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Output regulation of general discrete-time linear systems with saturation nonlinearities

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SUMMARY

This paper studies the classical problem of output regulation for linear discrete-time systems subject to actuator saturation and extends the recent results on continuous-time systems to discrete-time systems. The asymptotically regulatable region, the set of all initial conditions of the plant and the exosystem for which the asymptotic output regulation is possible, is characterized in terms of the null controllable region of the anti-stable subsystem of the plant. Feedback laws are constructed that achieve regulation on the asymptotically regulatable region. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: saturation nonlinearities; output regulation; asymptotically regulatable region; discrete-time systems

1. INTRODUCTION

There has been considerable research on the problem of stabilization and output regulation of linear systems subject to actuator saturation. The problem of stabilization involves issues ranging from the characterization of the null controllable region (or, asymptotically null controllable region), the set of all initial conditions that can be driven to the origin by the saturating actuators in some finite time (respectively, asymptotically), to the construction of feedback laws that achieve stabilization on the entire or a large portion of the asymptotically null controllable region. Recent years have witnessed extensive research that addresses these issues. In particular, for an open loop system that are stabilizable and have all its poles in the closed left-half plane, it was established in Reference [1] that the asymptotically null controllable region is the entire state space. For this reason, a linear system that is stabilizable in the usual

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linear sense and has all its poles in the closed left-half plane is referred to as asymptotically null controllable with bounded controls, or ANCBC. For ANCBC systems subject to actuator saturation, various feedback laws that achieve global or semi-global stabilization on the asymptotically null controllable region have been constructed (see, for example, References [2–5]). For exponentially unstable open-loop systems subject to actuator saturation, the asymptotically null controllable regions were recently characterized and feedback laws were constructed that achieve semi-global stabilization on the asymptotically null controllable region (see References [6–8]).

In comparison with the problem of stabilization, the problem of output regulation for linear systems subject to actuator saturation, however, has received relatively less attention. The few works that motivated our recent research [9] on continuous-time systems were References [10,5,11,2]. In References [5,2], the problem of output regulation was studied for ANCBC systems subject to actuator saturation. Necessary and sufficient conditions on the plant/exosystem and their initial conditions were derived under which output regulation can be achieved. Under these conditions, feedback laws that achieve output regulation were constructed based on the semi-global stabilizing feedback laws of Reference [4]. The recent work [10] made an attempt to address the problem of output regulation for exponentially unstable linear systems subject to actuator saturation. The attempt was to enlarge the set of initial conditions of the plant and the exosystem under which output regulation can be achieved. In particular, for plants with only one positive pole and exosystems that contain only one frequency component, feedback laws were constructed that achieve output regulation on what was later characterized in Reference [9] as the asymptotically regulatable region.

In Reference [9], we systematically studied the problem of output regulation for general continuous-time linear systems subject to actuator saturation. In particular, we characterized the regulatable region, the set of plant and exosystem initial conditions for which output regulation is possible with the saturating actuators. We then constructed feedback laws that achieve regulation on the regulatable region.

The objective of this paper is to extend the above results to discrete-time systems. In Section 2, we formulate the problem of output regulation for linear systems with saturating actuators. Section 3 characterizes the regulatable region. Sections 4 and 5, respectively, construct state feedback and error feedback laws that achieve output regulation on the regulatable region. Finally, Section 6 gives a brief concluding remark to our current work.

Throughout the paper, we will use standard notation. For a vector $u \in \mathbf{R}^m$, we use $|u|_\infty$ and $|u|_2$ to denote the vector ∞-norm and the 2-norm. For a vector sequence $u(k) \in \mathbf{R}^m$, $k = 0, 1, 2, \ldots$; we define $\|u\|_\infty = \sup_{k \geqslant 0} |u(k)|_\infty$. We use $\mathrm{sat}(\cdot)$ to denote the standard saturation function $\mathrm{sat}(s) = \mathrm{sgn}(s)\min\{1, |s|\}$. With a slight abuse of notation and for simplicity, for a vector $u \in \mathbf{R}^m$, we also use the same $\mathrm{sat}(u)$ to denote the vector saturation function, i.e. $\mathrm{sat}(u) = [\mathrm{sat}(u_1)\ \ \mathrm{sat}(u_2)\ \ \cdots\ \ \mathrm{sat}(u_m)]^{\mathrm{T}}$.

2. PRELIMINARIES AND PROBLEM STATEMENT

In this section, we state the discrete-time version of the classical results on the problem of output regulation for continuous-time linear systems [12] (see also Reference [13]). These results will motivate our formulation of as well as the solution to the problem of output regulation for discrete-time linear systems subject to actuator saturation.

2.1. Review of output regulation for linear systems

Consider a linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Pw(k) \\ w(k+1) &= Sw(k) \\ e(k) &= Cx(k) + Qw(k) \end{aligned} \tag{1}$$

The first equation of this system describes a plant, with state $x \in \mathbf{R}^n$ and input $u \in \mathbf{R}^m$, subject to the effect of a disturbance represented by Pw. The third equation defines the error $e \in \mathbf{R}^q$ between the actual plant output Cx and a reference signal $-Qw$ that the plant output is required to track. The second equation describes an autonomous system, often called the exosystem, with state $w \in \mathbf{R}^r$. The exosystem models the class to disturbances and references taken into consideration.

