic stabilization (on the asymptotically null controllable region) for linear systems subject to actuator saturation was introduced in References [13, 14]. The semi-global framework for stabilization requires feedback laws that yield a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes an *a priori* given (arbitrarily large) bounded subset of the asymptotically null controllable region. In References [13, 14], it was shown that, for linear ANCBC systems subject to actuator saturation, one can achieve semi-global asymptotic stabilization by using linear feedback laws.

In an effort to address closed-loop performances beyond large domain of attraction, [15] formulates and solves the problem of practical semi-global stabilization for ANCBC systems with saturating actuators. In particular, low-and-high gain feedback laws are constructed that not only achieve semi-global stabilization in the presence of input additive uncertainties but also have the ability to reject bounded input additive disturbance.

Despite the numerous results on linear ANCBC systems, the counterparts of the above-mentioned results for exponentially unstable linear systems are less understood. Recently, we made an attempt to systematically study issues related to the null controllable regions (or asymptotically null controllability regions) and the stabilizability for exponentially unstable linear systems subject to actuator saturation and gave a rather clear understanding of these issues [16]. Specifically, we gave a simple exact description of the null controllable region for a general anti-stable linear system in terms of a set of extremal trajectories of its time-reversed system. For a linear planar anti-stable system under a saturated linear stabilizing feedback law, we established that the boundary of the domain of attraction is the unique stable limit circle of its time-reversed system. Furthermore, we constructed feedback laws that semi-globally asymptotically stabilize any system with two anti-stable modes on its asymptotically null controllable region. This is in the sense that, for any *a priori* given set in the interior of the asymptotically null controllable region, there exists a saturated linear feedback law that yields a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes the given set.

The goal of this paper is to design feedback laws that, not only achieve semi-global stabilization on the asymptotically null controllable region, but also has the ability to reject bounded disturbance to an arbitrary level of accuracy. Our attention will be restricted to systems that have two anti-stable modes. Our problem formulation is motivated by its counterpart for ANCBC systems [15].

This paper is organized as follows. Section 2 formulates the problem and summarizes the main results. Sections 3 and 4 establish some fundamental properties of the behaviours of planar systems. These properties lead to the proof of the main results in Sections 5 and 6 . Section 7 uses an aircraft model to demonstrate the results obtained in this paper. Section 8 contains a brief concluding remark.

For a set X, we use ∂X, $\bar{X}$ and $\text{int}(X)$ to denote its boundary, closure and interior, respectively. For a measurable function, $w:[0, \infty) \rightarrow \mathbf{R}$, $\|w\|_\infty$ is its L_∞-norm. For a vector v, we use $(v)_i$ to denote its ith co-ordinate. For two bounded subsets X_1, X_2 of $\mathbf{R}^n$, their Hausdorff distance is defined as

$$d(X_1, X_2) := \max\{\vec{d}(X_1, X_2), \vec{d}(X_2, X_1)\}$$

where

$$\vec{d}(X_1, X_2) = \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} \|x_1 - x_2\|$$

Here the vector norm used is arbitrary.

2. PROBLEM STATEMENT AND THE MAIN RESULTS

2.1. Problem statement

Consider an open-loop system subject to both actuator saturation and disturbance,

$$\dot{x} = Ax + b\,\text{sat}(u + w) \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}$ is the control input, $w \in \mathbf{R}$ is the disturbance and $\text{sat}(s) = \text{sign}(s) \min\{1, |s|\}$ is the standard saturation function. Assume that (A, b) is stabilizable. We consider the following set of disturbances:

$$\mathscr{W} := \{w:[0, \infty) \rightarrow \mathbf{R},\ w \text{ is measurable and } \|w\|_\infty \leqslant D\},$$

where D is a known constant.

In addressing the practical stabilization problem, we need to describe the largest possible region in the state space that can be stabilized. For this purpose, we introduce the notions of null controllability and asymptotic null controllability.

Definition 1

Consider system (1) in the absence of the disturbance w. A state x_0 is said to be null controllable if there exist a $T \in [0, \infty)$ and a measurable control u such that the state trajectory $x(t)$ satisfies

$x(0) = x_0$ and $x(T) = 0$. The set of all null controllable states is called the null controllable region of the system and is denoted by $\mathscr{C}$.

Definition 2

Consider system (1) in the absence of the disturbance w. A state x_0 is said to be asymptotically null controllable if there exists a measurable control u such that the state trajectory $x(t)$ satisfies $x(0) = x_0$ and $\lim_{t\to\infty} x(t) = 0$. The set of all asymptotically null controllable states is called the asymptotic null controllable region of the system and is denoted by $\mathscr{C}_a$.

In this paper, the matrix A (or the corresponding linear system) is said to be anti-stable if all of its eigenvalues are in the open right-half-plane and semi-stable if all of its eigenvalues are in the closed left-half-plane.

Proposition 1

Assume that (A, b) is stabilizable.

(a) if A is semi-stable, then $\mathscr{C}_a = \mathbf{R}^n$.
(b) If A is anti-stable, then $\mathscr{C}_a = \mathscr{C}$ is a bounded convex open set containing the origin.
(c) If

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

with $A_1 \in \mathbf{R}^{n_1 \times n_1}$ anti-stable and $A_2 \in \mathbf{R}^{n_2 \times n_2}$ semi-stable, and b is partitioned as

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

accordingly, then $\mathscr{C}_a = \mathscr{C}_1 \times \mathbf{R}^{n_2}$ where $\mathscr{C}_1$ is the null controllable region of the anti-stable system $\dot{x}_1 = A_1 x_1 + b_1 \operatorname{sat}(u)$.

Note that if (A, b) is controllable, then $\mathscr{C}_a = \mathscr{C}$.

Proposition 1 follows from a similar result on the null controllable region in Reference [16] by further partitioning A_2 and b_2 as

$$A_2 = \begin{bmatrix} A_{20} & 0 \\ 0 & A_{2-} \end{bmatrix}, \quad b_2 = \begin{bmatrix} b_{20} \\ b_{2-} \end{bmatrix}$$

where A_{20} has all its eigenvalues on the imaginary axis and A_{2-} is Hurwitz. Let the state be partitioned accordingly as $x = [x_1^{\mathrm{T}}\ x_{20}^{\mathrm{T}}\ x_{2-}^{\mathrm{T}}]^{\mathrm{T}}$, with $x_{20} \in \mathbf{R}^{n_{20}}$, $x_{2-} \in \mathbf{R}^{n_{2-}}$. Then

$$\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_{20} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_{20} \end{bmatrix} \right)$$

is controllable and the null controllable region corresponding to the state $[x_1^{\mathrm{T}}\ x_{20}^{\mathrm{T}}]^{\mathrm{T}}$ is $\mathscr{C}_1 \times \mathbf{R}^{n_{20}}$ by Reference [16]. After $[x_1^{\mathrm{T}}\ x_{20}^{\mathrm{T}}]^{\mathrm{T}}$ is steered to the origin, the control can be removed and the state x_{2-} will approach the origin asymptotically.

Our objective is to design a family of feedback laws such that given any (arbitrarily large) set χ_0 in the interior of $\mathscr{C}_a$ and any (arbitrarily small) set χ_∞ containing the origin in its interior, there is a feedback law from this family such that any trajectory of the closed-loop system that starts from

χ_0 will enter χ_∞ in a finite time and remain there. A complete treatment of this problem was provided in Reference [15] for ANCBC systems. For such a system, a linear feedback can be designed so that the domain of attraction of a small neighbourhood of the origin includes any prescribed bounded set and the disturbance is rejected to an arbitrary level of accuracy. It should be noted that Reference [15] allows for multi-input and more general saturation functions but has the limitation that A has no exponentially unstable eigenvalues, i.e. A is semi-stable. Many earlier papers on control with saturating actuators also have this limitation. The main reason is that if A has exponentially unstable eigenvalues, the largest possible region that can be asymptotically stabilized, i.e. the null controllable region, was unknown.

To achieve our control objectives for exponentially unstable systems, we must know how to describe $\mathscr{C}_a$. In Reference [16], we gave some simple exact descriptions of $\mathscr{C}_a$, and constructed a family of switching saturated linear controllers for a system with two exponentially unstable modes that semi-globally stabilizes the system on $\mathscr{C}_a$. For easy reference, we give a brief review of the results in Reference [16] in the following subsection.

2.2. Background

Consider the system

$$\dot{x} = Ax + b\,\mathrm{sat}(u), \tag{2}$$

If A is anti-stable, then $\mathscr{C}_a = \mathscr{C}$ is a bounded convex open set. It was shown in Reference [16] that $\partial\mathscr{C}$ is composed of a set of extremal trajectories of the time reversed system of (2).

The second main result in Reference [16] is about the stability analysis of the following closed-loop system:

$$\dot{x} = Ax + b\,\mathrm{sat}(fx), \quad x \in \mathbf{R}^2 \tag{3}$$

where $A \in \mathbf{R}^{2\times 2}$ is anti-stable and $A + bf$ is Hurwitz. The time-reversed system of (3) is

$$\dot{z} = -Az - b\,\mathrm{sat}(fz) \tag{4}$$

Denote the state transition map of (3) by $\phi:(t, x_0) \mapsto x(t)$ and that of (4) by $\psi:(t, z_0) \mapsto z(t)$. Then the domain of attraction of the equilibrium $x_e = 0$ for (3) is defined by

$$\mathscr{S} := \left\{ x_0 \in \mathbf{R}^2 : \lim_{t\to\infty} \phi(t, x_0) = 0 \right\}$$

Proposition 2

$\mathscr{S}$ is convex and symmetric. $\partial\mathscr{S}$ is the unique limit cycle of systems (3) and (4), and has two intersections with each of the lines $fx = 1$ and $fx = -1$. Furthermore, $\cdot\mathscr{S}$ is the positive limit set of $\psi(\cdot, z_0)$ for all $z_0 \neq 0$.

It was also shown that $\mathscr{S}$ can be made arbitrarily close to $\mathscr{C}$ by suitably choosing f. Since A is anti-stable and (A, b) is controllable, the following Riccati equation

$$A'P + PA - Pbb'P = 0 \tag{5}$$

has a unique positive-definite solution $P > 0$. Let $f_0 = -b'P$. Then the origin is a stable equilibrium of the system

$$\dot{x} = Ax + b\,\mathrm{sat}(kf_0x), \quad x \in \mathbf{R}^2 \tag{6}$$

for all $k > 0.5$. Let $\mathscr{S}(k)$ be the domain of attraction of the equilibrium $x_e = 0$ for (6).

Proposition 3
$\lim_{k\to\infty} d(\mathscr{S}(k), \mathscr{C}) = 0$.

Hence, the domain of attraction can be made to include any compact subset of $\mathscr{C}$ by simply increasing the feedback gain. We say that the system is semi-globally stabilized (on its null controllable region) by the family of feedbacks $u = \text{sat}(kf_0x)$, $k > 0.5$. This result was then extended to construct a family of switching saturated linear feedback laws that semi-globally stabilizes a higher-order system with two anti-stable modes.

2.3. Main results of this paper

Given any (arbitrarily small) set that contains the origin in its interior, we will show that its domain of attraction can be made to include any compact subset of $\mathscr{C}_a$ in the presence of disturbances bounded by an (arbitrarily large) given number. More specifically, we will establish the following result on semi-global practical stabilization on the asymptotically null controllable region for system (1).

Theorem 1
Consider system (1) with A having two exponentially unstable eigenvalues. Given any set $\chi_0 \subset \text{int}(\mathscr{C}_a)$, any set χ_∞ such that $0 \in \text{int}(\chi_\infty)$, and any positive number D, there is a feedback law $u = F(x)$ such that any trajectory of the closed-loop system enters and remains in the set χ_∞ in a finite time as long as it starts from the set χ_0.

To prove Theorem 1, we need to establish some properties of planar linear systems, both in the absence and in the presence of actuator saturation.

3. PROPERTIES OF THE TRAJECTORIES OF SECOND-ORDER LINEAR SYSTEMS

We first consider the second-order anti-stable system

$$\dot{x} = Ax = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2 \end{bmatrix} x, \quad a_1, a_2 > 0 \tag{7}$$

We will examine its trajectories with respect to a horizontal line $kfx = 1$ where $f = [0\ 1]$, $k > 0$. On this line, $x_2 = 1/k$ and if $x_1 > -a_2/k$, then $\dot{x}_2 > 0$, i.e. the vector $\dot{x}$ points upward; if $x_1 < -a_2/k$, then $\dot{x}_2 < 0$, i.e. the vector $\dot{x}$ points downward. Above the line, $\dot{x}_1 < 0$, hence the trajectories all go leftward. Denote.

$$a_m = \begin{cases} -\frac{\lambda_1}{k} & \text{if } A \text{ has real eigenvalues } \lambda_1 \geqslant \lambda_2 > 0 \\ \infty & \text{if } A \text{ has a pair of complex eigenvalues} \end{cases}$$

Then we have

Lemma 1
Let $x_{11} \geqslant -a_2/k$ and

$$p = \begin{bmatrix} x_{11} \\ \frac{1}{k} \end{bmatrix}$$

be a point on the line $kfx = 1$. The trajectory $x(t) = e^{At}p, t \geqslant 0$ will return to this line if and only if $x_{11} < a_m$. Let T be the first time when it returns and

$$p' = \begin{bmatrix} y_{11} \\ \frac{1}{k} \end{bmatrix}$$

be the corresponding intersection, i.e. $p' = e^{AT}p$. This defines two functions: $x_{11} \to y_{11}$ and $x_{11} \to T$. Then for all $x_{11} \in (-a_2/k, a_m)$,

$$\frac{dy_{11}}{dx_{11}} < -1, \qquad \frac{d^2 y_{11}}{dx_{11}^2} < 0, \quad \frac{dT}{dx_{11}} > 0 \tag{8}$$

Proof. See Appendix A. □

It may be easier to interpret Lemma 1 by writing (8) as

$$\frac{d(-y_{11})}{dx_{11}} > 1, \qquad \frac{d^2(-y_{11})}{dx_{11}^2} > 0$$

An illustration of Lemma 1 is given in Figure 1, where p_1, p_2, p_3 are three points on $kfx = 1$,

$$p_i = \begin{bmatrix} x_{11}^i \\ \frac{1}{k} \end{bmatrix}, \quad x_{11}^i \in [-\tfrac{a_2}{k}, a_m), \quad i = 1, 2, 3,$$

and p'_1, p'_2 and p'_3 are the first intersections of the trajectories that start from p_1, p_2 and p_3. Then

$$\frac{\|p'_3 - p'_2\|}{\|p_3 - p_2\|} > \frac{\|p'_2 - p'_1\|}{\|p_2 - p_1\|} > 1 \tag{9}$$

