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Lyapunov characterization of forced oscillations[☆]

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Abstract

This paper develops a Lyapunov approach to the analysis of input–output characteristics for systems under the excitation of a class of oscillatory inputs. Apart from sinusoidal signals, the class of oscillatory inputs include multi-tone signals and periodic signals which can be described as the output of an autonomous system. The Lyapunov approach is developed for linear systems, homogeneous systems (differential inclusions) and nonlinear systems (differential inclusions), respectively. In particular, it is established that the steady-state gain can be arbitrarily closely characterized with Lyapunov functions if the output response converges exponentially to the steady-state. Other output measures that will be characterized include the peak of the transient response and the convergence rate. Tools based on linear matrix inequalities (LMIs) are developed for the numerical analysis of linear differential inclusions (LDIs). This paper's results can be readily applied to the evaluation of frequency responses of general nonlinear and uncertain systems by restricting the inputs to sinusoidal signals. Guided by the numerical result for a second order LDI, an interesting phenomenon is observed that the peak of the frequency response can be strictly larger than the L_2 gain.

Keywords: Forced oscillations; Lyapunov functions; Differential inclusions; Steady-state gain; Transient response; LMI

0. Basic definitions

- $|x|$: Euclidean norm of x.
- $\langle x_1, x_2 \rangle$: the inner product of x_1 and x_2.
- $\|X\|$: Matrix norm induced from the Euclidean norm.
- $\|u\|$: $\text{ess.sup}_{t\geqslant 0}|u(t)|$ for an essentially bounded function $u : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$.
- A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to belong to class $\mathscr{K}$ ($\alpha \in \mathscr{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to class $\mathscr{K}_\infty$ if, in addition, it is unbounded.
- A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to belong to class $\mathscr{KL}$ if, for each $t \geqslant 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+}\beta(s, t) = 0$, and for each $s \geqslant 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty}\beta(s, t) = 0$.

1. Introduction

1.1. Background

A large number of physical systems are excited by sinusoidal signals, multi-tone signals, periodic signals or general oscillatory inputs. The output response of a system under the excitation of sinusoidal signals can be characterized through both the time-domain approach and the frequency-domain approach. For linear systems, the steady-state responses are easily determined through the transfer

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functions and the transient responses can be analyzed with state-space descriptions. For nonlinear systems, the characterization of the output response is believed to be much harder because of various nonlinear phenomena, such as jump phenomena, subharmonic oscillations and frequency entrainmentas observed, for example, in Fukuma, Mattsubara, and Watanabe (1984), O'day and Hyde (1972). Some of the early attempts were made by using the describing function method to obtain an approximate characterization of the steady-state response, e.g., in Fukuma et al. (1984); Grensted (1955). For weakly nonlinear systems arising from electronic circuits, the steady-state responses are often investigated using the Volterra series theory (see, e.g., Chua & Tang (1982), Lang & Billings (2000), Sandberg (1984) and Swain & Billings (2001)).

In many communication circuits, the systems are driven by signals as a sum of sinusoids with different frequencies (called multiple inputs as in Lang & Billings (2000) and multi-tone signals as in Ushida & Chua, 1984). In mechanical systems such as a rotating machinery, the exogenous input may be a general periodic signal. All these oscillatory input signals can be characterized as the output of an autonomous system, the so-called exosystem as described in Francis (1977); Gilliam, Byrnes, Isidori, and Ramsey (2003); Isidori (1995, 1999); Isidori and Astolfi (1992); Isidori and Byrnes (1990). The output response of a system under the excitation of an input generated by such an exosystem is referred to as forced oscillation. Some steady-state behavior of forced oscillations, such as the existence of a steady-state solution and the structure of the steady-state solution set, has been studied in the aforementioned works.