The control action to the plant, u, can be provided either by state feedback or by error feedback. The objective is to achieve internal stability and output regulation. Internal stability means that if we disconnect the exosystem and set w equal to zero then the closed-loop system is asymptotically stable. Output regulation means that for any initial conditions of the closed-loop system, we have that $e(k) \to 0$ as $k \to \infty$.

The solution to these problems is based on the following three assumptions.

A1. The eigenvalues of S are on or outside of the unit circle;
A2. The pair (A, B) is stabilizable;
A3. The pair

$$\left([C \quad Q], \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$$

is detectable.

For continuous-time systems, complete solutions to the output regulation problems were established in Reference [12] by Francis. These solutions can be adapted for discrete-time systems as follows:

Proposition 1
Suppose Assumptions A1 and A2 hold. Then, the problem of output regulation by state feedback is solvable if and only if there exist matrices Π that Γ that solve the linear matrix equations

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P \\ C\Pi + Q &= 0 \end{aligned} \tag{2}$$

Moreover, if in addition Assumption A3 also holds, the solvability of the above linear matrix equations is also a necessary and sufficient condition for the solvability of the problem of output regulation by error feedback.

2.2. Output regulation for linear systems subject to actuator saturation

Motivated by the classical formulation of output regulation for linear systems, we consider the following plant and the exosystem:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Pw(k) \\ w(k+1) &= Sw(k) \\ e(k) &= Cx(k) + Qw(k) \end{aligned} \tag{3}$$

where u is the output of saturating actuators and is constrained by $\|u\|_\infty \leqslant 1$. A control u that satisfies this constraint is referred to as an *admissible control*. Because of the bound on the control input, both the plant and the exosystem cannot operate in the entire state space. For this reason, we assume that $(x_0, w_0) \in \mathcal{Y}_0$ for some $\mathcal{Y}_0 \subset \mathbf{R}^n \times \mathbf{R}^r$. Let

$$\mathcal{X}_0 = \{x_0 \in \mathbf{R}^n \colon (x_0, 0) \in \mathcal{Y}_0\}$$

The problem to be addressed in this paper is the following.

Problem 1

The problem of output regulation by state feedback for the system (3) is to find, if possible, a state feedback law $u = \phi(x, w)$, with $|\phi(x, w)|_\infty \leqslant 1$ and $\phi(0, 0) = 0$, such that

1. the equilibrium $x = 0$ of the system

$$x(k+1) = Ax(k) + B\phi(x(k), 0)$$

   is asymptotically stable with $\mathcal{X}_0$ contained in its domain of attraction;
2. for all $(x_0, w_0) \in \mathcal{Y}_0$, the interconnection of (3) and the feedback law $u = \phi(x, w)$ results in bounded state trajectories $x(k)$ and $\lim_{k\to\infty} e(k) = 0$.

If only the error e is available, the state (x, w) can be reconstructed after a finite number of steps if we further assume that the pair in A3 is observable. But the initial condition (x_0, w_0) might have to be constrained in a subset of $\mathcal{Y}_0$.

Our objective is to characterize the maximal set of initial conditions (x_0, w_0), the largest possible $\mathcal{Y}_0$, on which the above problem is solvable and to explicitly construct feedback law that actually solves the problem for $\mathcal{Y}_0$ as large as possible.

We will assume that (A, B) is stabilizable. We will also assume that S is neutrally stable and all its eigenvalues are on the unit circle. The stabilizability of (A, B) is clearly necessary for the stabilization of the plant. The assumption on S is without loss of generality. Since the components corresponding to the asymptotically stable modes of the exosystem will tend to zero, they will not affect the regulation of the output. On the other hand, if the exosystem has unstable modes, either the disturbance Pw or the signal Qw will go unbounded. It is generally impossible to drive the error e to zero asymptotically with a bounded control (see Reference [11]).

3. THE REGULATABLE REGION

In this section, we will characterize the set of all initial states of the plant and the exosystem on which the problem of output regulation is solvable under the restriction that $\|u\|_\infty \leqslant 1$. We will refer to this set as the asymptotically regulatable region.