It follows that

$$\frac{\|p'_2 - p'_1\|}{\|p'_3 - p'_2\|} < \frac{\|p_2 - p_1\|}{\|p_3 - p_2\|} \Rightarrow \frac{1 + \frac{\|p'_2 - p'_1\|}{\|p'_3 - p'_2\|}}{1 + \frac{\|p_2 - p_1\|}{\|p_3 - p_2\|}} < 1$$

Hence

$$\frac{\|p'_3 - p'_1\|}{\|p_3 - p_1\|} = \frac{\|p'_3 - p'_2\| + \|p'_2 - p'_1\|}{\|p_3 - p_2\| + \|p_2 - p_1\|} = \frac{\|p'_3 - p'_2\|}{\|p_3 - p_2\|} \frac{1 + \frac{\|p'_2 - p'_1\|}{\|p'_3 - p'_2\|}}{1 + \frac{\|p_2 - p_1\|}{\|p_3 - p_2\|}} < \frac{\|p'_3 - p'_2\|}{\|p_3 - p_2\|} \tag{10}$$

Also from (9)

$$\frac{\|p'_3 - p'_2\|}{\|p'_2 - p'_1\|} > \frac{\|p_3 - p_2\|}{\|p_2 - p_1\|} \Rightarrow \frac{1 + \frac{\|p'_3 - p'_2\|}{\|p'_2 - p'_1\|}}{1 + \frac{\|p_3 - p_2\|}{\|p_2 - p_1\|}} > 1$$

Hence

$$\frac{\|p'_3 - p'_1\|}{\|p_3 - p_1\|} = \frac{\|p'_3 - p'_2\| + \|p'_2 - p'_1\|}{\|p_3 - p_2\| + \|p_2 - p_1\|} = \frac{\|p'_2 - p'_1\|}{\|p_2 - p_1\|} \frac{1 + \frac{\|p'_3 - p'_2\|}{\|p'_2 - p'_1\|}}{1 + \frac{\|p_3 - p_2\|}{\|p_2 - p_1\|}} > \frac{\|p'_2 - p'_1\|}{\|p_2 - p_1\|} \tag{11}$$

Combining (10) and (11), we obtain

$$\frac{\|p'_3 - p'_2\|}{\|p_3 - p_2\|} > \frac{\|p'_3 - p'_1\|}{\|p_3 - p_1\|} > \frac{\|p'_2 - p'_1\|}{\|p_2 - p_1\|} > 1 \tag{12}$$

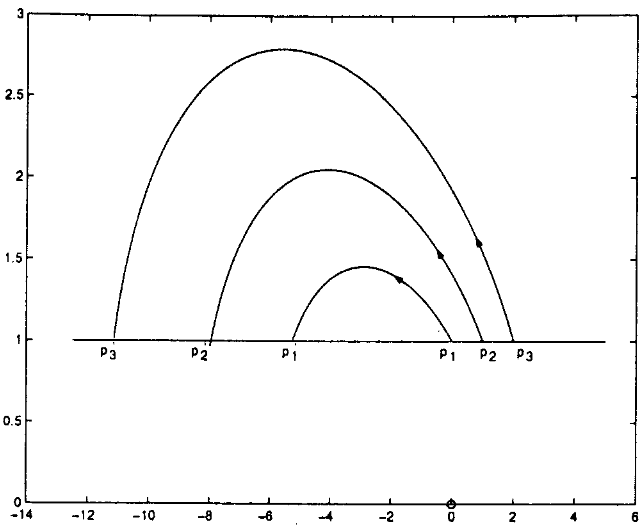

Figure 1. Illustration of Lemma 1.

We next consider a second-order stable linear system,

$$\dot{x} = Ax = \begin{bmatrix} 0 & -a_1 \\ 1 & -a_2 \end{bmatrix} x, \quad a_1, a_2 > 0 \tag{13}$$

We will study the trajectories of (13) with respect to two horizontal lines $kfx = 1$ and $kfx = -1$ where $f = [0 \ 1]$, $k > 0$. On the line $kfx = -1$, if $x_1 < -a_2/k$, the vector $\dot{x}$ points downward; if $x_1 > -a_2/k$, the vector $\dot{x}$ points upward.

Let

$$p_0 = \begin{bmatrix} -\frac{a_2}{k} \\ -\frac{1}{k} \end{bmatrix}$$

be a point on $kfx = -1$. There is a point p_0' on $kfx = 1$ and $T_d > 0$ such that $e^{AT_d} p_0' = p_0$, $|kfe^{At} p_0'| \leqslant 1$, $\forall t \in [0, T_d]$ (see Figure 2). Denote the first coordinate of p_0' as x_m, i.e.

$$p_0' = \begin{bmatrix} x_m \\ \frac{1}{k} \end{bmatrix}$$

Let

$$p' = \begin{bmatrix} x_{11} \\ \frac{1}{k} \end{bmatrix}, \ x_{11} \in (-\infty, x_m]$$

be a point on $kfx = 1$, then there is a unique

$$p = \begin{bmatrix} y_{11} \\ -\frac{1}{k} \end{bmatrix}$$

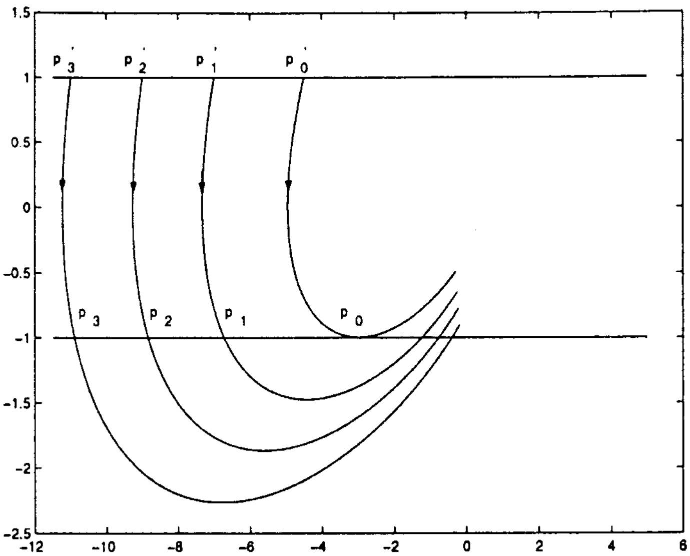

Figure 2. Illustration of Lemma 2.

on $kfx = -1$, where $y_{11} \in (-\infty, -a_2/k]$ and $T \in (0, T_d]$ such that

$$p = e^{AT} p', \qquad |kf e^{At} p'| \leqslant 1, \quad \forall t \in [0, T] \tag{14}$$

This defines two functions $x_{11} \to y_{11}$, and $x_{11} \to T$.

Lemma 2

For all $x_{11} \in (-\infty, x_m)$, we have $x_{11} < y_{11}$ and

$$\frac{dy_{11}}{dx_{11}} > 1, \qquad \frac{d^2 y_{11}}{dx_{11}^2} > 0, \quad \frac{dT}{dx_{11}} > 0$$

Proof. See Appendix B. □

This lemma is illustrated with Figure 2, where p'_1, p'_2, p'_3 are three points on $kfx = 1$ and p_1, p_2, p_3 are the three first intersections of $kfx = -1$ with the three trajectories starting from p'_1, p'_2, p'_3, respectively. Then

$$\frac{\|p_1 - p_2\|}{\|p'_1 - p'_2\|} > \frac{\|p_1 - p_3\|}{\|p'_1 - p'_3\|} > \frac{\|p_2 - p_3\|}{\|p'_2 - p'_3\|} > 1$$

4. PROPERTIES OF THE DOMAIN OF ATTRACTION

Consider the closed-loop system

$$\dot{x} = Ax + b\,\text{sat}(kfx), \quad x \in \mathbf{R}^2 \tag{15}$$

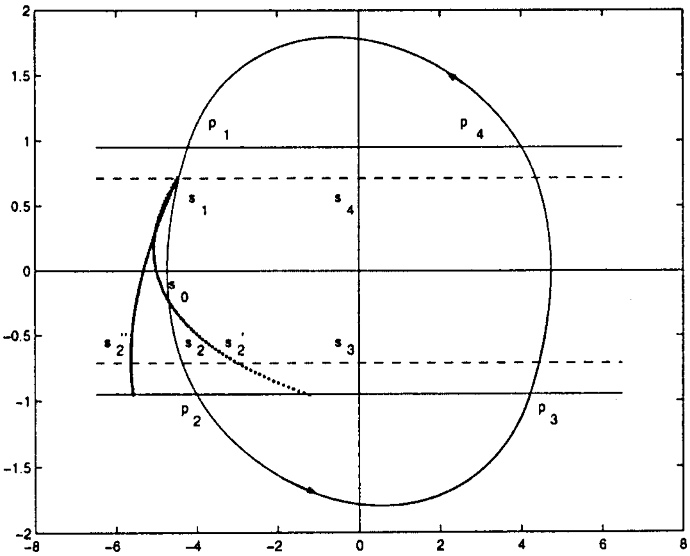

Figure 3. Illustration for the proof of Proposition 4.

where

$$A = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2 \end{bmatrix}, \quad b = \begin{bmatrix} -b_1 \\ -b_2 \end{bmatrix},$$

$a_1, a_2, b_2 > 0$, $b_1 \geqslant 0$, and $f = [0 \ 1]$. If $k > a_2/b_2$, then $A + kbf$ is Hurwitz and the origin is the unique equilibrium point of (15) and it is stable. Denote the domain of attraction of the origin as $\mathscr{S}(k)$, then by Proposition 2, $\partial\mathscr{S}(k)$ is the unique limit cycle of (15). We will further show that the domain of attraction $\mathscr{S}(k)$ increases as k is increased.

Consider $k_0 > a_2/b_2$. Denote the increment of k as δ_k. Proposition 2 says that $\partial\mathscr{S}(k_0)$ is symmetric with respect to the origin and has two intersections with each of the lines $k_0 fx = 1$ and $k_0 fx = -1$. In Figure 3, the closed curve is $\partial\mathscr{S}(k_0)$ and $p_1, p_2, p_3\ (= -p_1), p_4 (= -p_2)$ are the four intersections. Since at p_2, the trajectory goes downward, i.e. $\dot{x}_2 < 0$, so $(p_2)_1 < (a_2 - k_0 b_2)/k_0 < 0$. From Lemma 2, we have $(p_1)_1 < (p_2)_1 < 0$. Hence both p_1 and p_2 are on the left half plane. Define

$$\Delta(k_0) = -\frac{(p_2)_1}{b_2} k + \frac{a_2}{b_2} - k_0$$

Then $\Delta(k_0) > 0$ due to the fact that the trajectory goes downward at p_2.

Proposition 4

Suppose $k_0 > a_2/b_2$. Then for all $\delta_k \in (0, \Delta(k_0))$, $\mathscr{S}(k_0) \subset \mathscr{S}(k_0 + \delta_k)$.

Proof. Since $\delta_k > 0$, the two lines $(k_0 + \delta_k) fx = \pm 1$ lie in between $k_0 fx = \pm 1$. It follows that the vector field above $k_0 fx = 1$ and that below $k_0 fx = -1$ are the same for

$$\dot{x} = Ax + b\,\text{sat}\,(k_0 fx) \tag{16}$$

and

$$\dot{x} = Ax + b\,\text{sat}((k_0 + \delta_k)fx) \tag{17}$$

So, if a trajectory of (17) starts at p_4 (or p_2), it will go along $\partial\mathcal{S}(k_0)$ to p_1 (or p_3).

Claim

If a trajectory of (17) starts at a point on $\partial\mathcal{S}(k_0)$ between p_1 and p_2 and intersects the line $k_0fx = -1$, then the intersection must be inside $\mathcal{S}(k_0)$.

It follows from the claim that any trajectory of (17) that starts from $\partial\mathcal{S}(k_0)$ will stay inside of $\overline{\mathcal{S}(k_0)}$ when it returns to the lines $k_0fx = \pm 1$. So it is bounded and hence belongs to $\overline{\mathcal{S}(k_0 + \delta_k)}$. Note that any trajectory outside of $\mathcal{S}(k_0 + \delta_k)$ will diverge because the system has a unique limit cycle. Since the two sets are convex and open, we will have $\mathcal{S}(k_0) \subset \mathcal{S}(k_0 + \delta_k)$.

It remains to prove the claim.

Since $\mathcal{S}(k_0)$ is convex, $\angle(A + bk_0f)x$ from p_1 to p_2 along $\partial\mathcal{S}(k_0)$ is increasing. Let s_0 be the intersection of $\partial S(k_0)$ with the abscissa. Then at s_0, $\angle(A + bk_0f)x = -\pi/2$; from p_1 to s_0, $\angle(A + bk_0f)x \in (-\pi, -\pi/2)$; and from s_0 to p_2, $\angle(A + bk_0f)x \in (-\pi/2, 0)$. Now consider a point x along $\partial\mathcal{S}(k_0)$ between p_1 and p_2,

(1) If x is between p_1 and s_0, then $k_0fx \leqslant \text{sat}((k_0 + \delta_k)fx)$. If $\angle(A + bk_0f)x < \angle b$, then $\dot{x}$ of (17) directs inward of $\partial\mathcal{S}(k_0)$ and if $\angle(A + bk_0f)x) > \angle b$, then $\dot{x}$ of (17) directs outward of $\partial\mathcal{S}(k_0)$. Since $\angle(A + bk_0f)x$ is increasing, the vector $\dot{x}$ may direct outward of $\partial\mathcal{S}(k_0)$ for the whole segment or for a lower part of the segment.

(2) If x is between s_0 and p_2, then $k_0fx \geqslant \text{sat}((k_0 + \delta_k)fx)$. Since $\angle b \in (-\pi, -\pi/2)$, we have

$$\angle(A + bk_0f)x) \leqslant \angle(Ax + b\,\text{sat}((k_0 + \delta_k)fx))$$

i.e. the vector $\dot{x}$ of (17) directs inward of $\partial\mathcal{S}(k_0)$.

Let

$$s_1 = \begin{bmatrix} x_{11} \\ h \end{bmatrix}, \quad h > 0$$

be a point on $\partial\mathcal{S}(k_0)$ between p_1 and s_0 such that $\dot{x}$ of (17) at s_1 directs outward of $\partial\mathcal{S}(k_0)$.

Let

$$s_2 = \begin{bmatrix} y_{11} \\ -h \end{bmatrix}$$

be the intersection of $\partial\mathcal{S}(k_0)$ with $x_2 = -h$. Then by (1) the trajectory of (17) starting at s_1 will remain outside of $\partial\mathcal{S}(k_0)$ above the abscissa. We need to show that when the trajectory reaches the line $x_2 = -h$ at s_2', it must be inside $\partial\mathcal{S}(k_0)$.