In this paper, we will evaluate some quantitative measures of the output including the steady-state gain, the peak of the transient response and the convergence rate for systems excited by inputs described via the exosystem. Our objective is to propose a systematic Lyapunov approach to the evaluation of these quantities. A prominent feature of the Lyapunov approach is its capability of handling nonlinearities and uncertainties. Because of this, we will be able to evaluate these quantities for linear and nonlinear differential inclusions. A Lyapunov approach to the characterization of the input–output relationship for a general nonlinear system was proposed in Sontag and Wang (2001) where several notions of output stability introduced in Sontag and Wang (1997) were investigated. Consider nonlinear systems of the form

$$\dot{x} = f(x,u), \quad y = h(x), \tag{1}$$

where $f:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h:\mathbb{R}^n \to \mathbb{R}^q$ are both locally Lipschitz continuous. Given an initial state $x_\circ \in \mathbb{R}^n$ and an input u, let $x(\cdot, x_\circ, u)$ be the solution of the system and let $y(\cdot, x_\circ, u)$ be the corresponding output. Assume that for every $x_\circ$ and u, the solution $x(t, x_\circ, u)$ is defined for all $t \geqslant 0$. Then the system is said to be input to output stable if there exist a $\mathscr{KL}$-function β and a $\mathscr{K}$-function γ such that

$$|y(t,x_\circ,u)| \leqslant \beta(|x_\circ|, t) + \gamma(\|u\|), \quad \forall t > 0. \tag{2}$$

The function γ gives a bound for the asymptotic gain from the input u to the output y and the function β characterizes how the output approaches its asymptotic bound. Now suppose that the system is input to output stable. One interesting problem is to find a function γ that characterizes the asymptotic bound of the output as sharply as possible. Another problem is to find γ and β such that $\beta(|x_\circ|, 0) + \gamma(\|u\|)$ is minimized so that the peak norm of the output during the transient process can be estimated without too much conservatism. For linear systems, the problem of minimizing the asymptotic gain can be approached through reachable sets with unit-peak inputs, which are estimated with ellipsoids under the LMI framework (see Boyd, El Ghaoui, Feron, & Balakrishnan, 1994, p. 82). Most often, the input u is not an arbitrary signal and more detailed information about it may be exploited to obtain a weaker stability condition or a sharper bound on the output. For instance, bounds on the derivatives of the input are used in Angeli, Sontag, and Wang (2003, 2001) to obtain a weaker condition of stability. It is also notable that the notion of Cauchy gain was introduced in Sontag (2002) to describe the relation between the asymptotic amplitude of the input and that of the output.

In this paper, attention will be restricted to the class of input signals which can be described as the output of an aforementioned exosystem. Stability issues under such input signals are closely related to the measurement to error stability in Ingalls, Sontag, and Wang (2002) and can also be studied under the framework of Angeli et al. (2003, 2001). This paper attempts to evaluate some quantitative measures including the steady-state gain, the peak of the transient response and the convergence rate. It was expected that the exact description of the input signals with an autonomous exosystem would facilitate the characterization of these quantities. Indeed, we will show in this paper that the steady-state gain can be exactly (or arbitrarily closely) characterized through the construction of Lyapunov functions under certain assumption on the convergence property of the output to its steady-state, namely, that the output response converges exponentially fast to its steady-state.

It should be remarked that the Lyapunov approach can be readily applied to the evaluation of the frequency response of a system whose input involves one or a few harmonics—by varying the parameters of the exosystem.

This paper is organized as follows. In Section 1.2, the input is modeled as the output of an autonomous system and some input–output relationships are described. Section 2 contains the main results on the Lyapunov approach to the characterization of the steady-state gain. Section 3 presents a method for evaluating the peak of the output for initial conditions within a given set and the convergence rate of the output response. Section 4 develops numerical algorithms for the analysis of linear differential inclusions (LDIs). Section 5 uses an example to demonstrate an unexpected phenomenon in LDIs. Section 6 concludes this paper. Appendix A contain the proofs for the main results in Section 2.