To begin with, we observe from the classical output regulation theory (see Section 2.1) that for this problem to be solvable, there must exist matrices $\Pi \in \mathbf{R}^{n\times r}$ and $\Gamma \in \mathbf{R}^{m\times r}$ that solve the matrix equations (2). Given the matrices Π and Γ, we define a new state $z = x - \Pi w$ and rewrite the system equations as

$$\begin{aligned} z(k+1) &= Az(k) + Bu(k) - B\Gamma w(k) \\ w(k+1) &= Sw(k) \\ e(k) &= Cz(k) \end{aligned} \tag{4}$$

From these new equations, it is clear that $e(k)$ goes to zero asymptotically if $z(k)$ goes to zero asymptotically. The latter is possible only if (see Reference [11] for the continuous-time case)

$$\sup_{k\geqslant 0} |\Gamma S^k w_0|_\infty < 1 \tag{5}$$

For this reason, we will restrict our attention to exosystem initial conditions in the following compact set

$$\mathscr{W}_0 = \{w_0 \in \mathbf{R}^r\colon\ |\Gamma w(k)|_\infty = |\Gamma S^k w_0|_\infty \leqslant \rho,\ \ \forall k \geqslant 0\} \tag{6}$$

for some $\rho \in [0, 1)$. For later use, we also denote $\delta = 1 - \rho$. We note that the compactness of $\mathscr{W}_0$ can be guaranteed by the observability of (Γ, S). Indeed, if (Γ, S) is not observable, then the exosystem can be reduced to make it so.

We can now precisely define the notion of asymptotically regulatable region as follows.

Definition 1

1. Given $K > 0$, a pair $(z_0, w_0) \in \mathbf{R}^n \times \mathscr{W}_0$ is regulatable in K steps if there exists an admissible control u, such that the response of (4) satisfies $z(K) = 0$. The set of all (z_0, w_0) regulatable in K steps is denoted as $\mathscr{R}_g(K)$.
2. A pair (z_0, w_0) is regulatable if $(z_0, w_0) \in \mathscr{R}_g(K)$ for some $K < \infty$. The set of all regulatable (z_0, w_0) is referred to as the regulatable region and is denoted as $\mathscr{R}_g$.
3. The set of all (z_0, w_0) for which there exist admissible controls such that the response of (4) satisfies $\lim_{k\to\infty} z(k) = 0$ is referred to as the asymptotically regulatable region and is denoted as $\mathscr{R}_g^a$.

Remark 1

Note that the regulatable region is defined in terms of $\lim_{k\to\infty} z(k) = 0$ rather than $\lim_{k\to\infty} e(k) = 0$. Requiring the former instead of the latter will also guarantee the closed-loop stability in the absence of w. Like the continuous-time case [6,9], this will result in essentially the same description of the regulatable region.

We will describe $\mathscr{R}_g(K)$, $\mathscr{R}_g$ and $\mathscr{R}_g^a$ in terms of the asymptotically null controllable region of the plant

$$v(k+1) = Av(k) + Bu(k), \quad \|u\|_\infty \leqslant 1$$

Definition 2
The null controllable region at step K, denoted as $\mathscr{C}(K)$, is the set of $v_0 \in \mathbf{R}^n$ that can be driven to the origin in K steps and the null controllable region, denoted as $\mathscr{C}$, is the set of $v_0 \in \mathbf{R}^n$ that can be driven to the origin in finite number of steps by admissible controls. The asymptotically null controllable region, denoted as $\mathscr{C}^a$, is the set of all v_0 that can be driven to the origin asymptotically by admissible controls.

Clearly,

$$\mathscr{C} = \bigcup_{K \in [0,\infty)} \mathscr{C}(K)$$

and

$$\mathscr{C}(K) = \left\{ \sum_{i=0}^{K-1} A^{-i-1} Bu(i) : \|u\|_\infty \leqslant 1 \right\} \tag{7}$$

Some simple methods to describe $\mathscr{C}$ and $\mathscr{C}^a$ were developed in Reference [8] (see also Reference [6]).

To simplify the characterization of $\mathscr{R}_g$ and $\mathscr{R}_g^a$ and without loss of generality, let us assume that

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z_1 \in \mathbf{R}^{n_1}, \quad z_2 \in \mathbf{R}^{n_2}$$

and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{8}$$

where $A_1 \in \mathbf{R}^{n_1 \times n_1}$ is semi-stable (i.e. all its eigenvalues are on or inside the unit circle) and $A_2 \in \mathbf{R}^{n_2 \times n_2}$ is anti-stable (i.e. all its eigenvalues are outside of the unit circle). The anti-stable subsystem

$$\begin{aligned} z_2(k+1) &= A_2 z_2(k) + B_2 u(k) - B_2 \Gamma w(k) \\ w(k+1) &= Sw(k) \end{aligned} \tag{9}$$

is of crucial importance. Denote its regulatable regions as $\mathscr{R}_{g_2}(K)$ and $\mathscr{R}_{g_2}$, and the null controllable regions for the system

$$v_2(k+1) = A_2 v_2(k) + B_2 u(k)$$

as $\mathscr{C}_2(K)$ and $\mathscr{C}_2$. Then, the asymptotically null controllable region of the system

$$v(k+1) = Av(k) + Bu(k)$$

is given by $\mathscr{C}^a = \mathbf{R}^{n_1} \times \mathscr{C}_2$ [14], where $\mathscr{C}_2$ is a bounded convex open set. Denote the closure of $\mathscr{C}_2$ as $\bar{\mathscr{C}}_2$, then

$$\bar{\mathscr{C}}_2 = \left\{ \sum_{i=0}^{\infty} A_2^{-i-1} B_2 u(i) : \|u\|_\infty \leqslant 1 \right\}$$