Let

$$s_3 = \begin{bmatrix} 0 \\ -h \end{bmatrix}, \qquad s_4 = \begin{bmatrix} 0 \\ h \end{bmatrix}$$

(see Figure 3). Denote the region enclosed by $s_1s_2s_3s_4s_1$ as G_0, where the part s_1s_2 is on $\partial\mathcal{S}(k_0)$ and the other parts are straight lines. Since this region lies between $k_0fx = \pm 1$, the vector field of

(16) on this region is

$$\dot{x}_1 = -(a_1 + k_0 b_1)x_2 =: f_1(x)$$
$$\dot{x}_2 = x_1 + (a_2 - k_0 b_2)x_2 =: f_2(x)$$

Applying Green's Theorem to system (16) on G_0, we get

$$\oint_{\partial G_0} f_2\,\mathrm{d}x_1 - f_1\,\mathrm{d}x_2 = -\iint_{G_0}\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\mathrm{d}x_1\,\mathrm{d}x_2 \tag{18}$$

Note that the left-hand side integral from s_1 and s_2 and that from s_3 to s_4 are zero. Denote the area of G_0 as Q_0, then from (18), we have

$$\tfrac{1}{2}x_{11}^2 + (a_2 - k_0 b_2)hx_{11} - \tfrac{1}{2}y_{11}^2 + (a_2 - k_0 b_2)hy_{11} = -(a_2 - k_0 b_2)Q_0 \tag{19}$$

Clearly $Q_0 > -h(x_{11} + y_{11})$ by the convexity of $\mathscr{S}(k_0)$ and the region G_0.

On the other hand, we consider a trajectory of (17) starting at s_1 but cross the line $x_2 = -h$ at

$$s_2' = \begin{bmatrix} y_{11} + \delta_{y11} \\ -h \end{bmatrix}$$

Firstly, we assume that s_1 lies between $(k_0 + \delta_k)fx = \pm 1$. Apply Green's Theorem to (17) on the region enclosed by $s_1 s_2' s_3 s_4 s_1$, where the part $s_1 s_2'$ is on a trajectory of (17). Denote the area of the region as $Q_0 + \delta_Q$. Similarly,

$$\begin{aligned}&\tfrac{1}{2}x_{11}^2 + (a_0 - k_0 b_2 - \delta_k b_2)hx_{11} - \tfrac{1}{2}(y_{11} + \delta_{y11})^2 + (a_2 - k_0 b_2 - \delta_k b_2)h(y_{11} + \delta_{y11})\\ &\quad = -(a_2 - k_0 b_2 - \delta_k b_2)(Q_0 + \delta_Q)\end{aligned} \tag{20}$$

Subtracting (19) from (20), we obtain

$$-[y_{11} - (a_2 - k_0 b_2 - \delta_k b_2)h]\delta_{y11} = (k_0 b_2 - a_2)\delta_Q + \delta_k b_2(Q_0 + hx_{11} + hy_{11}) + \tfrac{1}{2}\delta_{y11}^2 + \delta_k b_2 \delta_Q \tag{21}$$

Note that $Q_0 + hx_{11} + hy_{11} > 0$ and $k_0 b_2 - a_2 > 0$.

From the definition of $\Delta(k_0)$, we have

$$(p_2)_1 - (a_2 - k_0 b_2 - \delta_k b_2)\frac{1}{k_0} < 0$$

for all $\delta_k \in [0, \Delta(k_0))$. Since $y_{11} < (p_2)_1$, $h < 1/k_0$ and $-(a_2 - k_0 b_2 - \delta_k b_2) > 0$, it follows that

$$y_{11} - (a_2 - k_0 b_2 - \delta_k b_2)h < 0, \quad \forall\, \delta_k \in [0, \Delta(k_0))$$

Now, suppose that $\delta_k \in [0, \Delta(k_0))$. If $\delta_{y11} < 0$, then s_2' is outside of $\partial\mathscr{S}(k_0)$ and we must have $\delta_Q > 0$. In this case the left-hand side of (21) is negative and the right-hand side is positive. A contradiction. This shows that δ_{y11} must be positive and s_2' must be inside $\partial\mathscr{S}(k_0)$. By (2), the vector $\dot{x}$ of system (17) directs inward of $\partial\mathscr{S}(k_0)$ from s_2 to p_2, we know that when the trajectory reaches $k_0 fx = -1$, it must be to the right of p_2, i.e. still inside $\partial\mathscr{S}(k_0)$.

Now suppose s_1 lies between $(k_0 + \delta_k)fx = 1$ and $k_0 fx = 1$. Then by applying Green's Theorem, we get exactly the same equation as (21), although we need to partition the region enclosed by $s_1 s_2' s_3 s_4 s_1$ into 3 parts. And similar argument applies. Thus we conclude that for all $\delta_k \in [0, \Delta(k_0\,))$, $\mathscr{S}(k_0) \subset \mathscr{S}(k_0 + \delta_k)$. $\square$

Proposition 5
Consider

$$\dot{x} = Ax + b\,\text{sat}(fx), \quad x \in \mathbf{R}^2 \tag{22}$$

where $A \in \mathbf{R}^{2\times 2}$, $b \in \mathbf{R}^{2\times 1}$ are constant matrices, A is anti-stable and $f \in \mathbf{R}^{1\times 2}$ is a variable. Denote the domain of attraction of the origin for (22) as $\mathscr{S}(f)$. Then, at any f such that $A + bf$ is Hurwitz and has distinct eigenvalues, $\overline{\mathscr{S}(f)}$ is continuous.

Proof. We only need to show that $\partial\mathscr{S}(f)$ is continuous. Recall from Proposition 2 that $\partial\mathscr{S}(f)$ is a closed trajectory and has four intersections with $fx = \pm 1$. Since the vector $\dot{x} = Ax + b\,\text{sat}(fx)$ is continuous in f at each x, it suffices to show that one of the intersections is continuous in f. Actually, we can show that the intersections are also differentiable in f. For simplicity and for direct use of Lemmas 1 and 2, we apply a state-space transformation, $\hat{x} = V(f)x$, to system (22), such that

$$V(f)AV^{-1}(f) = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2 \end{bmatrix} =: \hat{A}, \qquad V(f)b = \begin{bmatrix} b_1(f) \\ b_2(f) \end{bmatrix} =: \hat{b}(f), \qquad fV^{-1}(f) = [0 \;\; 1] =: \hat{f} \tag{23}$$

Such a transformation always exists. To see this, assume that A is already in this form. Since A is anti-stable and $A + bf$ is stable, (f, A) must be observable. So

$$V(f) = \begin{bmatrix} fA - a_2 f \\ f \end{bmatrix}$$

in non-singular and it can be verified that this $V(f)$ is the desired transformation matrix. Moreover, $V(f)$, $V^{-1}(f)$, $b_1(f)$ are all analytic in f. Now consider the transformed system

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}(f)\,\text{sat}(\hat{f}\hat{x}) \tag{24}$$

Note that $\hat{A}$ and $\hat{f}$ are both independent of f. Under the state transformation, $\mathscr{S}(f)$ is transformed into $\hat{\mathscr{S}}(f) = \{V(f)x : x \in \mathscr{S}(f)\}$, the domain of attraction for (24) and $\partial\hat{\mathscr{S}}(f)$ is its unique limit cycle. Let

$$p_1 = \begin{bmatrix} \hat{x}_{11} \\ 1 \end{bmatrix}$$

be a point on $\hat{f}\hat{x} = 1$ such that a trajectory starting at p_1 will go above the line and return to the line (for the first time) at

$$p_1' = \begin{bmatrix} \hat{y}_{11} \\ 1 \end{bmatrix}$$

Let T_1 be the time for the trajectory to go from p_1 to p_1', then

$$e^{\hat{A}T_1}(p_1 + \hat{A}^{-1}\hat{b}(f)) = (p_1' + \hat{A}^{-1}\hat{b}(f))$$

or equivalently,

$$e^{\hat{A}T_1} \begin{bmatrix} \frac{\hat{x}_{11} + (\hat{A}^{-1}\hat{b}(f))_1}{1 + (\hat{A}^{-1}\hat{b}(f))_2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\hat{y}_{11} + (\hat{A}^{-1}\hat{b}(f))_1}{1 + (\hat{A}^{-1}\hat{b}(f))_2} \\ 1 \end{bmatrix}$$

where $(\cdot)_i$, $i = 1, 2$, denotes the ith coordinate of a vector. It can be verified from the stability of $\hat{A} + \hat{b}(f)\hat{f}$ that $1 + (\hat{A}^{-1}\hat{b}(f))_2 > 0$. So Lemma 1 applies here with a changing of variables. We can write $\hat{y}_{11} = \hat{y}_{11}(f, \hat{x}_{11})$. By Lemma 1, $\hat{y}_{11}$ is continuously differentiable in $\hat{x}_{11}$. It is easy to see that $\hat{y}_{11}$ is also continuously differentiable in f.

Suppose that the trajectory continuous from p'_1 and intersects the line $\hat{f}\hat{x} = -1$ at a non-zero angle. Let

$$p''_1 = \begin{bmatrix} \hat{z}_{11} \\ -1 \end{bmatrix}$$

be the first intersection of the trajectory with $\hat{f}\hat{x} = -1$. Note that between $\hat{f}\hat{x} = 1$ and $\hat{f}\hat{x} = -1$, the vector field of (24) is

$$\dot{\hat{x}} = (\hat{A} + \hat{b}(f)\hat{f})\hat{x} = \begin{bmatrix} 0 & -a_1 + b_1(f) \\ 1 & a_2 + b_2(f) \end{bmatrix}\hat{x}$$

and that $\hat{A} + \hat{b}(f)\hat{f}$ is Hurwitz, so Lemma 2 applies and we know that $\hat{z}_{11}$ is continuously differentiable in $\hat{y}_{11}$. To see that $\hat{z}_{11}$ is also continuously differentiable in f, recall we have assumed that $A + bf$ has distinct eigenvalues, so the eigenvalues are analytic in f. From (48) in the proof of Lemma 2, we see that T is continuously differentiable in λ_1, λ_2 and hence in f for $T < T_{\rm d}$. Thus $\hat{z}_{11}$ is also continuously differentiable in f. (Here $\hat{z}_{11}$ corresponds to y_{11} in (B2) and $\hat{y}_{11}$ to x_{11} in (B1).) In summary, we can write

$$\hat{z}_{11} = \hat{z}_{11}(f, \hat{x}_{11})$$

where $\hat{z}_{11}$ is continuously differentiable in f and $\hat{x}_{11}$. Now suppose

$$p_1 = \begin{bmatrix} \hat{x}_{11} \\ 1 \end{bmatrix}$$

is a point in the limit cycle $\partial\hat{\mathcal{S}}(f)$, then we must have $\hat{z}_{11} = -\hat{x}_{11}$, i.e.,

$$\hat{z}_{11}(f, \hat{x}_{11}) + \hat{x}_{11} = 0 \tag{25}$$

due to the symmetry of $\partial\hat{\mathcal{S}}(f)$. We write $g(f, \hat{x}_{11}) = \hat{z}_{11}(f, \hat{x}_{11}) + \hat{x}_{11} = 0$.

By the uniqueness of the limit cycle, $\hat{x}_{11}$ is uniquely determined by f. By Lemmas 1 and 2, we know $\partial\hat{z}_{11}/\partial\hat{x}_{11} = (\partial\hat{z}_{11}/\partial\hat{y}_{11})\partial\hat{y}_{11}/\partial\hat{x}_{11} < -1$, so $\partial g/\partial\hat{x}_{11} \neq 0$ and by the implicit function theorem, $\hat{x}_{11}$ is differentiable in f. Recall that

$$p_1 = \begin{bmatrix} \hat{x}_{11} \\ 1 \end{bmatrix}$$

is a point in the vector field of (24). The corresponding intersection in the original system (22) is

$$V^{-1}(f)\begin{bmatrix} \hat{x}_{11} \\ 1 \end{bmatrix}$$

Clearly, it is also differentiable in f. □

Combining Propositions 4 and 5, we have

Corollary 1

Consider system (15) with A, b and f in the specified form. Given k_1 and k_2, $k_2 > k_1 > a_2/b_2$. Suppose that $A + kbf$ has distinct eigenvalues for all $k \in [k_1, k_2]$. Then $\mathscr{S}(k) \subset \mathscr{S}(k + \delta_k)$ for all $k \in [k_1, k_2]$, $\delta_k \in [0, k_2 - k]$.

Proof. By proposition 5, $\partial\mathscr{S}(k)$ is continuous in k for all $k \in [k_1, k_2]$. So $(p_2)_1$ and hence the function $\Delta(k)$ are also continuous in k. If follows that $\min\{\Delta(k)\colon k \in [k_1, k_2]\} > 0$. By applying Proposition 4, we have the corollary. □

It can be seen that there exists a $k_0 > 0$ such that $A + kbf$ has distinct eigenvalues for all $k > k_0$. Thus by Corollary 1, $\mathscr{S}(k)$ will be continuous and monotonically increasing for all $k > k_0$.

5. PROOF OF THEOREM 1: THE SECOND-ORDER CASE

We will prove the theorem by explicit construction of a family of feedback laws that solve the problem. To this end, let us first establish some preliminary results for a general system (1), not necessarily second order or anti-stable. Let $P(\varepsilon)$ be the positive definite solution of the Riccati equation.

$$A'P + PA - Pbb'P + \varepsilon I = 0 \tag{26}$$

It is known that $P(\varepsilon)$ is continuous for $\varepsilon \geqslant 0$. Let $f(\varepsilon) = -b'P(\varepsilon)$. With $u = kf(\varepsilon)x$, we have the closed-loop system

$$\dot{x} = Ax + b\,\mathrm{sat}(kf(\varepsilon)x + w) \tag{27}$$

Clearly, $A + kbf(\varepsilon)$ is Hurwitz for all $k \geqslant 0.5$. For $x(0) = x_0$, $w \in \mathscr{W}$, denote the state trajectory of (27) as $\psi(t, x_0, w)$.

Lemma 3

Consider system (27). Let $\varepsilon > 0$ be given. Let $c_\infty = \sigma_{\max}(P(\varepsilon))D^2/\varepsilon(2k-1)$, $c_0 = 4/b'P(\varepsilon)b$. Suppose k is sufficiently large such that $c_\infty < c_0$. Denote

$$\mathscr{S}_p(\varepsilon) := \{x\colon x'P(\varepsilon)x \leqslant c_0\}$$

$$\mathscr{S}_\infty(\varepsilon, k) := \{x\colon x'P(\varepsilon)x \leqslant c_\infty\}$$

Then, $\mathscr{S}_p(\varepsilon)$ and $\mathscr{S}_\infty(\varepsilon, k)$ are invariant sets, and, for any $w \in \mathscr{W}$, $x_0 \in \mathscr{S}_p(\varepsilon)$, $\psi(t, x_0, w)$ will enter $\mathscr{S}_\infty(\varepsilon, k)$ in a finite time and remain there.