1.2. Problem statement

1.2.1. Input model and quantitative measures of output response

For system (1), consider the class of input u which can be modeled as

$$u = k(w), \quad \dot{w} = g(w), \tag{3}$$

where $w \in \mathbb{R}^\ell$ and $\langle w, g(w)\rangle = 0$ for all $w \in \mathbb{R}^\ell$. When both k and g are linear functions, we have

$$u = \Gamma w, \quad \dot{w} = Sw, \tag{4}$$

with $S+S^{\mathrm{T}}=0$. These input models are inherited from the exosystem description in Francis (1977), Gilliam et al. (2003), Isidori (1995, 1999), Isidori and Astolfi (1992), Isidori and Byrnes (1990). The assumption $\langle w, g(w)\rangle = 0$ is extended from the linear condition $S + S^{\mathrm{T}} = 0$ as originally assumed in Francis (1977) and is not as general as Poisson stability as assumed in other works. This assumption facilitates Lyapunov characterization of the input–output relationship but retains the capability of describing a wide class of oscillatory input signals such as sinusoidal, multi-tone or periodic signals. For example, a vector sinusoidal signal u with $u_i(t) = \bar{u}_i \sin(\phi t + \theta_i)$ can be modeled as

$$u = \Gamma w = [I \;\; 0]w, \quad \dot{w} = Sw = \begin{bmatrix} 0 & -\phi I \\ \phi I & 0 \end{bmatrix} w, \tag{5}$$

where the phase and magnitude of u_i can be generated by choosing appropriate $w_\circ$. In particular, we have $|\bar{u}| = |w_\circ|$, where $\bar{u} = [\bar{u}_1 \;\; \bar{u}_2 \;\; \ldots \;\; \bar{u}_m]^{\mathrm{T}}$. This shows that the magnitude of the vector sinusoidal signal equals the Euclidean norm of $w_\circ$. Also, a multi-tone signal can be described by (4) with all the different frequencies contained in S. The assumption that $\langle w, g(w)\rangle = 0$ for all w, which is equivalent to $|w(t)| \equiv |w_\circ|$, can be replaced with a seemingly weaker one such as $|q(w(t))| \equiv |q(w_\circ)|$, where $q : \mathbb{R}^\ell \mapsto \mathbb{R}^\ell$ is continuously differentiable with a continuously differentiable inverse. More discussion about the relaxation of the assumption is contained in Remark 2 in Section 2.3.

Combining system (1) with the input described by (3), we have the autonomous system

$$\dot{x} = f(x, k(w)), \quad \dot{w} = g(w), \quad y = h(x). \tag{6}$$

We observe that the original input $u = k(w)$ is completely determined by the initial condition $w_\circ$. For this reason, we denote the state response and the output response as $x(\cdot, x_\circ, w_\circ)$ and $y(\cdot, x_\circ, w_\circ)$, respectively. Assume that the functions k and g are carefully chosen such that the magnitude of u is closely reflected by $w_\circ$, for instance, as in (5). Then the input–output relationship can be indirectly characterized through the relationship between $w_\circ$ and y. In this paper, we would like to evaluate the following quantity

$$\gamma_{\mathrm{ss}}(\Delta) := \sup\left\{ \lim_{T\to\infty} \|y_{[T,\infty]}\| : x_\circ \in \mathbb{R}^n, |w_\circ| \leqslant \Delta \right\}, \tag{7}$$

where $y_{[T,\infty]}(t) = y(t)$ for $t \geqslant T$ but 0 otherwise. The quantity $\gamma_{\mathrm{ss}}(\Delta)$ denotes the maximal magnitude of the steady-state output under the restriction $|w_\circ| \leqslant \Delta$ and the ratio $\gamma_{\mathrm{ss}}(\Delta)/\Delta$ can be referred to as the steady state gain from $w_\circ$ to y. In the case that u is a sinusoidal signal as described in (5), $\gamma_{\mathrm{ss}}(\Delta)/\Delta$ depends on the frequency ϕ and can be used to measure the magnitude frequency response for a given Δ. For transient analysis, we would also like to determine how fast the output response converges to its steady-state and the maximal value of $|y(t)|$ during the transient process, i.e., the quantity

$$y_{\mathrm{M}}(\Delta, X_\circ) := \sup\{\|y\| : |w_\circ| \leqslant \Delta, x_\circ \in X_\circ\}, \tag{8}$$

where $X_\circ$ is the set of possible initial conditions for x.