Theorem 1
Let $V_2 \in \mathbf{R}^{n_2 \times r}$ be the unique solution to the matrix equation

$$-A_2 V_2 + V_2 S = -B_2 \Gamma \tag{10}$$

and let

$$V(K) = V_2 - A^{-K} V_2 S^K$$

Then,

(a) $\mathscr{R}_{g_2}(K) = \{(z_2, w) \in \mathbf{R}^{n_2} \times \mathscr{W}_0 : z_2 - V(K)w \in \mathscr{C}_2(K)\}$ (11)

(b) $\mathscr{R}_{g_2} = \{(z_2, w) \in \mathbf{R}^{n_2} \times \mathscr{W}_0 : z_2 - V_2 w \in \mathscr{C}_2\}$ (12)

(c) $\mathscr{R}_g^a = R^{n_1} \times \mathscr{R}_{g_2}$ (13)

Proof
(a) Given $(z_{20}, w_0) \in \mathbf{R}^{n_2} \times \mathscr{W}_0$ and an admissible control u, the solution of (9) at $k = K$ is

$$z_2(K) = A^K \left(z_{20} + \sum_{i=0}^{K-1} A_2^{-i-1} B_2 u(i) - \sum_{i=0}^{K-1} A_2^{-i-1} B_2 \Gamma S^i w_0 \right) \tag{14}$$

Applying (10), we have

$$\begin{aligned} -\sum_{i=0}^{K-1} A_2^{-i-1} B_2 \Gamma S^i &= \sum_{i=0}^{K-1} A_2^{-i-1} (-A_2 V_2 + V_2 S) S^i \\ &= \sum_{i=0}^{K-1} (-A_2^{-i} V_2 S^i + A_2^{-i-1} V_2 S^{i+1}) \\ &= -V_2 + A_2^{-K} V_2 S^K \\ &= -V(K) \end{aligned} \tag{15}$$

where the third "=" is simply obtained by expanding the terms in the summation and cancelling all the middle terms. Thus,

$$A^{-K} z_2(K) = z_{20} - V(K) w_0 + \sum_{i=0}^{K-1} A^{-i-1} B_2 u(i)$$

By setting $z_2(K) = 0$, we immediately obtain (a) from the definition of $\mathscr{R}_g(K)$ and (7).

(b) and (c). The proof is lengthy and can be found in Reference [6]. □

Remark 2
Given (z_0, w_0), there exists an admissible control u such that $\lim_{k \to \infty} z(k) = 0$ if and only if $(z_0, w_0) \in \mathscr{R}_g^a$. Recalling that $z = x - \Pi w$, we observe that, for a given pair of initial states in the

original co-ordinates, (x_0, w_0), there is an admissible control u such that $\lim_{k\to\infty}(x(k) - \Pi w(k)) = 0$ if and only if $x_{20} - (\Pi_2 + V_2)w_0 \in \mathscr{C}_2$, where $\Pi_2 = [0\;\; I_{n_2}]\Pi$.

4. STATE FEEDBACK CONTROLLER

In this section, we will construct a feedback law that solves the problem of output regulation by state feedback for linear systems subject to actuator saturation. Our feedback law will be based on a stabilizing feedback law $u = f(v)$, $|f(v)|_\infty \leqslant 1$ for all $v \in \mathbf{R}^n$, which makes the system

$$v(k+1) = Av(k) + Bf(v(k)) \tag{16}$$

have an asymptotically stable equilibrium at the origin. Actually, any feedback of the form $u = f(v) = \mathrm{sat}(Fu)$ will stabilize the system locally at the origin as long as $A + BF$ is asymptotically stable. In References [6–8], we presented some methods for designing $f(v)$ to enlarge the domain of attraction of the origin. Here, we assume that a stabilizing feedback law $u = f(v)$ has been designed and the equilibrium $v = 0$ of the closed-loop system (16) has a domain of attraction $\mathscr{S} \subset \mathscr{C}^a$.

Now consider the system (4). Given a state feedback $u = g(z, w), |g(z, w)|_\infty \leqslant 1$ for all $(z, w) \in \mathbf{R}^n \times \mathscr{W}_0$, we have the closed-loop system

$$\begin{aligned} z(k+1) &= Az(k) + Bg(z(k), w(k)) - B\Gamma w(k) \\ w(k+1) &= Sw(k) \end{aligned} \tag{17}$$

Denote the time response of $z(k)$ to the initial state (z_0, w_0) as $z(k, z_0, w_0)$ and define

$$\mathscr{S}_{zw} := \left\{ (z_0, w_0) \in \mathbf{R}^n \times \mathscr{W}_0 : \lim_{k\to\infty} z(k, z_0, w_0) = 0 \right\}$$

Since $\mathscr{R}_g^a$ is the set of all (z_0, w_0) for which $z(k)$ can be driven to the origin asymptotically, we must have $\mathscr{S}_{zw} \subset \mathscr{R}_g^a$. Our objective is to design a control law $u = g(z, w)$ such that $\mathscr{S}_{zw}$ is as large as possible, or as close to $\mathscr{R}_g^a$ as possible.