Proof. Let $V(x) = x'P(\varepsilon)x$. It suffices to show that for all $x \in \mathscr{S}_p(\varepsilon)\backslash\mathscr{S}_\infty(\varepsilon, k)$ and for all $|w| \leqslant D$, $\dot{V} < 0$. In the following, we simply write $P(\varepsilon)$ as P and $f(\varepsilon)$ as f, since in this lemma, ε is fixed. Note that

$$\dot{V} = x'(A'P + PA)x + 2x'Pb\,\mathrm{sat}(kfx + w)$$

We will consider the case where $x'Pb \geqslant 0$. The case where $x'Pb \leqslant 0$ is similar.

If $kfx + w \leqslant -1$, then

$$\begin{aligned}\dot{V} &= x'(A'P + PA)x - 2x'Pb \\ &= x'Pbb'Px - 2x'Pb - \varepsilon x'x \\ &= x'Pb(x'Pb - 2) - \varepsilon x'x\end{aligned}$$

Since $x'Px \leqslant c_0 = 4/b'Pb$, we have $b'Px \leqslant \|b'P^{1/2}\| \, \|P^{1/2}x\| \leqslant 2$, and hence $\dot{V} < 0$.

If $kfx + w > -1$, then $\mathrm{sat}(kfx + w) \leqslant kfx + w$, and,

$$\begin{aligned}\dot{V} &\leqslant x'(A'P + PA)x + 2x'Pb(kfx + w) \\ &= -(2k-1)x'Pbb'Px - \varepsilon x'x + 2x'Pbw \\ &= -\left(\sqrt{2k-1}\,x'Pb - \frac{w}{\sqrt{2k-1}}\right)^2 + \frac{w^2}{2k-1} - \varepsilon x'x\end{aligned}$$

Since $x'Px > c_\infty = \sigma_{\max}(P)D^2/\varepsilon(2k-1)$, we have $x'x > D^2/\varepsilon(2k-1)$. It follows that $\dot{V} < 0$. □

It is clear from Lemma 3 that as k goes to infinity, $\mathscr{S}_\infty(\varepsilon, k)$ converges to the origin. In particular, there exists a k such that $\mathscr{S}_\infty(\varepsilon, k) \subset \chi_\infty$.

For any ANCBC system, as $\varepsilon \to 0$, $P(\varepsilon) \to 0$, and $c_0 \to \infty$. Thus $\mathscr{S}_p(\varepsilon)$ can be made arbitrarily large; and with a fixed ε, we can increase k to make c_∞ arbitrarily small. So the proof of Theorem 1 would have been completed here. However, for exponentially unstable systems, $\mathscr{S}_p(\varepsilon)$ is a quite small subset of $\mathscr{C}_a$ as $\varepsilon \to 0$ [16] and hence considerable work needs to be carried out before completing the proof.

Define the domain of attraction of the origin in the absence of disturbance as

$$\mathscr{S}(\varepsilon, k) := \left\{ x_0 : \lim_{t \to \infty} \psi(t, x_0, 0) = 0 \right\}$$

and in the presence of disturbance, define the domain of attraction of the set $\mathscr{S}_\infty(\varepsilon, k)$ as

$$\mathscr{S}_D(\varepsilon, k) := \left\{ x_0 : \lim_{t \to \infty} d(\psi(t, x_0, w), \mathscr{S}_\infty(\varepsilon, k)) = 0, \forall w \in \mathscr{W} \right\}$$

where $d(\psi(t, x_0, w), \mathscr{S}_\infty(\varepsilon, k))$ is the distance between the point $\psi(t, x_0, w)$ and the set $\mathscr{S}_\infty(\varepsilon, k)$. Our objective is to choose ε and k such that $\chi_0 \subset \mathscr{S}_D(\varepsilon, k)$ and $\mathscr{S}_\infty(\varepsilon, k) \subset \chi_\infty$.

Clearly $\mathscr{S}_p(\varepsilon) \subset \mathscr{S}_{\mathrm{D}}(\varepsilon, k) \subset \mathscr{S}(\varepsilon, k)$. By using the Lyapunov function $V(x) = x'P(\varepsilon)x$, we can only determine a subset $\mathscr{S}_p(\varepsilon)$ of $\mathscr{S}_{\mathrm{D}}(\varepsilon, k)$. As ε decreases, $P(\varepsilon)$ decreases. It was shown in Reference [17] that if $\varepsilon_1 < \varepsilon_2$, then $\mathscr{S}_p(\varepsilon_2) \subset \mathscr{S}_p(\varepsilon_1)$. So by decreasing ε, we can enlarge $\mathscr{S}_p(\varepsilon)$. However, since $\lim_{\varepsilon \to 0} \mathscr{S}_p(\varepsilon)$ can be much smaller than $\mathscr{C}_a$, we are unable to prove that $\mathscr{S}_{\mathrm{D}}(\varepsilon, k)$ is close to $\mathscr{C}_a$ by simply enlarging $\mathscr{S}_p(\varepsilon)$ as was done in Reference [15]. For this reason, we will resort to the detailed investigation on the vector field of (27) in the presence of the disturbance.

We now continue with the proof of the theorem and focus on the second order systems. Also assume that A is anti-stable. In this case $\mathscr{C}_a = \mathscr{C}$.

We will prove the theorem by showing that, given any $\chi_0 \subset \mathrm{int}(\mathscr{C})$, any (arbitrarily small) χ_∞ such that $0 \in \mathrm{int}(\chi_\infty)$, and any $D > 0$, there exist an $\varepsilon > 0$ and a $k \geqslant 0.5$ such that $\chi_0 \subset \mathscr{S}_{\mathrm{D}}(\varepsilon, k)$ and $\mathscr{S}_\infty(\varepsilon, k) \subset \chi_\infty$.

Proposition 3 applies to the case where $\varepsilon = 0$. It means that $\lim_{k\to\infty} d(\mathscr{S}(0,k),\mathscr{C}) = 0$. But when $\varepsilon = 0$, it is impossible to achieve disturbance rejection by increasing the value of k even if there is no saturation. We can first let $\varepsilon = 0$, choose k_0 sufficiently large so that $A + k_0 bf(\varepsilon)$ has distinct eigenvalues and $\chi_0 \subset \operatorname{int}(\mathscr{S}(0,k_0))$. Then by the continuity of the domain of attraction stated in Proposition 5 and the continuity of the solution of the Ricatti equation, we can fix this k_0 and choose ε sufficiently small so that $\chi_0 \subset \operatorname{int}(\mathscr{S}(\varepsilon,k_0))$. By Corollary 1, we know that $\mathscr{S}(\varepsilon,k)$ is non-decreasing, so $\chi_0 \subset \operatorname{int}(\mathscr{S}(\varepsilon,k))$ for all $k \geqslant k_0$. What remains to be shown is that for any given positive number D and a fixed ε, we can choose k sufficiently large so that $d(\mathscr{S}_D(\varepsilon,k),\mathscr{S}(\varepsilon,k))$ is arbitrarily small. Then we will have $\chi_0 \subset \mathscr{S}_D(\varepsilon,k)$ for some k.

Now, let us fix an ε such that $\chi_0 \subset \operatorname{int}(\mathscr{S}(\varepsilon,k))$, $\forall k \geqslant k_0$. Since ε is fixed, we can assume that a state transformation $\hat{x} = Vx$ like (23) is performed so that

$$\hat{f} = -b'P(\varepsilon)V^{-1} = [0 \;\; 1] \tag{28}$$

$$\hat{A} = VAV^{-1} = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2 \end{bmatrix}, \quad \hat{b} = Vb = \begin{bmatrix} -b_1 \\ -b_2 \end{bmatrix}, \qquad a_1, a_2, b_1, b_2 > 0 \tag{29}$$

where $a_1, a_2 > 0$ is from the anti-stability of A and $b_1, b_2 > 0$ follows from the fact that an LQ controller has infinite gain margin and $\varepsilon \neq 0$. ($b_1 = 0$ iff $\varepsilon = 0$). Under this state transformation, the sets $\mathscr{S}_p(\varepsilon)$, $\mathscr{S}_D(\varepsilon,k)$, $\mathscr{S}(\varepsilon,k)$, $\mathscr{S}_\infty(\varepsilon,k)$, $\mathscr{C}$, χ_0 and χ_∞ are transformed, respectively, into $\hat{\mathscr{S}}_p(\varepsilon)$, $\hat{\mathscr{S}}_D(\varepsilon,k)$, $\hat{\mathscr{S}}(\varepsilon,k)$, $\hat{\mathscr{S}}_\infty(\varepsilon,k)$, $\hat{\mathscr{C}}$, $\hat{\chi}_0$ and $\hat{\chi}_\infty$, all defined in an obvious way. For example, $\hat{\mathscr{C}} = \{Vx : x \in \mathscr{C}\}$. Let $\hat{P}(\varepsilon) = (V^{-1})'P(\varepsilon)V^{-1}$. Since ε is now fixed, we denote $\hat{P}(\varepsilon)$, $\hat{\mathscr{S}}_p(\varepsilon)$, $\hat{\mathscr{S}}_D(\varepsilon,k)$, $\hat{\mathscr{S}}(\varepsilon,k)$, and $\hat{\mathscr{S}}_\infty(\varepsilon,k)$, as $\hat{P}$, $\hat{\mathscr{S}}_p$, $\hat{\mathscr{S}}_D(k)$, $\hat{\mathscr{S}}(k)$ and $\hat{\mathscr{S}}_\infty(k)$, respectively.

Now we consider

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}\,\mathrm{sat}(k\hat{f}\hat{x} + w) \tag{30}$$

This standard form fits very well into Corollary 1, so we can be sure that $\hat{\mathscr{S}}(k)$ increases as k is increased. It follows that

$$\hat{\mathscr{S}}(k_0) \subset \hat{\mathscr{S}}(k), \quad \forall k > k_0$$

To satisfy the design requirement, it is necessary that no point in $\hat{\chi}_0 \backslash \hat{\chi}_\infty$ can be made stationary with any $|w| \leqslant D$. Let us first exclude this possibility by appropriate choice of k.

For a constant w, there are three candidate equilibrium points, $\hat{x}_e^+ = -\hat{A}^{-1}\hat{b}$, $\hat{x}_e^- = \hat{A}^{-1}\hat{b}$ and $\hat{x}_e^w = -(\hat{A} + k\hat{b}\hat{f})^{-1}\hat{b}w$, corresponding to $\mathrm{sat}(k\hat{f}\hat{x} + w) = 1$, $\mathrm{sat}(k\hat{f}\hat{x} + w) = -1$ and $\mathrm{sat}(k\hat{f}\hat{x} + w) = k\hat{f}\hat{x} + w$, respectively. For each of them to be an actual equilibrium point, we must have

$$k\hat{f}\hat{x}_e^+ + w \geqslant 1, \; k\hat{f}\hat{x}_e^- + w \leqslant -1 \text{ or } |k\hat{f}\hat{x}_e^w + w| \leqslant 1$$

respectively.

Here we have

$$\hat{x}_e^+ = \frac{1}{a_1}\begin{bmatrix} a_2 b_1 + a_1 b_2 \\ -b_1 \end{bmatrix}, \qquad \hat{x}_e^- = -\hat{x}_e^+, \qquad \hat{x}_e^w = \frac{1}{a_1 + b_1 k}\begin{bmatrix} a_2 b_1 + a_1 b_2 \\ -b_1 \end{bmatrix} w$$

If $\hat{A}$ has no complex eigenvalues, then $\hat{x}_e^+, \hat{x}_e^- \in \partial\hat{\mathscr{C}}$ [16], so $\hat{x}_e^+, \hat{x}_e^- \notin \hat{\chi}_0$ for any $\hat{\chi}_0 \subset \operatorname{int}(\hat{\mathscr{C}})$. But if $\hat{A}$ has a pair of complex eigenvalues, $\hat{x}_e^+, \hat{x}_e^- \in \operatorname{int}(\hat{\mathscr{C}})$ and will be in $\hat{\chi}_0$ if $\hat{\chi}_0$ is close enough to $\hat{\mathscr{C}}$. So, it is desirable that $\hat{x}_e^+$ and $\hat{x}_e^-$ cannot be made stationary by any $|w| \leqslant D$. This requires

$$k\hat{f}\hat{x}_e^+ + w < 1, \qquad k\hat{f}\hat{x}_e^- + w > -1, \quad \forall |w| \leqslant D$$

which is equivalent to $k(b_1/a_1) + w > -1, \forall |w| \leqslant D$. If $D \leqslant 1$, this is satisfied for all k; if $D > 1$, we need to choose k such that

$$k > \frac{a_1}{b_1}(D - 1)$$

Note that this will be impossible if $b_1 = 0$, which corresponds to the case where $\varepsilon = 0$. This is one reason that ε should be non-zero.

Finally, as $k \to \infty$, $\hat{x}_e^w \to 0$ for all $|w| \leqslant D$. So k can be chosen large enough such that $\hat{x}_e^w \notin \hat{\chi}_0 \backslash \hat{\chi}_\infty$.

In summary, from the above analysis, we will restrict ourselves to k such that

$$k > \frac{a_1}{b_1}(D - 1), \qquad \frac{D}{a_1 + b_1 k}\begin{bmatrix} a_2 b_1 + a_1 b_2 \\ -b_1 \end{bmatrix} \in \chi_\infty \tag{31}$$

To study the vector field of (30), we rewrite it as

$$\dot{\hat{x}}_1 = -a_1 \hat{x}_2 - b_1 \operatorname{sat}(k\hat{f}\hat{x} + w)$$
$$\dot{\hat{x}}_2 = \hat{x}_1 + a_2 \hat{x}_2 - b_2 \operatorname{sat}(k\hat{f}\hat{x} + w)$$

The vector field is much complicated by the presence of the disturbance. However, it still exhibits some properties which we will make use in our construction of the desired controller:

- Above the line $k\hat{f}\hat{x} = D + 1$, $k\hat{f}\hat{x} + w \geqslant 1$ for all $|w| \leqslant D$, so $\operatorname{sat}(k\hat{f}\hat{x} + w) = 1$, i.e. the vector $\dot{\hat{x}}$ is independent of w and is affine in $\hat{x}$. Similarly, below $k\hat{f}\hat{x} = -(D + 1)$, $\operatorname{sat}(k\hat{f}\hat{x} + w) = -1$.
- In the ellipsoid $\hat{\mathscr{S}}_p$, we have shown that all the trajectories will converge to $\hat{\mathscr{S}}_\infty(k)$, which can be made arbitrarily small by increasing the value of k.