1.2.2. Characterization of steady-state and transient output responses

Depending on how the output response converges to its steady-state, different approaches may be derived for the characterization of the steady-state and transient responses. A general convergence may be much harder to describe than one with some desired property such as uniform asymptotic convergence

$$|y(t, x_\circ, w_\circ)| \leqslant \beta\left(\left| \begin{matrix} x_\circ \\ w_\circ \end{matrix} \right|, t \right) + \gamma_1(|w_\circ|), \quad \forall t > 0, \tag{9}$$

where $\beta \in \mathscr{KL}$, $\gamma_1 \in \mathscr{K}$, and exponential convergence

$$|y(t, x_\circ, w_\circ)| \leqslant K \left| \begin{matrix} x_\circ \\ w_\circ \end{matrix} \right| \mathrm{e}^{-\eta t} + \gamma_2(|w_\circ|), \quad \forall t > 0, \tag{10}$$

where $K, \eta > 0$ and $\gamma_2 \in \mathscr{K}$. If (9) or (10) is satisfied, then $\gamma_{\mathrm{ss}}(\Delta) \leqslant \gamma_1(\Delta)$ or $\gamma_{\mathrm{ss}}(\Delta) \leqslant \gamma_2(\Delta)$ and we can use γ_1 or γ_2 to evaluate an upper bound for γ_{ss}. Just as general stability does not imply uniform asymptotic stability or exponential stability, one can construct examples such that the steady-state of the output exists but neither (9) nor (10) is satisfied. Thus it is of interest to know for what type of systems these convergence properties are equivalent. Our first conclusion is that they are the same for linear systems. This conclusion can be drawn from some discussions in Francis (1977), Gilliam et al. (2003) and Isidori (1999), which are summarized as follows.

Claim 1. *Consider the linear system*

$$\dot{x} = Ax + B\Gamma w, \quad \dot{w} = Sw, \quad y = Cx. \tag{11}$$

Assume that A is Hurwitz (its eigenvalues have negative real parts) and that $S + S^{\mathrm{T}} = 0$. *Let* Π *be the solution to*

$$A\Pi - \Pi S = B\Gamma, \tag{12}$$

and let $\gamma_* = \|C\Pi\|$. *Then* $\gamma_{\mathrm{ss}}(\Delta) = \gamma_* \Delta$ *and* γ_* *is the least positive number* γ *such that there exist* $K > 0$ *and* $\eta > 0$

satisfying

$$|y(t)| \leqslant K \left| \begin{matrix} x_\circ \\ w_\circ \end{matrix} \right| \mathrm{e}^{-\eta t} + \gamma |w_\circ| \quad \forall t \geqslant 0, \quad x_\circ \in \mathbb{R}^n, \quad w_\circ \in \mathbb{R}^\ell.$$

In the case that $\Gamma = [1\ \ 0]$ *and* $S = \begin{bmatrix} 0 & -\phi \\ \phi & 0 \end{bmatrix}$, *we have* $u(t) = \Gamma w(t) = |w_\circ| \sin(\phi t + \theta)$ *and* $\gamma_* = |C(j\phi I - A)^{-1} B|$.

By Claim 1, for a linear system, the convergence of the output response to a steady-state magnitude $\gamma_{ss}(\Delta)$ implies its exponential convergence (and hence uniform asymptotic convergence) to the same steady-state magnitude. Other types of systems where the convergence properties are possibly equivalent are LDIs and homogeneous of degree one systems. For these systems, it has been confirmed that local asymptotic stability implies global uniform asymptotic and exponential stability (see Hahn (1967)). However, this paper will not pursue this issue and will be focused on characterizing the steady-state gain and transient response through the exponential convergence property (10). It will be shown that when the output converges to its steady-state exponentially as in (10), the steady-state gain can be characterized through Lyapunov functions. In other words, a Lyapunov function can be constructed to reflect the steady-state gain. Our converse Lyapunov theorems are based on those for stability with respect to two measures for differential inclusions given in Teel and Praly (2000). Those results assume uniform asymptotic convergence rather than exponential convergence. So, for nonhomogeneous nonlinear systems that satisfy a bound like (9) (not (10)), the tools needed to address the existence of Lyapunov functions are in place. However, for the particular Lyapunov structure we use in this paper, it is somewhat cumbersome to clarify the regularity of the Lyapunov function characterizing the steady-state gain. For this reason, in order to keep the presentation simple, we will restrict our attention to the exponential decay case in this paper.