First we need a mild assumption which can be removed by modifying the controller (see Reference [6]). Assume that there exists a matrix $V \in \mathbf{R}^{n\times r}$ such that

$$-AV + VS = -B\Gamma \tag{18}$$

This will be the case if A and S have no common eigenvalues (see e.g. p. 26 of Reference [16]). With the decomposition in (8), if we partition V accordingly as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

then V_2 satisfies $-A_2V_2 + V_2S = -B_2\Gamma$. Denote

$$D_{zw} := \{(z, w) \in \mathbf{R}^n \times \mathscr{W}_0 : z - Vw \in \mathscr{S}\} \tag{19}$$

on which the following observation can be made.

Observation 1
(a) The set D_{zw} increases as $\mathscr{S}$ increases, and if $\mathscr{S} = \mathscr{C}^a$, then $D_{zw} = \mathscr{R}_g^a$; (b) In the absence of w, $x_0 \in \mathscr{S} \Rightarrow (z_0, 0) \in D_{zw}$.

Proof
The fact that D_{zw} increases as $\mathscr{S}$ increases is easy to see. To see the rest of (a), we note that, for a general plant, $\mathscr{C}^a = \mathbf{R}^{n_1} \times \mathscr{C}_2$. If $\mathscr{S} = \mathscr{C}^a$, then $\mathscr{S} = \mathbf{R}^{n_1} \times \mathscr{C}_2$, and

$$\begin{aligned} D_{zw} &= \{(z, w) \in \mathbf{R}^n \times \mathscr{W}_0 : z - Vw \in \mathbf{R}^{n_1} \times \mathscr{C}_2\} \\ &= \{(z_1, z_2, w) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \mathscr{W}_0 : z_1 - V_1 w \in \mathbf{R}^{n_1}, z_2 - V_2 w \in \mathscr{C}_2\} \\ &= \mathbf{R}^{n_1} \times \mathscr{R}_{g_2} = \mathscr{R}_g^a \end{aligned}$$

Part (b) is also clear if we note that $z_0 = x_0 - \Pi w_0 = x_0$ for $w_0 = 0$. □

With this observation, we see that our objective of enlarging $\mathscr{S}_{zw}$ is simply to design a feedback law such that $D_{zw} \subset \mathscr{S}_{zw}$. We will reach this objective through a series of technical lemmas.

Lemma 1
Let $u = f(z - Vw)$. Consider the closed-loop system

$$\begin{aligned} z(k+1) &= Az(k) + Bf(z(k) - Vw(k)) - B\Gamma w(k) \\ w(k+1) &= Sw(k) \end{aligned} \tag{20}$$

For this system, D_{zw} is an invariant set and for all $(z_0, w_0) \in D_{zw}, \lim_{k\to\infty} (z(k) - Vw(k)) = 0$.

Proof
Substitute (18) into system (20), we obtain

$$\begin{aligned} z(k+1) &= Az(k) + Bf(z(k) - Vw(k)) - AVw(k) + VSw(k) \\ &= A(z(k) - Vw(k)) + Bf(z(k) - Vw(k)) + Vw(k+1) \end{aligned}$$

Define the new state $v := z - Vw$, we have

$$v(k+1) = Av(k) + Bf(v(k))$$

which has a domain of attraction $\mathscr{S}$. This also implies that $\mathscr{S}$ is an invariant set for the v-system.
If $(z_0, w_0) \in D_{zw}$, then $v_0 = z_0 - Vw_0 \in \mathscr{S}$. It follows that

$$v(k) = z(k) - Vw(k) \in \mathscr{S}$$

for all $k \geqslant 0$ and $\lim_{k\to\infty} (z(k) - Vw(k)) = \lim_{k\to\infty} v(k) = 0$. □

Lemma 1 says that, in the presence of w, the simple feedback $u = f(z - Vw)$ will cause $z(k) - Vw(k)$ to approach zero and $z(k)$ to approach $Vw(k)$, which is bounded. Our next step is to construct a finite sequence of controllers

$$u = f_\ell(z, w, \alpha), \quad \ell = 0, 1, 2, \ldots, N,$$

all parameterized in $\alpha \in (0,1)$. By judiciously switching between these controllers, we can cause $z(k)$ to approach $\alpha^\ell Vw(k)$ for any ℓ. By choosing N large enough, $z(k)$ will become arbitrarily small in a finite number of steps. Once $z(k)$ becomes small enough, we will use the controller

$$u = \Gamma w + \delta \operatorname{sat}\left(\frac{Fz}{\delta}\right)$$

(F to be specified later) to make $z(k)$ converge to the origin.