Suppose that k is sufficiently large such that the boundary of $\hat{\mathscr{S}}_p$ intersects with the lines $k\hat{f}\hat{x} = \pm(D + 1)$. Denote the region between $k\hat{f}\hat{x} = (D + 1)$ and $k\hat{f}\hat{x} = -(D + 1)$, and to the left of $\hat{\mathscr{S}}_p$ as $Q(k)$, see the shaded region in Figure 4. Let

$$\hat{x}_m(k) = -\max\{\hat{x}_1 : \hat{x} \in Q(k)\}$$

If k is sufficiently large, then $Q(k)$ lies entirely in the left-half-plane, so $\hat{x}_m(k) > 0$. Choose K such that

$$-x_m(K) + a_2 \frac{D + 1}{K} < 0, \qquad \frac{-x_m(K) + a_2(D + 1)/K}{-a_1(D + 1)/K} > \frac{b_2}{b_1} \tag{32}$$

(Note that $x_m(k)$ increases as k is increased.) Then the vector field in $Q(k)$ has the following property:

Lemma 4

Suppose $k > K$. Then for all $\hat{x} \in Q(k)$, $|w| \leqslant D$,

$$\tan^{-1}\left(\frac{b_2}{b_1}\right) - \pi < \angle\left(\hat{A}\begin{bmatrix} -x_m(k) \\ \frac{D+1}{k} \end{bmatrix} + b\right) \leqslant \angle\dot{\hat{x}} < \tan^{-1}\left(\frac{b_2}{b_1}\right) \tag{33}$$

This implies that for any straight line E with slope b_2/b_1, if $\hat{x} \in E \cap Q(k)$, then the vector $\dot{\hat{x}}$ points to the right of E for all $|w| \leqslant D$.

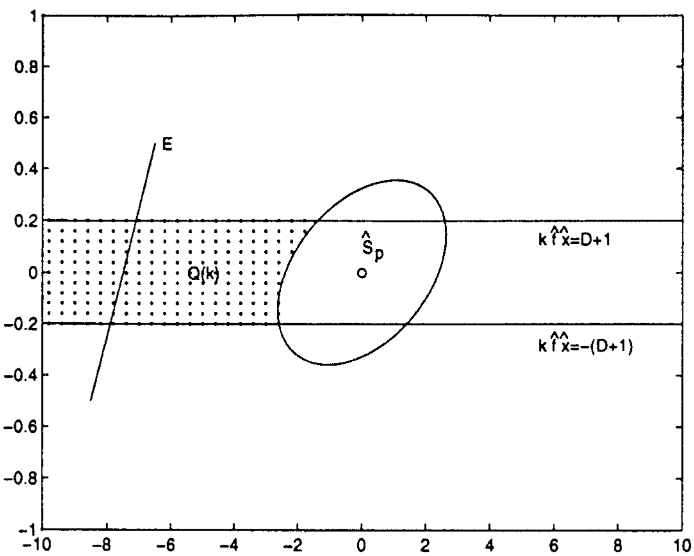

Figure 4. The vector field of system (30).

Proof. Between the lines $k\hat{f}\hat{x} = D + 1$ and $k\hat{f}\hat{x} = -(D+1)$, $\text{sat}(k\hat{f}\hat{x} + w)$ takes values in $[-1, 1]$ and hence,

$$\dot{\hat{x}} \in \left\{ \begin{bmatrix} -a_1\hat{x}_2 \\ \hat{x}_1 + a_2\hat{x}_2 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} : \lambda \in [-1, 1] \right\} \tag{34}$$

For $\hat{x} \in Q(k)$, if

$$\tan^{-1}\left(\frac{b_2}{b_1}\right) - \pi < \angle \hat{A}\hat{x} < \tan^{-1}\left(\frac{b_2}{b_1}\right)$$

then

$$\tan^{-1}\left(\frac{b_2}{b_1}\right) - \pi < \angle(\hat{A}\hat{x} + \lambda\hat{b}) < \tan^{-1}\left(\frac{b_2}{b_1}\right), \quad \forall \lambda \in [-1, 1] \tag{35}$$

Since $x_m(k)$ is increasing, we see from (32) that for all $k > K$,

$$-x_m(k) + a_2\frac{D+1}{k} < 0, \quad \frac{-x_m(k) + a_2\frac{D+1}{k}}{-a_1\frac{D+1}{k}} > \frac{b_2}{b_1}.$$

It follows that

$$\tan^{-1}\left(\frac{b_2}{b_1}\right) - \pi < \angle\left(\hat{A}\begin{bmatrix} -x_m(k) \\ \frac{D+1}{k} \end{bmatrix}\right) = \angle \begin{bmatrix} -a_1\frac{D+1}{k} \\ -x_m(k) + a_2\frac{D+1}{k} \end{bmatrix} < -\frac{\pi}{2} < \tan^{-1}\left(\frac{b_2}{b_1}\right)$$

For all $\hat{x} \in Q(k)$, we have $\hat{x}_1 \leqslant -x_m(k)$, $|\hat{x}_2| \leqslant (D+1)/k$. So

$$\angle\left(\hat{A}\begin{bmatrix} -x_m(k) \\ \frac{D+1}{k} \end{bmatrix}\right) \leqslant \angle\hat{A}\hat{x} \leqslant 0 \Rightarrow \tan^{-1}\left(\frac{b_2}{b_1}\right) - \pi < \angle\hat{A}\hat{x} < \tan^{-1}\left(\frac{b_2}{b_1}\right)$$

Hence by (35),

$$\tan^{-1}\left(\frac{b_2}{b_1}\right) - \pi < \angle\dot{\hat{x}} < \tan^{-1}\left(\frac{b_2}{b_1}\right)$$

for all $\hat{x} \in Q(k)$ and $|w| \leqslant D$. It can be further verified that

$$\min\{\angle \hat{A}\hat{x} + \lambda\hat{b} : \hat{x} \in Q(k),\ \lambda \in [-1, 1]\} \geqslant \angle\left(\hat{A}\begin{bmatrix} -x_m(k) \\ \frac{D+1}{k} \end{bmatrix} + b\right) > \tan^{-1}\left(\frac{b_2}{b_1}\right) - \pi$$

so (33) follows. □

This lemma means that any trajectory of (30) starting from inside of $Q(k)$ and to the right of E will remain to the right of E before it leaves $Q(k)$.

Based on Lemma 4, we can construct an invariant set $\hat{\mathscr{S}}_I(k) \subset \hat{\mathscr{S}}(k)$ and show that it is also a subset of $\hat{\mathscr{S}}_D(k)$. Moreover, it can be made arbitrarily close to $\hat{\mathscr{S}}(k)$.

Lemma 5

(a) if $k > K$ satisfies (31) and

$$\left(b_2 - \frac{a_2(D+1)}{k}\right) > \frac{b_1(D+1)}{kb_2} \tag{36}$$

then there exist unique $p_1,\ p_2 \in \hat{\mathscr{S}}(k)$ on the line $k\hat{f}\hat{x} = D + 1$ such that the trajectory of (30) starting at p_1 goes upward, returns to the line at p_2 and the line from p_2 to $-p_1$ has slope b_2/b_1 (see Figure 5, where the outer closed curve is $\partial\hat{\mathscr{S}}(k)$).

(b) Denote the region enclosed by the trajectories from $\pm p_1$ to $\pm p_2$, and the straight lines from $\pm p_2$ to $\mp p_1$ as $\hat{\mathscr{S}}_I(k)$. (In Figure 5, the region enclosed by the inner closed curve.) Then

$$\lim_{k\to\infty} d(\hat{\mathscr{S}}_I(k), \hat{\mathscr{S}}(k)) = 0$$

(c) $\hat{\mathscr{S}}_I(k)$ is an invariant set and $\hat{\mathscr{S}}_I(k) \subset \hat{\mathscr{S}}_D(k)$, i.e., it is inside the domain of attraction of $\hat{\mathscr{S}}_\infty(k)$.

Proof. Recall that $\partial\hat{\mathscr{S}}(k)$ is a closed trajectory of (30) with $w \equiv 0$. Denote the intersections of $\partial\hat{\mathscr{S}}(k)$ with $k\hat{f}\hat{x} = D + 1$ as s_1 and s_2 (see Figure 5). Let

$$p_0 = \begin{bmatrix} b_2 - \frac{a_2(D+1)}{k} \\ \frac{D+1}{k} \end{bmatrix}$$

then $\dot{\hat{x}}_2 = 0$ at p_0 and to the left (right) of p_0, $\dot{\hat{x}}_2 < 0(>0)$. Let p_1 be a point on $k\hat{f}\hat{x} = D + 1$ between p_0 and s_1, then a trajectory starting at p_1 goes upward and will return to $k\hat{f}\hat{x} = D + 1$ at some p_2 between p_0 and s_2. p_2 is uniquely determined by p_1. We then draw a straight line from p_2 with slope b_2/b_1. Let the intersection of the line with $k\hat{f}\hat{x} = -(D + 1)$ be p_3. Clearly, p_2 and p_3 depends on p_1 continuously. And the quantity

$$r(p_1) := \frac{(p_3 - (-s_1))_1}{(s_1 - p_1)_1}$$

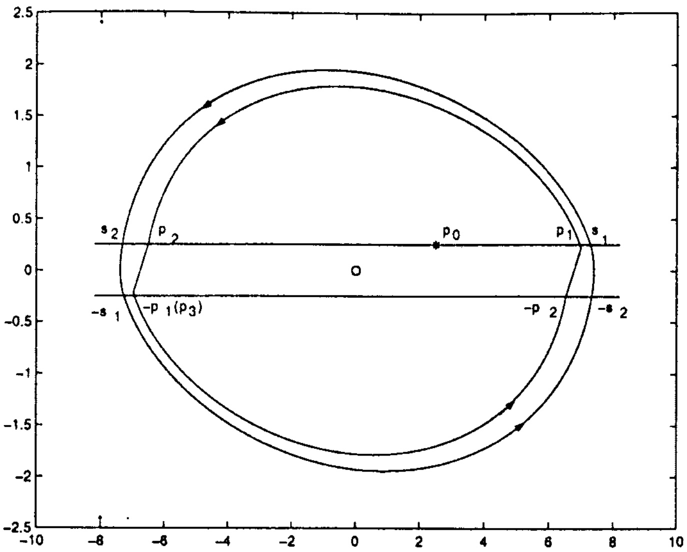

Figure 5. Illustration for Lemma 5.

also depends on p_1 continuously. If $p_1 = s_1$, then $p_2 = s_2$. Note that the trajectories above the line $k\hat{f}\hat{x} = D + 1$ are independent of w and hence are the same with those with $w = 0$. Since s_2 and $-s_1$ are on a trajectory of (30) with $w = 0$, so by Lemma 4, $-s_1$ must be to the right of the straight line with slope b_2/b_1 that passes s_2. This shows $-s_1$ is to the right of p_3 (with $p_1 = s_1$) and hence $\lim_{p_1 \to s_1} r(p_1) = -\infty$. If $p_1 = p_0$, then $p_2 = p_0$ and

$$p_3 = \begin{bmatrix} b_2 - \frac{a_2(D+1)}{k} - \frac{2(D+1)b_1}{kb_2} \\ -\frac{D+1}{k} \end{bmatrix}$$

So

$$r(p_1 = p_0) = \frac{(s_1)_1 + b_2 - \frac{a_2(D+1)}{k} - \frac{2(D+1)b_1}{kb_2}}{(s_1)_1 - (b_2 - \frac{a_2(D+1)}{k})}$$

And by condition (36), $r(p_1 = p_0) > 1$. Since $\lim_{p_1 \to s_1} r(p_1) = -\infty$, by the continuity of $r(p_1)$, there exists a p_1 between s_1 and p_0 such that $r(p_1) = 1$, i.e. $p_3 = -p_1$ and hence the line from p_2 to $-p_1$ has slope b_2/b_1. This shows the existence of (p_1, p_2) in (a). Suppose on the contrary that such pair (p_1, p_2) is not unique and there exists (p'_1, p'_2) with the same property, say, p'_1 to the left of p_1 and p'_2 to the right of p_2, by Lemma 1, $\|p_2 - p'_2\| > \|p_1 - p'_1\|$. But $\|(-p_1) - (-p'_1)\| = \|p_2 - p'_2\|$ since the line from p_2 to $-p_1$ and that from p'_2 to $-p'_1$ have the same slope. This is a contradiction.

(b) We see that $\dot{\hat{x}}_2 = 0$ at p_0, so by applying Lemma 1 with a shifting of the origin,

$$\frac{\|p_2 - s_2\|}{\|p_1 - s_1\|} > \frac{\|p_0 - s_2\|}{\|p_0 - s_1\|} > 1$$

(refer to (12)). As $k \to \infty$, $s_2 + s_1 \to 0$, and

$$p_0 \to \begin{bmatrix} b_2 \\ 0 \end{bmatrix}$$

Since s_1 and s_2 are restricted to the null controllable region $\hat{\mathscr{C}}$, there exist some $K_1 > 0$, $\gamma > 0$, such that for all $k > K_1$,

$$\frac{\|p_0 - s_2\|}{\|p_0 - s_1\|} \geqslant 1 + \gamma \Rightarrow \frac{\|p_2 - s_2\|}{\|p_1 - s_1\|} > 1 + \gamma \tag{37}$$

From Figure 5, we see that

$$\|p_2 - s_2\| = \|p_1 - s_1\| + (-s_1 - s_2)_1 + (p_2 - (-p_1))_1$$

As $k \to \infty$, $2(D+1)/k \to 0$, so $(p_2 - (-p_1))_1 \to 0$. Since $s_1 + s_2 \to 0$, we have

$$\|p_2 - s_2\| - \|p_1 - s_1\| \to 0$$

From (37), $\|p_2 - s_2\| - \|p_1 - s_1\| > \gamma \|p_1 - s_1\|$. So we must have $\|p_1 - s_1\| \to 0$ and hence $\|p_2 - s_2\| \to 0$. Therefore, $\lim_{k\to\infty} d(\hat{\mathscr{S}}_I(k), \hat{\mathscr{S}}(k)) = 0$.

(c) First we show that $\hat{\mathscr{S}}_I(k)$ is an invariant set. Note that $\partial\hat{\mathscr{S}}_I(k)$ from p_1 to p_2 and that from $-p_1$ to $-p_2$ are trajectories of (30) under any $|w| \leqslant D$. At any point on the line from p_2 to $-p_1$, Lemma 4 says that $\dot{\hat{x}}$ directs to the right side of the line, i.e. no trajectory can cross the line from p_2 to $-p_1$ leftward, symmetrically, no trajectory can cross the line from $-p_2$ to p_1 rightward. These show that no trajectory can cross $\partial\hat{\mathscr{S}}_I(k)$ outward, thus $\hat{\mathscr{S}}_I k)$ is an invariant set. Since $\hat{\mathscr{S}}_p$ is also an invariant set and any trajectory that starts from inside of it will converge to $\hat{\mathscr{S}}_\infty(k)$, it suffices to show that any trajectory that starts from inside of $\hat{\mathscr{S}}_I(k)$ will enter $\hat{\mathscr{S}}_p$. We will do this by contradiction.