2. Steady-state analysis: main results

This section provides a general framework for the characterization of the steady-state gain through Lyapunov functions. The results will be presented for linear systems, homogeneous differential inclusions and nonlinear differential inclusions. The linear system is a special case of (6). The differential inclusions are extended from (6). For linear systems, as we recall from Claim 1, the steady-state gain can be simply computed from the solution to a Sylvester equation. The alternative Lyapunov approach is not intended to improve or replace this simple method but to suggest a framework for more general systems.

Standing assumption: All systems considered are forward complete, i.e., from each initial condition there is no finite escape time.

2.1. Linear systems

Consider the linear system

$$\dot{x} = Ax + Ew, \quad \dot{w} = Sw, \quad y = Cx, \tag{13}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^q$.

Assumption 1. $S + S^{\mathrm{T}} = 0$ and A is Hurwitz.

Under Assumption 1, we have $|w(t)| = |w_\circ|$ for all $t \geqslant 0$. Define

$$A_L := \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}. \tag{14}$$

We consider matrix $P \in \mathbb{R}^{(n+\ell)\times(n+\ell)}$, $P = P^{\mathrm{T}} > 0$ and numbers $\gamma > 0$, $\eta > 0$ satisfying

$$\begin{bmatrix} C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix} \leqslant P \tag{15a}$$

$$A_L^{\mathrm{T}} P + P A_L \leqslant -2\eta \left(P - \gamma^2 \begin{bmatrix} 0 & 0 \\ 0 & I_\ell \end{bmatrix} \right). \tag{15b}$$

Theorem 1. *Suppose Assumption* 1 *holds and let* $\bar{\gamma} > 0$ *be given. The following statements are equivalent*:

(a) *For each* $\gamma > \bar{\gamma}$ *there exist a matrix* P *and* $\eta > 0$ *satisfying* (15a) and (15b);
(b) *For each* $\gamma > \bar{\gamma}$ *there exist* $K > 0, \bar{\eta} > 0$ *such that*

$$|y(t)| \leqslant K \left| \begin{matrix} x_\circ \\ w_\circ \end{matrix} \right| \mathrm{e}^{-\bar{\eta} t} + \gamma |w_\circ| \quad \forall t \geqslant 0, \quad x_\circ \in \mathbb{R}^n, \quad w_\circ \in \mathbb{R}^\ell. \tag{16}$$

Proof. See Appendix A. □

By Claim 1, the least γ such that there exist $K > 0, \bar{\eta}$ satisfying (16) equals the steady-state gain from $|w_\circ|$ to y, and it can be computed as $\gamma_* = \|C\Pi\|$ with Π solved from (12). Theorem 1 provides an alternative approach to compute γ_* as the minimal γ satisfying the matrix inequalities (15a) and (15b). This approach may not be as efficient as the other one but it suggests a new framework that can be extended to general nonlinear uncertain systems. If we associate the matrix P with a quadratic function $W(\xi) = \xi^{\mathrm{T}} P \xi$ and interpret the inequalities (15a) and (15b) in terms of W, then a Lyapunov approach takes shape.

2.2. Homogeneous systems

Let $M : \mathbb{R}^n \to$ (subsets of $\mathbb{R}^n$) be a set-valued map. We say that M is homogeneous of degree p if $M(\lambda x) = \lambda^p M(x)$ for all $\lambda \geqslant 0$ and $x \in \mathbb{R}^n$.

Consider the system

$$\dot{x} \in A(x, w), \quad \dot{w} \in G(w), \quad y = h(x), \tag{17}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^q$. $A : \mathbb{R}^n \times \mathbb{R}^\ell \to$ (subsets of $\mathbb{R}^n$) and $G : \mathbb{R}^\ell \to$ (subsets of $\mathbb{R}^\ell$) are set-valued maps. Define

$$\xi := \begin{bmatrix} x \\ w \end{bmatrix},$$
$$F(\xi) := \left\{ \begin{bmatrix} a \\ g \end{bmatrix} : a \in A(x,w),\, g \in G(w) \right\}. \tag{18}$$

Assumption 2. $\langle w, g\rangle = 0$ for all $g \in G(w)$. The set-valued map F and the function h are homogeneous of degree one and globally Lipschitz with nonempty compact, convex values.