Let $F \in \mathbf{R}^{m\times n}$ be such that

$$v(k+1) = Av(k) + B\operatorname{sat}(Fv(k)) \tag{21}$$

is asymptotically stable. Let $X > 0$ be such that

$$(A+BF)^{\mathrm{T}}X(A+BF) - X < 0$$

and the ellipsoid $\mathscr{E} := \{v \in \mathbf{R}^n : v^{\mathrm{T}}Xv \leqslant 1\}$ be in the linear region of the saturation function, i.e. $|Fv|_\infty \leqslant 1$ for all $v \in \mathscr{E}$. Then $\mathscr{E}$ is an invariant set and is in the domain of attraction for the closed-loop system (21). Similar to the continuous-time case, we have the following lemma.

Lemma 2

Suppose that $D \subset \mathbf{R}^n$ is an invariant set in the domain of attraction for the system

$$v(k+1) = Av(k) + Bf(v(k)) \tag{22}$$

then for any $\alpha > 0, \alpha D$ is an invariant set in the domain of attraction for the system

$$v(k+1) = Av(k) + \alpha Bf\left(\frac{v(k)}{\alpha}\right) \tag{23}$$

For any $\alpha \in (0,1)$, there exists a positive integer N such that

$$\alpha^N |X^{1/2}Vw|_2 < \delta, \quad \forall w \in \mathscr{W}_0 \tag{24}$$

i.e. $\alpha^N Vw \in \delta\mathscr{E}$, for all $w \in \mathscr{W}_0$. Define a sequence of subsets in $\mathbf{R}^n \times \mathscr{W}_0$ as

$$D_{zw}^\ell = \{(z,w) \in \mathbf{R}^n \times \mathscr{W}_0 : z - \alpha^\ell Vw \in \alpha^\ell \mathscr{E}\}, \quad \ell = 0,1,\ldots,N$$

$$D_{zw}^{N+1} = \{(z,w) \in \mathbf{R}^n \times \mathscr{W}_0 : z \in \delta\mathscr{E}\}$$

and, on each of these sets, define a state feedback law as follows:

$$f_\ell(z,w,\alpha) = (1-\alpha^\ell)\Gamma w + \alpha^\ell \operatorname{sat}\left(\frac{F(z-\alpha^\ell Vw)}{\alpha^\ell}\right), \quad \ell = 0,1,\ldots,N$$

$$f_{N+1}(z,w) = \Gamma w + \delta \operatorname{sat}\left(\frac{Fz}{\delta}\right)$$

It can be verified that, for each $\ell = 0,1,\ldots,N+1$, $|f_\ell(z,w,\alpha)|_\infty \leqslant 1$ for all $(z,w) \in \mathbf{R}^n \times \mathscr{W}_0$.

Lemma 3
Let $u = f_\ell(z, w, \alpha)$. Consider the closed-loop system

$$\begin{aligned} z(k+1) &= Az(k) + Bf_\ell(z(k), w(k), \alpha) - B\Gamma w(k) \\ w(k+1) &= Sw(k) \end{aligned} \tag{25}$$

For this system, D^ℓ_{zw} is an invariant set. Moreover, if $\ell = 0, 1, \ldots, N$, then for all $(z_0, w_0) \in D^\ell_{zw}$, $\lim_{k\to\infty} (z(k) - \alpha^\ell Vw(k)) = 0$; if $\ell = N+1$, then, for all $(z_0, w_0) \in D^{N+1}_{zw}$, $\lim_{k\to\infty} z(k) = 0$.

Proof
With $u = f_\ell(z, w, \alpha)$, $\ell = 0, 1, \ldots, N$, we have

$$\begin{aligned} z(k+1) &= Az(k) + (1-\alpha^\ell)B\Gamma w(k) + \alpha^\ell B \operatorname{sat}\left(\frac{F(z(k) - \alpha^\ell Vw(k))}{\alpha^\ell}\right) - B\Gamma w(k) \\ &= Az(k) + \alpha^\ell B \operatorname{sat}\left(\frac{F(z(k) - \alpha^\ell Vw(k))}{\alpha^\ell}\right) - \alpha^\ell B\Gamma w(k) \end{aligned} \tag{26}$$

Let $v_\ell = z - \alpha^\ell Vw$, then by (18)

$$v_\ell(k+1) = Av_\ell(k) + \alpha^\ell B \operatorname{sat}\left(\frac{Fv_\ell(k))}{\alpha^\ell}\right) \tag{27}$$

It follows from Lemma 2 that $\alpha^\ell \mathscr{E}$ is an invariant set in the domain of attraction for the v_ℓ-system. Hence D^ℓ_{zw} is invariant for the system (25) and if $(z_0, w_0) \in D^\ell_{zw}$, i.e.

$$v_{\ell 0} = z_0 - \alpha^\ell Vw_0 \in \alpha^\ell \mathscr{E}$$

then

$$\lim_{k\to\infty} (z(k) - \alpha^\ell Vw(k)) = \lim_{k\to\infty} v_\ell(k) = 0$$