Suppose that there exist an $\hat{x}_0 \in \hat{\mathscr{S}}_I(k)\backslash\hat{\mathscr{S}}_p$ and a $w \in \mathscr{W}$ such that $\psi(t, \hat{x}_0, w) \in \hat{\mathscr{S}}_I(k)\backslash\hat{\mathscr{S}}_p$ for all $t > 0$, then there must be a point $\hat{x}^* \in \hat{\mathscr{S}}_I(k)\backslash\hat{\mathscr{S}}_p$ either

(1) $\lim_{t\to\infty} \psi(t, \hat{x}_0, w) = \hat{x}^*$; or
(2) there exists a sequence $t_1, t_2, \ldots, t_i, \ldots$ such that $\lim_{i\to\infty} \psi(t_i, \hat{x}_0, w) = \hat{x}^*$ and there is an $\varepsilon > 0$ such that for any $T > 0$, there exists $t > T$ satisfying $\|\psi(t, \hat{x}_0, w) - \hat{x}^*\| > \varepsilon$.

Item (1) implies that $\hat{x}^*$ can be made stationary by some $w \in \mathscr{W}$. This is impossible as we have shown that k has been chosen such that all the stationary points are inside $\hat{\mathscr{S}}_\infty(k)$. Item (2) implies that there is a closed trajectory with length greater than 2ε that passes through $\hat{x}^*$. There are two possibilities here: the closed trajectory encloses $\hat{\mathscr{S}}_p$ or it does not enclose $\hat{\mathscr{S}}_p$. We will show that none of the cases is possible.

Suppose that there is a closed trajectory that encloses $\hat{\mathscr{S}}_p$. Let q_1, q_2, q_3, q_4 be the four intersections of the closed trajectory with $k\hat{f}\hat{x} = \pm(D+1)$ as shown in Figure 6. By Lemma 1

$$\|p_2 - q_2\| > \|p_1 - q_1\|, \ \|q_4 - (-p_2)\| > \|q_3 - (-p_1)\|$$

and by Lemma 4,

$$\|q_3 - (-p_1)\| \geqslant \|p_2 - q_2\|, \quad \|p_1 - q_1\| \geqslant \|q_4 - (-p_2)\|$$

So we have

$$\|p_2 - q_2\| > \|p_1 - q_1\| \geqslant \|q_4 - (-p_2)\| > \|q_3 - (-p_1)\| \geqslant \|p_2 - q_2\|$$

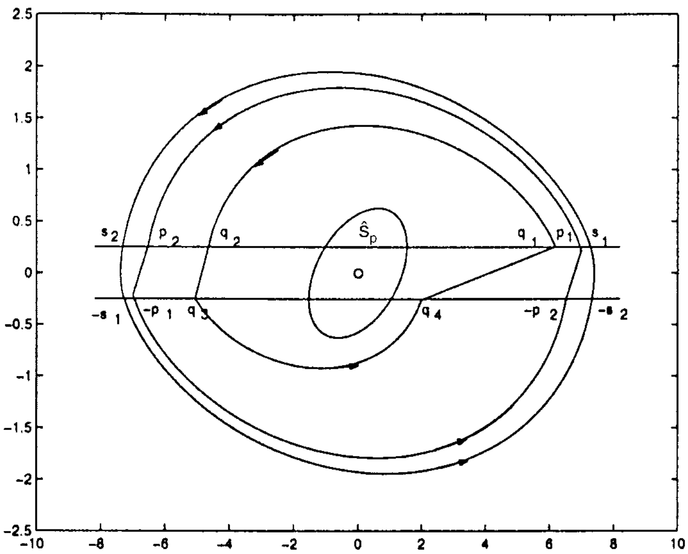

Figure 6. Illustration of the proof.

A contradiction. Therefore, there exists no closed trajectory that encloses $\hat{\mathscr{S}}_p$. We next exclude the other possibility.

Clearly, there can be no closed trajectory that is completely above $k\hat{f}\hat{x} = D + 1$ or below $k\hat{f}\hat{x} = -(D+1)$. So if there is a closed trajectory, it must intersect $k\hat{f}\hat{x} = D + 1$ or $k\hat{f}\hat{x} = -(\bar{D}+1)$ to the left (or to the right) of $\hat{\mathscr{S}}_p$ at least twice, or lies completely within $Q(k)$. We assume it is to the left of $\hat{\mathscr{S}}_p$. Since $k > K$ satisfies (36), so $x_m(k) > 0$ and $b_2 - a_2(D+1)/k > 0$. Hence for all points on the line $k\hat{f}\hat{x} = D + 1$ to the left of $\hat{\mathscr{S}}_p$, $\dot{\hat{x}}_2 < 0$, so no closed trajectory lying between $\hat{\mathscr{S}}_I(k)$ and $\hat{\mathscr{S}}_p$ will cross this piece of straight line twice. On the line $k\hat{f}\hat{x} = -(D+1)$ to the left of $\hat{\mathscr{S}}_p$, $\dot{\hat{x}}_1 > 0$. Since no trajectory in $Q(k)$ will cross a line that is parallel to the line from p_2 to $-p_1$ leftward, there will be no closed trajectory crossing the line $k\hat{f}\hat{x} = -(D+1)$ to the left of $\hat{\mathscr{S}}_p$ twice. In view of Lemma 4, there exists no closed trajectory completely inside $Q(k)$. These show that there exist no closed trajectory that does not enclose $\hat{\mathscr{S}}_p$ either.

In conclusion, for every $\hat{x}_0 \in \mathscr{S}_I(k)$, there must be a $T < \infty$ such that $\psi(T, \hat{x}_0, w) \in \hat{\mathscr{S}}_p$. And since $\hat{\mathscr{S}}_p$ is in the domain of attraction of $\hat{\mathscr{S}}_\infty(k)$, it follows that $\hat{x}_0 \in \hat{\mathscr{S}}_D(k)$ and hence $\hat{\mathscr{S}}_I(k) \subset \hat{\mathscr{S}}_D(k)$. □

The proof of Theorem 1 can be completed by invoking Lemmas 3 and 5. For clarity, we organize it as follows, including a constructive method to choose the parameters ε and k.

Proof of Theorem 1. Given $\chi_0 \subset \text{int}(\mathscr{C})$, χ_∞ such that $0 \in \text{int}\,\chi_\infty$ and $D > 0$, we need to choose ε and k such that $\chi_0 \subset \mathscr{S}_D(\varepsilon, k)$ and $\mathscr{S}_\infty(\varepsilon, k) \subset \chi_\infty$.

Step 1: Let $\varepsilon = 0$ and find k_0 such that $\chi_0 \subset \text{int}(\mathscr{S}(0, k_0))$. This is guaranteed by Proposition 3. Increase k_0, if necessary, such that $A + k_0 b f(\varepsilon)$ has distinct eigenvalues.

Step 2: Find $\varepsilon > 0$ such that $\chi_0 \subset \text{int}(\mathscr{S}(\varepsilon, k_0))$. This is guaranteed by Proposition 5 that $\mathscr{S}(\varepsilon, k_0)$ is continuous in $f(\varepsilon)$ and $f(\varepsilon)$ is continuous in ε.

Step 3: Fix ε and perform state transformation $\hat{x} = Vx$ such that $(\hat{f}, \hat{A}, \hat{b})$ is in the form of (28) and (29). Also perform this transformation to the sets χ_0, χ_∞ to get $\hat{\chi}_0$, $\hat{\chi}_\infty$. We do not need to transform $\mathscr{S}(\varepsilon, k_0)$ to $\hat{\mathscr{S}}(k_0)$ but should remember that $\hat{\chi}_0 \subset \mathrm{int}(\hat{\mathscr{S}}(k_0))$.

Step 4: Find $k > K$ satisfying (31) and (36) such that $\hat{\chi}_0 \subset \hat{\mathscr{S}}_I(k)$. Since $\hat{\chi}_0 \subset \mathrm{int}(\hat{\mathscr{S}}(k_0))$, we have $\hat{\chi}_0 \subset \mathrm{int}(\hat{\mathscr{S}}(k))$ for all $k > k_0$. So by Lemma 5, $\hat{\chi}_0 \subset \hat{\mathscr{S}}_I(k) \subset \hat{\mathscr{S}}_D(k)$ for some $k > 0$. It follows that $\chi_0 \subset \mathscr{S}_D(\varepsilon, k)$.

Step 5: Increase k, if necessary, so that $\mathscr{S}_\infty(\varepsilon, k) \subset \chi_\infty$. This is possible due to Lemma 3. □

6. PROOF OF THEOREM 1: HIGHER-ORDER SYSTEMS

As with the stabilization problem in Reference [16], where the disturbance is absent, the main idea in this section is first to bring those exponentially unstable states to a 'safe set' by using partial state feedback, then to switch to a full state feedback that steers all the states to a neighbourhood of the origin. The first step control is justified in the last section and the second step control is guaranteed by the property of the solution of the Riccati equation and Lemma 3, which allow the states that are not exponentially unstable to grow freely.

Without loss of generality, assume that the matrix pair (A, b) in system (1) is in the form of

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $A_1 \in \mathbf{R}^{2\times 2}$ is anti-stable and $A_2 \in \mathbf{R}^n$ is semi-stable. Assume that (A, b) is stabilizable. Denote the null controllable region of the subsystem

$$\dot{x}_1 = A_1 x_1 + b_1 \mathrm{sat}(u)$$

as $\mathscr{C}_1$. Then the asymptotically null controllable region of (1) is $\mathscr{C}_a = \mathscr{C}_1 \times \mathbf{R}^n$. Given any $\gamma_1 \in (0, 1)$, and $\gamma_2 > 0$, denote

$$\Omega_1(\gamma_1) := \{\gamma_1 x_1 \in \mathbf{R}^2 : x_1 \in \bar{\mathscr{C}}_1\}, \qquad \Omega_2(\gamma_2) := \{x_2 \in \mathbf{R}^n : \|x_2\| \leqslant \gamma_2\} \tag{38}$$

For any compact subset χ_0 of $\mathscr{C}_a = \mathscr{C}_1 \times \mathbf{R}^n$, there exist γ_1 and γ_2 such that $\chi_0 \subset \Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$. For this reason, we assume, without loss of generality, that $\chi_0 = \Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$.

For $\varepsilon > 0$, let

$$P(\varepsilon) = \begin{bmatrix} P_1(\varepsilon) & P_2(\varepsilon) \\ P_2'(\varepsilon) & P_3(\varepsilon) \end{bmatrix} \in \mathbf{R}^{(2+n)\times(2+n)}$$

be the unique positive definite solution to the ARE

$$A'P + PA - Pbb'P + \varepsilon I = 0 \tag{39}$$

Clearly, as $\varepsilon \downarrow 0$, $P(\varepsilon)$ decreases. Hence $\lim_{\varepsilon\to 0} P(\varepsilon)$ exists.

Let P_1 be the unique positive definite solution to the ARE

$$A_1' P_1 + P_1 A_1 - P_1 b_1 b_1' P_1 = 0$$

Then by the continuity property of the solution of the Riccati equation [18],

$$\lim_{\varepsilon\to 0} P(\varepsilon) = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let $f(\varepsilon) := -b'P(\varepsilon)$. Let us first study the following closed-loop system

$$\dot{x} = Ax + b\,\mathrm{sat}(kf(\varepsilon)x + w) \tag{40}$$

Recall from Lemma 3, the invariant set $\mathscr{S}_p(\varepsilon)$ is a domain of attraction of the set $\mathscr{S}_\infty(\varepsilon, k)$.

Lemma 6
Denote

$$r_1(\varepsilon) := \frac{1}{\|P_1^{1/2}(\varepsilon)\|\,\|b'P^{1/2}(\varepsilon)\|}$$

$$r_2(\varepsilon) := \frac{-\|P_2(\varepsilon)\| + \sqrt{\|P_2(\varepsilon)\|^2 + 3\|P_1(\varepsilon)\|\,\|P_3(\varepsilon)\|}}{\|P_3(\varepsilon)\|}\, r_1(\varepsilon)$$

Then

$$D_1(\varepsilon) := \{x \in \mathbf{R}^{2+n} : \|x_1\| \leqslant r_1(\varepsilon),\ \|x_2\| \leqslant r_2(\varepsilon)\} \subset \mathscr{S}_p(\varepsilon)$$

Moreover, $\lim_{\varepsilon \to 0} r_2(\varepsilon) = \infty$, and $r_1(\varepsilon)$ increases with an upper bound as ε tends to zero.

Proof. Similar to the proof of Lemma 4.4.1 in Reference [16]. □

Proof of Theorem 1. Denote $\chi_{10} = \Omega(\gamma_1)$, then $\chi_{10} \subset \mathrm{int}(\mathscr{C}_1)$. Given $\varepsilon_0 > 0$, let $\chi_{1\infty} = \{x_1 \in \mathbf{R}^2 : \|x_1\| \leqslant r_1(\varepsilon_0)\}$. By the result of the second-order case, there exists a controller $u = f_1 x_1$ such that any trajectory of

$$\dot{x}_1 = A_1 x_1 + b_1\,\mathrm{sat}(f_1 x_1 + w) \tag{41}$$

that starts from within χ_{10} will converge to $\chi_{1\infty}$ at a finite time and stay there. Denote the trajectory of (41) that starts at x_{10} as $\psi_1(t, x_{10}, w)$ and define

$$T_M := \max_{x_{10} \in \partial\chi_{10},\, w \in \mathscr{W}} \min\{t > 0 : \psi_1(t, x_{10}, w) \in \chi_{1\infty}\}$$

(An upper bound on T_M can be obtained by estimating the largest possible length of a trajectory $\psi_1(t, x_{10}, w)$, $x_{10} \in \chi_{10}$ before it enters $\chi_{1\infty}$ from Lemma 1 and (33), and the minimal $\|\dot{x}_1\|$ outside of $\chi_{1\infty}$. To apply (33), we can construct a region similar to $Q(k)$ by using $\chi_{1\infty}$ instead of $\hat{\mathscr{S}}_p$.) Let

$$\gamma = \max_{t \in [0, T_M]} \|\mathrm{e}^{A_2 t}\|\,\gamma_2 + \int_0^{T_M} \|\mathrm{e}^{A_2(T_M - \tau)} b_2\|\,\mathrm{d}\tau \tag{42}$$

then by Lemma 6, there exists an $\varepsilon < \varepsilon_0$ such that $r_1(\varepsilon) \geqslant r_1(\varepsilon_0)$, $r_2(\varepsilon) \geqslant \gamma$ and

$$D_1(\varepsilon) = \{x \in \mathbf{R}^{2+n} : \|x_1\| \leqslant r_1(\varepsilon),\ \|x_2\| \leqslant r_2(\varepsilon)\} \subset \mathscr{S}_p(\varepsilon)$$

lies in the domain of attraction of $\mathscr{S}_\infty(\varepsilon, k)$.