We consider continuously differentiable functions $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$ and numbers $\gamma > 0$, $\eta > 0$, $p > 1$ satisfying

$$W(0) = 0, \quad |h(x)|^p \leqslant W(\xi), \tag{19a}$$

$$\max_{f \in F(\xi)} \langle \nabla W(\xi), f\rangle \leqslant -p\eta(W(\xi) - \gamma^p |w|^p). \tag{19b}$$

Theorem 2. *Suppose Assumption* 2 *holds and let* $\overline{\gamma} > 0$ *be given. The following statements are equivalent*:

(a) *There exists* $\eta > 0$ *such that, for each* $\gamma > \overline{\gamma}$ *and* $p > 1$, (19a) and (19b) *have a continuously differentiable solution W that is homogeneous of degree p*;
(b) *There exists* $\overline{\eta} > 0$ *and for each* $\gamma > \overline{\gamma}$ *there exists* $K > 0$ *such that*

$$|y(t)| \leqslant K|\xi_\circ| e^{-\overline{\eta}t} + \gamma |w_\circ| \;\; \forall t \geqslant 0, \quad \xi_\circ \in \mathbb{R}^{n+\ell}. \tag{20}$$

Proof. See Appendix A. □

Remark 1. From the proof of the theorem, we see that statement (a) can be replaced with a seemingly weaker one: There exist $\eta > 0$ and $p > 1$ such that, for each $\gamma > \overline{\gamma}$, (19a) and (19b) has a continuously differentiable solution W that is homogeneous of degree p. The equivalence of these conditions implies that we can restrict our attention to a fixed $p > 1$, such as an even integer.

For homogeneous of degree one systems, the steady-state gain $\gamma_{ss}(\Delta)/\Delta$ is independent of Δ. Theorem 2 says that if the output response converges exponentially to its steady-state, then this gain can be determined through the Lyapunov functions W satisfying (19a) and (19b). For the case where the original input u is sinusoidal as given by (5), the steady-state gain depends on the frequency ϕ. If we perform the computation for all $\phi \geqslant 0$, a frequency response of the magnitude is generated.

2.3. Nonlinear systems

Consider the system

$$\dot{x} \in A(x,w), \quad \dot{w} \in G(w), \quad y = h(x), \tag{21}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^q$, and A and G are set-valued maps. Let ξ and F be defined as in (18).

Assumption 3. $\langle w, g\rangle = 0$ for all $g \in G(w)$, and the set-valued map F is locally Lipschitz with nonempty compact, convex values.

Given $\overline{\gamma} \in \mathcal{K}$ locally Lipschitz, we consider locally Lipschitz functions $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$ and numbers $\eta > 0$, $\varepsilon > 0$, $p > 1$ satisfying

$$W(0) = 0, \quad |h(x)|^p \leqslant W(\xi), \tag{22a}$$

$$\max_{f \in F(\xi)} \langle \nabla W(\xi),\; f\rangle \leqslant -p\eta(W(\xi) - (1+\varepsilon)^p \overline{\gamma}(|w|)^p) \quad \text{a.e.} \tag{22b}$$

Theorem 3. *Suppose Assumption* 3 *holds and let* $\overline{\gamma} \in \mathcal{K}$ *be given and locally Lipschitz. The following statements are equivalent*:

(a) *There exists* $\eta > 0$ *such that, for each* $\varepsilon > 0$ *and* $p > 1$, (22a) and (22b) *have a locally Lipschitz solution W*;
(b) *There exists* $\overline{\eta} > 0$ *and for each* $\varepsilon > 0$ *there exists* $\alpha_\varepsilon \in \mathcal{K}_\infty$ *such that*

$$|y(t)| \leqslant \alpha_\varepsilon(|\xi_\circ|) e^{-\overline{\eta}t} + (1+\varepsilon)\overline{\gamma}(|w_\circ|) \quad \forall t \geqslant 0, \quad \xi_\circ \in \mathbb{R}^{n+\ell}. \tag{23}$$

Proof. See Appendix A. □

Remark 2. The assumption $\langle w, g\rangle = 0$ can be replaced with $\langle \nabla\kappa(w), g\rangle = 0$ for a more general measure, $\kappa : \mathbb{R}^\ell \to \mathbb{R}_{\geqslant 0}$. In that case, we have $\kappa(w(t)) \equiv \kappa(w_\circ)$ instead of $|w(t)| \equiv |w_\circ|$ and Theorem 3 remains true if $|w|$ and $|w_\circ|$ are replaced with $\kappa(w)$ and $\kappa(w_\circ)$.