With $u = f_{N+1}(z, w) = \Gamma w + \delta \operatorname{sat}(Fz/\delta)$, we have

$$z(k+1) = Az(k) + \delta B \operatorname{sat}\left(\frac{Fz(k)}{\delta}\right)$$

and the same argument applies. $\square$

Based on the technical lemmas established above, we construct our final state feedback law as follows:

$$\begin{aligned} u &= g(z, w, \alpha, N) \\ &= \begin{cases} f_{N+1}(z, w) & \text{if } (z, w) \in \Omega^{N+1} := D^{N+1}_{zw} \\ f_\ell(z, w, \alpha) & \text{if } (z, w) \in \Omega^\ell := D^\ell_{zw} \Big\backslash \bigcup_{j=\ell+1}^{N+1} D^j_{zw}, \quad \ell = 0, 1, \ldots, N \\ f(z - Vw) & \text{if } (z, w) \in \Omega := \mathbf{R}^n \times \mathscr{W}_0 \Big\backslash \bigcup_{j=0}^{N+1} D^j_{zw} \end{cases} \end{aligned} \tag{28}$$

Since $\Omega, \Omega^0, \ldots, \Omega^{N+1}$ are disjoint and their union is $\mathbf{R}^n + \mathscr{W}_0$, the controller is well defined on $\mathbf{R}^n \times \mathscr{W}_0$. What remains to be shown is that this controller will accomplish our objective if the parameter α is properly chosen.

Let

$$\alpha_0 = \max_{w \in \mathscr{W}_0} \frac{|X^{1/2} Vw|_2}{|X^{1/2} Vw|_2 + 1}$$

It is obvious that $\alpha_0 \in (0, 1)$.

Theorem 2

Choose any $\alpha \in (\alpha_0, 1)$ and let N be specified as in (24). Then for all $(z_0, w_0) \in D_{zw}$, the solution of the closed-loop system

$$\begin{aligned} z(k+1) &= Az(k) + Bg(z(k), w(k), \alpha, N) - B\Gamma w(k) \\ w(k+1) &= Sw(k) \end{aligned} \tag{29}$$

satisfies $\lim_{k \to \infty} z(k) = 0$, i.e. $D_{zw} \subset \mathscr{S}_{zw}$

Proof

The control $u = g(z, w, \alpha, N)$ is executed by choosing one from $f_\ell(z, w, \alpha)$, $\ell = 0, 1, \ldots, N+1$, and $f(z - Vw)$. The crucial point is to guarantee that (z, w) will move successively from Ω, to Ω^0, $\Omega^1, \ldots,$ finally entering Ω^{N+1}, in which $z(k)$ will converge to the origin.

Without loss of generality, we assume that $(z_0, w_0) \in \Omega \cap D_{zw}$, so the control $u = f(z - Vw)$ is in effect at the beginning. By Lemma 1, $\lim_{k \to \infty} (z(k) - Vw(k)) = 0$. Hence there is a finite step $k_0 \geqslant 0$ such that $z(k_0) - Vw(k_0) \in \mathscr{E}$, i.e. $(z(k_0), w(k_0)) \in D^0_{zw}$. The condition $(z(k), w(k)) \in D^\ell_{zw}$, $\ell > 0$, might be satisfied at a smaller step $k_1 \leqslant k_0$. In any case, there is a finite step $k_1 \geqslant 0$ such that

$$(z(k_1), w(k_1)) \in \Omega^\ell = D^\ell_{zw} \Big\backslash \bigcup_{j=\ell+1}^{N+1} D^j_{zw}$$

for some $\ell = 0, 1, \ldots, N+1$. After that, the control $u = f_\ell(z, w, \alpha)$ will be in effect.

We claim that, for any $(z(k_1), w(k_1)) \in \Omega^\ell$, under the control $u = f_\ell(z, w, \alpha)$, there is a finite integer $k_2 > k_1$ such that $(z(k_2), w(k_2)) \in D^{\ell+1}_{zw}$.

Since $\Omega^\ell \subset D^\ell_{zw}$, by Lemma 3, we have that, under the control $u = f_\ell(z, w, \alpha)$,

$$\lim_{k \to \infty} (z(k) - \alpha^\ell Vw(k)) = 0$$

Since $\alpha \in (\alpha_0, 1)$, we have

$$(1 - \alpha)|X^{1/2} Vw| < \alpha, \quad \forall w \in \mathscr{W}_0$$

Therefore, for $\ell < N$

$$\begin{aligned} |X^{1/2}(z - \alpha^{\ell+1} Vw)| &\leqslant |X^{1/2}(z - \alpha^\ell Vw)| + \alpha^\ell (1 - \alpha)|X^{1/2} Vw| \\ &< |X^{1/2}(z - \alpha^\ell Vw)| + \alpha^{\ell+1} \end{aligned} \tag{30}$$

Since the first term on the right-hand side goes to zero asymptotically, there exists a finite $k_2 > k_1$ such that

$$|X^{1/2}(z(k_2) - \alpha^{\ell+1} Vw(k_2))| \leqslant \alpha^{\ell+1}$$

This implies that $z(k_2) - \alpha^{\ell+1} Vw(k_2) \in \alpha^{\ell+1}\mathscr{E}$, i.e. $(z(k_2), w(k_2)) \in D_{zw}^{\ell+1}$.