Choose k such that $\mathscr{S}_\infty(\varepsilon, k) \subset \chi_\infty$, and let the combined controller be

$$u(t) = \begin{cases} f_1 x_1(t), & x \notin \mathscr{S}_p(\varepsilon) \\ kf(\varepsilon)x(t), & x \in \mathscr{S}_p(\varepsilon) \end{cases} \tag{43}$$

and consider an initial state of the closed-loop system of (1) with (43), $x_0 \in \Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$. If $x_0 \in \mathscr{S}_p(\varepsilon)$, then $x(t)$ will enter $\mathscr{S}_\infty(\varepsilon, k) \subset \chi_\infty$. If $x_0 \notin \mathscr{S}_p(\varepsilon)$, we conclude that $x(t)$ will enter $\mathscr{S}_p(\varepsilon)$ at some $T_1 \leqslant T_M$ under the control $u = f_1 x_1$. Observe that under this control, $x_1(t)$ goes along a trajectory of (41). If there is no switch, $x_1(t)$ will hit the ball $\chi_{1\infty}$ at $T_1 \leqslant T_M$ and at this instant $\|x_2(T_1)\| \leqslant \gamma \leqslant r_2(\varepsilon)$, so $x(T_1) \in D_1(\varepsilon)$. Thus we see that if there is no switch, $x(t)$ will be in $D_1(\varepsilon)$ at T_1. Since $D_1(\varepsilon) \subset \mathscr{S}_p(\varepsilon)$, $x(t)$ must have entered $\mathscr{S}_p(\varepsilon)$ at some earlier time $T \leqslant T_1 \leqslant T_M$. So we have the conclusion. With the switching control applied, once $x(t)$ enters the invariant set $\mathscr{S}_p(\varepsilon)$, it will converge to $\mathscr{S}_\infty(\varepsilon, k)$ and remain there. $\square$

7. EXAMPLE

In this section, we will use an aircraft model to demonstrate the results obtained in this paper. Consider the longitudinal dynamics of the TRANS3 aircraft under certain flight condition [19],

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 14.3877 & 0 & -31.5311 \\ -0.0012 & -0.4217 & 1.0000 & -0.0284 \\ 0.0002 & -0.3816 & -0.4658 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 4.526 \\ -0.0337 \\ -1.4566 \\ 0 \end{bmatrix} v$$

The states z_1, z_2, z_3 and z_4 are the velocity, the angle of attack, the pitch rate and the Euler angle rotation of aircraft about the inertial y-axis, respectively. The control v is the elevator input, which is bounded by 10°, or 0.1745 rad. With a state transformation of the form $x = Tz$ and the input normalization such that the control is bounded by 1, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \mathrm{sat}(u + w)$$

where

$$A_1 = \begin{bmatrix} 0.0212 & 0.1670 \\ -0.1670 & 0.0212 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.4650 & 0.6247 \\ -0.6247 & -0.4650 \end{bmatrix}$$

and

$$b_1 = \begin{bmatrix} 8.2856 \\ -2.4303 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 0.7584 \\ -1.8562 \end{bmatrix}$$

The system has two stable modes -0.4650 ± 0.6247i and two anti-stable ones, 0.0212 ± 0.1670i. Suppose that w is bounded by $|w| \leqslant D = 2$.

For the anti-stable x_1-subsystem, we take $\gamma_1 = 0.9$. With the technique in Section 5, we obtain a feedback $u = f_1 x_1$, where $f_1 = [-0.4335 \;\; 0.2952]$, such that $\Omega_1(\gamma_1)$ (as defined in (38)) is inside some invariant set $\mathscr{S}_I$. Moreover, for all initial $x_{10} \in \mathscr{S}_I$, under the control $u = f_1 x_1$, $x_1(t)$ will enter a ball $\chi_1 = \{x_1 \in \mathbf{R}^2 : \|x_1\|_2 \leqslant 29.8501\}$. In Figure 7, the outermost dotted closed curve is the boundary of the null controllable region $\partial\mathscr{C}_1$, the inner dash-dotted closed curve is $\partial\mathscr{S}_I$, the dashed closed curve is $\partial\Omega_1(\gamma_1)$, and the innermost solid closed curve is $\partial\chi_1$.

The x_2-subsystem is exponentially stable. Under the saturated control, it can be shown that for any initial value $x_{20} \in \mathbf{R}^2$, there exists a $T > 0$ such that $x_2(t)$ will enter a bounded ball at time T and remain there. The bounded ball is computed as

$$\chi_2 = \{x_2 \in \mathbf{R}^2 \colon \|x_2\|_2 \leqslant 4\}.$$

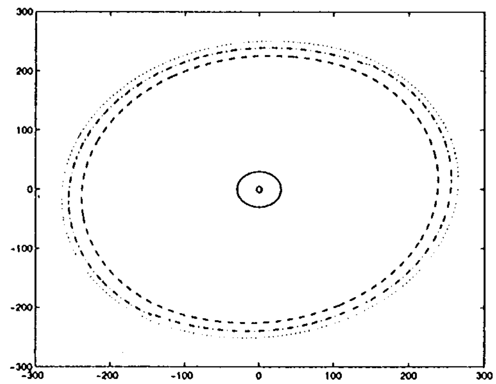

Figure 7. Design partial feedback $u = f_1 x_1$ such that $\Omega_1(\gamma_1) \subset \mathscr{S}_I$.

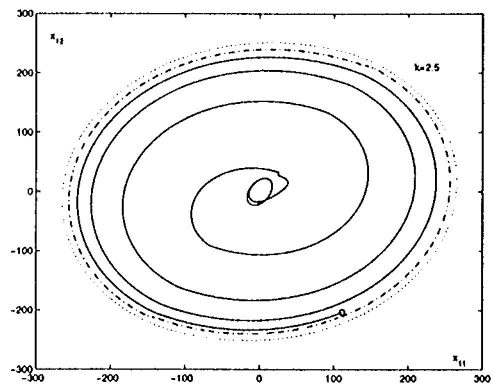

Figure 8. A trajectory of x_1 with $\varepsilon = 0.03$, $k = 2.5$.

We see that, for any $(x_{10}, x_{20}) \in \mathscr{S}_I \times \mathbf{R}^2$, under the partial feedback control $u = f_1 x_1$, the state (x_1, x_2) will enter the set $\chi_1 \times \chi_2$ in a finite time and remain there. The next step is to design a full state feedback to make the set $\chi_1 \times \chi_2$ inside the domain of attraction of an arbitrarily small set.

Choose $\varepsilon = 0.03$, we get

$$P(\varepsilon) = 0.001 \times \begin{bmatrix} 0.9671 & 0.0005 & -0.0686 & 0.0375 \\ 0.0005 & 0.9664 & 0.0345 & -0.0410 \\ -0.0686 & 0.0345 & 4.1915 & -0.7462 \\ 0.0375 & -0.0410 & -0.7462 & 11.3408 \end{bmatrix}$$

$f(\varepsilon) = 0.001 \times [-0.0729 \;\; 1.408 \;\; -36.4271 \;\; 33.6402]$, and $\mathscr{S}_p(\varepsilon) = \{x \in \mathbf{R}^4 : x' P(\varepsilon) x \leqslant 10.3561\}$. It can be verified that $\chi_1 \times \chi_2 \subset \mathscr{S}_p(\varepsilon)$. This implies that under the control $u = f_1 x_1$, the state will enter $\mathscr{S}_p(\varepsilon)$ at a finite time. If k is sufficiently large, then under the control $u = k f(\varepsilon) x$, $\mathscr{S}_p(\varepsilon)$ will be an invariant set. In this case, the switching controller (43) is well defined.

The final step is to choose k sufficiently large such that the state will converge to an arbitrarily small subset. We illustrate this point by simulation results for different values of k. In the

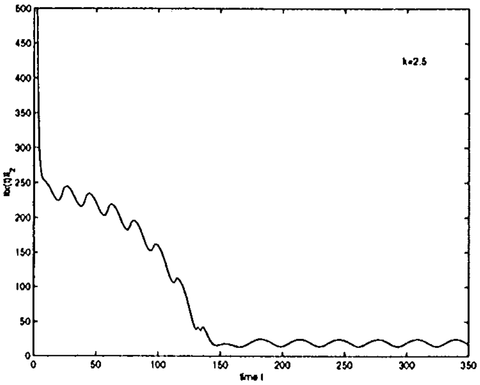

Figure 9. Time response of $\|x(t)\|_2$ with $\varepsilon = 0.03$, $k = 2.5$.

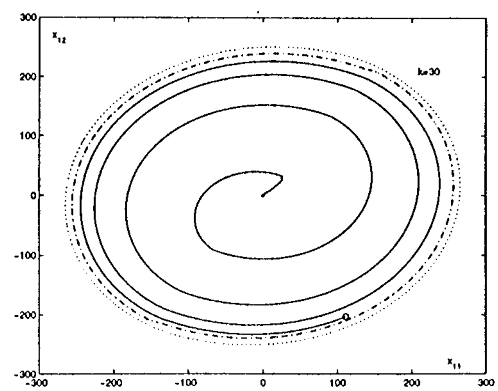

Figure 10. A trajectory of x_1 with $\varepsilon = 0.03$, $k = 30$.

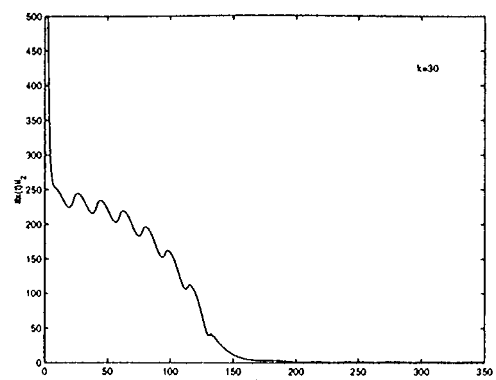

Figure 11. Time response of $\|x(t)\|_2$ with $\varepsilon = 0.03$, $k = 30$.

simulation, we choose $w(t) = 2\sin(0.1t)$ and x_{10} to be a point very close to the boundary of $\mathscr{S}_I$, see the point market with 'o' in Figures 8 and 10. We also set $x_{20} = [1000\ 1000]^{\mathrm{T}}$, which is very far away from the origin. When $k = 2.5$, the disturbance is not satisfactorily rejected (see Figure 8 for a trajectory of x_1 and Figure 9 for the time response of $\|x(t)\|_2$). When $k = 30$, the disturbance is rejected to a much higher level of accuracy (see Figures 10 and 11).

8. CONCLUSIONS

For linear exponentially unstable systems subject to actuator saturation and input additive disturbance, we have solved the problem of semi-global practical stabilization. We have assumed that the open-loop system has only two anti-stable modes and our results generalized the existing results on systems that do not have any exponentially unstable poles. Our analysis relies heavily on limit cycle theory and vector fields analysis of the exponentially unstable subsystem. It is not expected that our results can be further extended in a direct way to systems with more than two exponentially unstable open-loop poles.

APPENDIX A: PROOF OF LEMMA 1

Since at the intersection p', the trajectory goes downward, so $y_{11} < -a_2/k$. Using the fact that $fp = fp' = 1/k$ and $p' = \mathrm{e}^{AT}p$, we have

$$[0\ k]\mathrm{e}^{AT}\begin{bmatrix} x_{11} \\ \frac{1}{k} \end{bmatrix} = 1 \tag{A1}$$

$$[0\ k]\mathrm{e}^{-AT}\begin{bmatrix} y_{11} \\ \frac{1}{k} \end{bmatrix} = 1 \tag{A2}$$

From (A1) and (A2), x_{11} and y_{11} can be expressed as functions of T. In other words, x_{11} and y_{11} are related to each other through the parameter T. Since the domain of valid x_{11} can be finite or infinite depending on the location of the eigenvalues of A, it is necessary to break the proof for different cases. We will see later that the relation among x_{11}, y_{11} and T are quite different for different cases.

Case 1:

$$A = \begin{bmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{bmatrix}$$

has two different real eigenvalues $\lambda_1, \lambda_2 > 0$. Assume that $\lambda_1 > \lambda_2$.
Let

$$V = \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix}$$

then

$$\mathrm{e}^{AT} = V\begin{bmatrix} \mathrm{e}^{\lambda_1 T} & 0 \\ 0 & \mathrm{e}^{\lambda_2 T} \end{bmatrix}V^{-1}$$

From (A1) and (A2) we have

$$x_{11}(T) = \frac{1}{k}\frac{\lambda_1 - \lambda_2 + \lambda_2 e^{\lambda_2 T} - \lambda_1 e^{\lambda_1 T}}{e^{\lambda_1 T} - e^{\lambda_2 T}} \quad \text{(A3)}$$

$$y_{11}(T) = \frac{1}{k}\frac{\lambda_1 - \lambda_2 + \lambda_2 e^{-\lambda_2 T} - \lambda_1 e^{-\lambda_1 T}}{e^{-\lambda_1 T} - e^{-\lambda_2 T}} \quad \text{(A4)}$$

Due to the uniqueness of the trajectory, T is also uniquely determined by x_{11}. So, $x_{11} \leftrightarrow T$, $x_{11} \leftrightarrow y_{11}$, $y_{11} \leftrightarrow T$ are all one to one maps. From the above two equations, we know that $x_{11}(T)$ and $y_{11}(T)$ are analytic on $(0, \infty)$. It can be verified from (A3) that

$$\lim_{T \to 0} x_{11} = -\frac{\lambda_1 + \lambda_2}{k} = -\frac{a_2}{k}, \quad \lim_{T \to \infty} x_{11} = -\frac{\lambda_1}{k} = a_m$$

so we know the valid domain of x_{11} is $(-a_2/k, a_m)$. It can also be verified that $dx_{11}/dT > 0$, or $dT/dx_{11} > 0$.

Denote $g(T) := -dy_{11}/dx_{11}$, then

$$g(T) = \frac{\lambda_1 - \lambda_2 + \lambda_2 e^{\lambda_1 T} - \lambda_1 e^{\lambda_2 T}}{\lambda_1 - \lambda_2 + \lambda_2 e^{-\lambda_1 T} - \lambda_1 e^{-\lambda_2 T}}$$

It can be verified that $\lim_{T \to 0} g(T) = 1$ and

$$\frac{dg}{dT} = \frac{\lambda_1 \lambda_2 (e^{\lambda_1 T} - e^{\lambda_2 T})}{(\lambda_1 - \lambda_2 + \lambda_2 e^{-\lambda_1 T} - \lambda_1 e^{-\lambda_2 T})^2} h(T)$$

where $h(T) = (\lambda_1 - \lambda_2)(1 - e^{-(\lambda_1 + \lambda_2)T}) + (\lambda_1 + \lambda_2)(e^{-\lambda_1 T} - e^{-\lambda_2 T})$. Since $h(0) = 0$ and

$$\frac{dh}{dT} = (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)}(\lambda_1 - \lambda_2 + \lambda_2 e^{\lambda_1 T} - \lambda_1 e^{\lambda_2 T}) > 0$$

for all $T > 0$, we have $h(T) > 0$, hence $dg/dT > 0$. This shows $g(T) > 1$ for all $T > 0$, i.e. $dy_{11}/dx_{11} < -1$. Since $dg/dT = dg(T)/dx_{11} \cdot dx_{11}/dT = -d^2 y_{11}/dx_{11}^2 \cdot dx_{11}/dT$, and $dg/dT > 0$, $dx_{11}/dT > 0$, it follows that

$$d^2 y_{11}/dx_{11}^2 < 0$$

Case 2:

$$A = \begin{bmatrix} 0 & -\lambda^2 \\ 1 & 2\lambda \end{bmatrix}$$

has two identical real eigenvalues $\lambda > 0$.
Let

$$V = \begin{bmatrix} -\lambda & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$e^{AT} = V \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} V^{-1} e^{\lambda T}$$

In this case

$$x_{11}(T) = -\frac{1}{kT}(1 + \lambda T - e^{-\lambda T})$$

$$y_{11}(T) = \frac{1}{kT}(1 - \lambda T - e^{\lambda T})$$

Similar to Case 1, it can be shown that

$$\lim_{T\to 0} x_{11} = -\frac{2\lambda}{k} = -\frac{a_2}{k}, \qquad \lim_{T\to\infty} x_{11} = -\frac{\lambda}{k} = a_m$$

so the valid domain of x_{11} is $(-a_2/k, a_m)$. It can also be verified that $\mathrm{d}x_{11}/\mathrm{d}T > 0$. Denote $g(T) := -\mathrm{d}y_{11}/\mathrm{d}x_{11}$, then

$$g(T) = \frac{1 - \mathrm{e}^{\lambda T} + \lambda T \mathrm{e}^{\lambda T}}{1 - \mathrm{e}^{-\lambda T} - \lambda T \mathrm{e}^{-\lambda T}}, \qquad \lim_{T\to\infty} g(T) = 1$$

and

$$\frac{\mathrm{d}g}{\mathrm{d}T} = \frac{\lambda^2 T}{(1 - \mathrm{e}^{-\lambda T} - \lambda T \mathrm{e}^{-\lambda T})^2} h(T)$$

where $h(T) = \mathrm{e}^{\lambda T} - \mathrm{e}^{-\lambda T} - 2\lambda T$. It can be shown that $h(T) > 0$, hence $\mathrm{d}g/\mathrm{d}T > 0$. The remaining part is similar to Case 1.

Case 3:

$$A = \begin{bmatrix} 0 & -(\alpha^2 + \beta^2) \\ 1 & 2\alpha \end{bmatrix}$$

has two complex eigenvalues $\alpha \pm \mathrm{j}\beta$, $\alpha, \beta > 0$.

Let

$$V = \begin{bmatrix} \beta & -\alpha \\ 0 & 1 \end{bmatrix}$$

then

$$\mathrm{e}^{AT} = V \begin{bmatrix} \cos\beta T & -\sin\beta T \\ \sin\beta T & \cos\beta T \end{bmatrix} V^{-1} \mathrm{e}^{\alpha T}$$

From (A1) and (A2) we have,

$$x_{11}(T) = \frac{1}{k\sin\beta T}(-\beta\cos\beta T - \alpha\sin\beta T + \beta\mathrm{e}^{-\alpha T})$$

$$y_{11}(T) = \frac{1}{k\sin\beta T}(\beta\cos\beta T - \alpha\sin\beta T - \beta\mathrm{e}^{\alpha T})$$

The valid domain of T is $(0, \pi/\beta)$, this can be obtained directly from the vector field and also from the above equations. Notice that

$$\lim_{T\to 0} x_{11}(T) = -\frac{2\alpha}{k} = -\frac{a_2}{k}, \qquad \lim_{T\to\pi/\beta} x_{11}(T) = \infty$$

So we have $a_m = \infty$ in this case.

Define $g(T)$ similarly as in Case 1, we have

$$g(T) = \frac{\beta + (\alpha\sin\beta T - \beta\cos\beta T)\mathrm{e}^{\alpha T}}{\beta - (\alpha\sin\beta T + \beta\cos\beta T)\mathrm{e}^{-\alpha T}}, \quad \lim_{T\to 0} g(T) = 1$$

and

$$\frac{\mathrm{d}g}{\mathrm{d}T} = \frac{(\alpha^2 + \beta^2)\sin\beta T}{(\beta - (\alpha\sin\beta T + \beta\cos\beta T)\mathrm{e}^{-\alpha T})^2} h(T)$$

where $h(T) = \beta e^{\alpha T} - \beta e^{-\alpha T} - 2\alpha \sin \beta T$. It can be verified that $h(0) = 0$, $dh/dT > 0$, thus $dg/dT > 0$ for all $T \in (0, \pi/\beta)$. The remaining part is similar to case 1. □

APPENDIX B: PROOF OF LEMMA 2

Similar to the proof of Lemma 1, from (14), we can express x_{11} and y_{11} as functions of T, $x_{11}(T)$ and $y_{11}(T)$. Clearly these functions are analytic. Denote

$$g(T) := \frac{dy_{11}(T)/dT}{dx_{11}(T)/dT}$$

It suffices to show that $dx_{11}/dT > 0$, $g(T) > 1$, and $dg/dT > 0$. We need to break the proof into three different cases.

Case 1:

$$A = \begin{bmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & -(\lambda_1 + \lambda_2) \end{bmatrix}$$

has two different real eigenvalues $-\lambda_1, -\lambda_2 < 0$. Assume that $\lambda_1 > \lambda_2$.
Let

$$V = \begin{bmatrix} \lambda_2 & \lambda_1 \\ 1 & 1 \end{bmatrix}$$

then

$$e^{AT} = V \begin{bmatrix} e^{-\lambda_1 T} & 0 \\ 0 & e^{-\lambda_2 T} \end{bmatrix} V^{-1}$$

From (14) and the fact that $kfp' = 1$, $kfp = -1$, we have

$$x_{11}(T) = \frac{1}{k} \frac{\lambda_2 - \lambda_1 + \lambda_2 e^{-\lambda_2 T} - \lambda_1 e^{-\lambda_1 T}}{e^{-\lambda_2 T} - e^{-\lambda_1 T}} \tag{B1}$$

$$y_{11}(T) = \frac{1}{k} \frac{\lambda_2 - \lambda_1 + \lambda_2 e^{\lambda_2 T} - \lambda_1 e^{\lambda_1 T}}{e^{\lambda_1 T} - e^{\lambda_2 T}} \tag{B2}$$

and

$$g(T) = \frac{\lambda_2 - \lambda_1 + \lambda_2 e^{-\lambda_1 T} - \lambda_1 e^{-\lambda_2 T}}{\lambda_2 - \lambda_1 + \lambda_2 e^{\lambda_1 T} - \lambda_1 e^{\lambda_2 T}}, \quad T \in (0, T_d)$$

By the definition of T_d, $y_{11}(T_d) = a_2/k = -(\lambda_1 + \lambda_2)/k$. It can be shown that as $T \to T_d$, $g(T) \to \infty$. Since $g(0) = 1$ and

$$\frac{dg}{dT} = \frac{2\lambda_1\lambda_2}{(\lambda_2 - \lambda_1 + \lambda_2 e^{\lambda_1 T} - \lambda_1 e^{\lambda_2 T})^2} \times \{(\lambda_1 + \lambda_2)[\mathrm{ch}(\lambda_1 T - \lambda_2 T) - 1] + (\lambda_2 - \lambda_1)[\mathrm{ch}(\lambda_2 T) - \mathrm{ch}(\lambda_1 T)]\} > 0$$

where $\mathrm{ch}(a) = (e^a + e^{-a})/2 \geqslant 1$ is monotonously increasing, we have that $g(T) > 1$ for all $T \in (0, T_d)$.

It can also be verified that $dx_{11}/dT > 0$. The remaining proof is similar to Appendix A.

Case 2:

$$A = \begin{bmatrix} 0 & -\lambda^2 \\ 1 & -2\lambda \end{bmatrix}$$

has two identical real eigenvalues.
Let

$$V = \begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$e^{AT} = V \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} V^{-1} e^{-\lambda T}$$

In this case,

$$x_{11}(T) = -\frac{1}{kT}(1 - \lambda T + e^{\lambda T})$$

$$y_{11}(T) = -\frac{1}{kT}(1 + \lambda T + e^{-\lambda T})$$

and

$$g(T) = \frac{1 + \lambda T e^{-\lambda T} + e^{-\lambda T}}{1 - \lambda T e^{\lambda T} + e^{\lambda T}}$$

Since $g(0) = 1$ and

$$\frac{dg}{dT} = \frac{\lambda^2 T(2\lambda T + e^{\lambda T} - e^{-\lambda T})}{(1 - \lambda T e^{\lambda T} + e^{\lambda T})^2} > 0$$

we have $g(T) > 1$ for all $T \in (0, T_d)$. It can be verified that $dx_{11}/dT > 0$.

Case 3:

$$A = \begin{bmatrix} 0 & -(\alpha^2 + \beta^2) \\ 1 & -2\alpha \end{bmatrix}$$

has two complex eigenvalues $-\alpha \pm j\beta$, $\alpha, \beta > 0$.
Let

$$V = \begin{bmatrix} \beta & \alpha \\ 0 & 1 \end{bmatrix}$$

then

$$e^{AT} = V \begin{bmatrix} \cos\beta T & -\sin\beta T \\ \sin\beta T & \cos\beta T \end{bmatrix} V^{-1} e^{-\alpha T}$$

In this case,

$$x_{11}(T) = -\frac{1}{k\sin\beta T}(\beta\cos\beta T - \alpha\sin\beta T + \beta e^{\alpha T})$$

$$y_{11}(T) = -\frac{1}{k\sin\beta T}(\beta\cos\beta T + \alpha\sin\beta T + \beta e^{-\alpha T})$$

and

$$g(T) = \frac{\beta + (\beta\cos\beta T + \alpha\sin\beta T)e^{-\alpha T}}{\beta + (\beta\cos\beta T - \alpha\sin\beta T)e^{\alpha T}}, \quad T \in (0, T_d)$$

Since $g(0) = 1$ and

$$\frac{\mathrm{d}g}{\mathrm{d}T} = \frac{\alpha^2 + \beta^2}{(\beta + (\beta\cos\beta T - \alpha\sin\beta \mathrm{T})\mathrm{e}^{\alpha T})^2}[2\alpha\sin^2\beta T + \beta(\mathrm{e}^{\alpha T} - \mathrm{e}^{-\alpha T})\sin\beta T] > 0$$

we have $g(T) > 1$ for all $T \in (0, T_{\mathrm{d}})$. It can also be verified that $\mathrm{d}x_{11}/\mathrm{d}T > 0$.

For all the above three cases, Since $g(T) > 1$, i.e. $\mathrm{d}y_{11}/\mathrm{d}T > \mathrm{d}x_{11}/\mathrm{d}T$ for all T and $\lim_{T\to 0} x_{11}(T)/y_{11}(T) = 1$, we finally have $y_{11} > x_{11}$.

REFERENCES

1. Macki J, Strauss M. *Introduction to Optimal Control*. Springer: Berlin, 1982.
2. Schmitendorf WE, Barmish BR. Null controllability of linear systems with constrained controls. *SIAM Journal on Control and Optimization* 1980; **18**:327–345.
3. Sontag ED. An algebraic approach to bounded controllability of linear systems. *International Journal of Control* 1984; **39**:181–188.
4. Sontag ED, Sussmann HJ. Nonlinear output feedback design for linear systems with saturating controls. *Proceedings of the 29th IEEE Conference on Decision and Control*, 1990; 3414–3416.
5. Teel AR. Global stabilization and restricted tracking for multiple integrators with bounded controls. *System and Control Letters* 1992; **18**:165–171.
6. Sussmann HJ, Sontag ED, Yang Y. A general result on the stabilization of linear systems using bounded controls. *IEEE Transactions on Automatic Control*. 1994; **39**:2411–2425.
7. Teel AR. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Transactions on Automatic Control* 1996; **41**:1256–1270.
8. Megretski A. L_2 BIBO output feedback stabilization with saturated control. *Proceedings of the 13th IFAC World Congress*, vol. D, 1996; pp. 435–440.
9. Suarez R, Alvarez-Ramirez J, Solis-Daun J. Linear systems with bounded inputs: global stabilization with eigenvalue placement. *International Journal of Robust and Nonlinear Control* 1997; **7**:835–845.
10. Teel AR. Linear systems with input nonlinearities: global stabilization by scheduling a family of H_∞-type controllers. *International Journal of Robust and Nonlinear Control* 1995; **5**:399–441.
11. Fuller AT. In-the-large stability of relay and saturating control systems with linear controller. *International Journal of Control* 1969; **10**:457–480.
12. Sussmann HJ, Yang Y. On the stabilizability of multiple integrators by means of bounded feedback controls. *Proceedings of the 30th IEEE Conference on Decision and Control*, 1991; 70–72.
13. Lin Z, Saberi A. Semi-global exponential stabilization of linear systems subject to 'input saturation' via linear feedbacks. *Systems and Control Letters* 1993; **21**:225–239.
14. Lin Z. *Low Gain Feedback*, Lecture Notes in Control and Information Sciences, vol. 240. Springer: London, 1998.
15. Saberi A, Lin Z, Teel AR. Control of linear systems with saturating actuators. *IEEE Transactions Automatic Control* 1996; **41**:368–378.
16. Hu T, Lin Z. Control Systems with Actuator Saturation: Analysis and Design, Birkhäuser, Boston, 2001.
17. Wredenhagen GF, Belanger PR. Piecewise-linear LQ control for systems with input constraints. *Automatica* 1994; **30**:403–416.
18. Willems JC. Least squares stationary optimal control and the algebraic Riccati equations. *IEEE Transactions on Automatic Control* 1971; **16**:621–634.
19. Junkins J, Valasek J, Ward D. *Report on ONR UCAV Modeling Effort*, Department of Aerospace Engineering, Texas A&M University, 1999.