If $\ell = N$, then, by (24)

$$|X^{1/2}z| \leqslant |X^{1/2}(z - \alpha^N Vw)| + \alpha^N |X^{1/2} Vw|$$

$$< |X^{1/2}(z - \alpha^N Vw)| + \delta$$

Also, the first term goes to zero asymptotically, so there exists a finite integer k_2 such that $|X^{1/2}z(k_2)| \leqslant \delta$, i.e. $(z(k_2), w(k_2)) \in D_{zw}^{N+1}$.

Just as before, (z, w) might have entered $D_{zw}^{\ell+q}$, $q > 1$, before it enters $D_{zw}^{\ell+1}$. In any case, there is a finite k such that

$$(z(k), w(k)) \in \Omega^{\ell+q} = D_{zw}^{\ell+q} \Big\backslash \bigcup_{j=\ell+q+1}^{N+1} D_{zw}^j$$

for some $q \geqslant 1$. After that, the controller will be switched to $f_{\ell+q}(z, w, \alpha)$.

It is also important to note that, by Lemma 3, D_{zw}^{ℓ} is invariant under the control $u = f_\ell(z, w, \alpha)$. Once $(z, w) \in \Omega^\ell \subset D_{zw}^\ell$, it will never go back to Ω^q, $q < \ell$ (or Ω) since Ω^q, $q < \ell$ and Ω have no intersection with D_{zw}^ℓ, (but Ω^q, $q > \ell$, might have intersection with D_{zw}^ℓ). In summary, for any $(z_0, w_0) \in D_{zw}$, suppose $(z_0, w_0) \in \Omega^\ell$, the control will first be $f_\ell(z, w, \alpha)$ and then switch successively to $f_{\ell_1}, f_{\ell_2}, \ldots$, with $\ell_1, \ell_2, \ldots$, strictly increasing until (z, w) enters D_{zw}^{N+1} and remains there. Hence, $\lim_{k\to\infty} z(k) = 0$. $\square$

From the proof of Theorem 2, we see that for all $(z_0, w_0) \in D_{zw}$, the number of switches is at most $N + 2$.

5. ERROR FEEDBACK

In the continuous-time case [9], the set of initial conditions on which $\lim_{k\to\infty} z(k) = 0$ is achieved by an error feedback can be made arbitrarily close to that by a state feedback. This is because the observer error can be made arbitrarily small in an arbitrarily short time interval. However, for discrete-time systems, this is impossible. Suppose that the pair in Assumption A3 is observable, then there is a minimal number of steps for any observer to reconstruct all the states. Let this minimal number of steps be n_0. We also assume that a stabilizing state feedback law $u = f(v)$, $|f(v)|_\infty \leqslant 1$ for all $v \in \mathbf{R}^n$, has been designed such that the origin of the closed-loop system (16) has a domain of attraction $S \subset \mathscr{C}^a$. By using the design of Section 4, the set

$$D_{zw} := \{(z, w) \in \mathbf{R}^n \times \mathscr{W}_0 : z - Vw \in \mathscr{S}\}$$

can be made a subset of $\mathscr{S}_{zw}$ with a state feedback $u = g(z, w, \alpha, N)$.

Now for the case where only the tracking error e is available for feedback, a simple strategy is to let the control u be zero before the states are completely recovered, and after that we let $u = g(z, w, \alpha, N)$ as in (28) i.e.

$$u = \begin{cases} 0 & \text{if } k < n_0 \\ g(z, w, \alpha, N) & \text{if } k \geqslant n_0 \end{cases} \tag{31}$$

The question is: what is the set of initial states on which $\lim_{k\to\infty} z(k) = 0$ under the control of (31)? The answer is very simple. With $u = 0$, we have

$$z(k+1) = Az(k) - B\Gamma w(k)$$

$$w(k+1) = Sw(k)$$

By applying (18), we have

$$(z - Vw)(k+1) = A(z - Vw)(k)$$

Hence,

$$(z - Vw)(n_0) = A^{n_0}(z - Vw)(0) = A^{n_0}(z_0 - Vw_0)$$

For (z_0, w_0) to be in $\mathcal{S}_{zw}$, it suffices to have $(z(n_0), w(n_0)) \in D_{zw}$, i.e. $z(n_0) - Vw(n_0) \in \mathcal{S}$. This is in turn equivalent to

$$(z_0, w_0) \in \hat{D}_{zw} := \{(z, w) \in \mathbf{R} \times \mathcal{W}_0 : A^{n_0}(z - Vw) \in \mathcal{S}\}$$

In summary, we have $\hat{D}_{zw} \subset \mathcal{S}_{zw}$ under the control of (31).

The set $\hat{D}_{zw}$ is close to D_{zw} if A^{n_0} is close to the identity matrix.

6. CONCLUSIONS

In this paper, we have studied the problem of output regulation for linear discrete-time systems subject to actuator saturation. The plants considered here are general and can be exponentially unstable. We first characterized the regulatable region, the set of initial conditions of the plant and the exosystem for which output regulation can be achieved. We then constructed feedback laws, of both state feedback and error feedback type, that achieve output regulation on the regulatable region.

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