The state-output description in (23) is a global relation. For some systems, such a relation may only be valid for initial states in a subset of the state space. To characterize a regional state-output relation, we need to identify some invariant set in $\mathbb{R}^{n+\ell}$. Suppose that (22a) and (22b) are satisfied. Then for any $\rho > 0$,

$$\Lambda(\rho) := \{\xi \in \mathbb{R}^{n+\ell} : W(\xi) \leqslant \rho,\, (1+\varepsilon)^p \overline{\gamma}(|w|)^p \leqslant \rho\}$$

is an invariant set. As can be seen from the last inequality in (22a) and (22b), if $\xi_\circ \in \Lambda(\rho)$, then $W(\xi)$ will never exceed ρ and $\xi(t) \in \Lambda(\rho)$ for all $t \geqslant 0$ (see the proof of Theorem 4 for more explanation). On the other hand, if (22a) and (22b) are only satisfied for $\xi \in \Lambda(\rho)$, we can still conclude the existence of $\alpha_\varepsilon \in \mathcal{K}_\infty$ such that

$$|y(t)| \leqslant \alpha_\varepsilon(|\xi_\circ|) e^{-\overline{\eta}t} + (1+\varepsilon)\overline{\gamma}(|w_\circ|) \quad \forall t \geqslant 0, \quad \xi_\circ \in \Lambda(\rho).$$

As the bound $\overline{\gamma}$ gets tighter and as $\varepsilon \to 0$, the structure of W may become more complicated. When applying the Lyapunov approach to characterize the output response of a particular system, we may construct a function W of a simple structure for the estimation of an upper bound. To compensate the restricted freedom in choosing a Lyapunov function of a specific structure, we may replace the constant η in (22a) and (22b) with a scalar function $\eta(\xi)$. Here we note that if certain $\eta > 0$ satisfies (22a) and (22b), it does not imply that $\eta_1 \in (0, \eta)$ satisfies (22a) and (22b). The following corollary provides a method for the estimation of an upper bound and can be proved with arguments similar to the proof of (a) implies (b) in Theorem 3.

Corollary 1. *Given $\overline{\gamma} \in \mathscr{K}$ locally Lipschitz. Suppose there exist locally Lipschitz function $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$ and a function $\eta : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$, and numbers $p > 1$, $\rho > 0$, $\overline{\eta} > 0$ such that $W(0) = 0$ and for all*

$$\xi \in \Lambda(\rho) = \{\xi \in \mathbb{R}^{n+\ell} : W(\xi) \leqslant \rho, \overline{\gamma}(|w|)^p \leqslant \rho\}$$

the following is satisfied:

$$\begin{aligned} &\eta(\xi) \geqslant \overline{\eta}, \\ &|h(x)|^p \leqslant W(\xi), \\ &\max_{f\in F(\xi)} \langle \nabla W(\xi), f\rangle \leqslant -p\eta(\xi)(W(\xi) - \overline{\gamma}(|w|)^p) \quad \text{a.e.}\ . \end{aligned} \tag{24}$$

Then $\Lambda(\rho)$ is an invariant set and there exists $\alpha \in \mathscr{K}_\infty$ such that,

$$|y(t)| \leqslant \alpha(|\xi_\circ|)e^{-\overline{\eta}t} + \overline{\gamma}(|w_\circ|) \quad \forall\, \xi_\circ \in \Lambda(\rho).$$

In the following, we use a simple example to illustrate the application of the Lyapunov approach to the estimation of the steady-state gain. This example was used in Angeli et al. (2001) to show that the bound on the derivative of the input signal may help to weaken the condition for input–output stability.

Example 1. Consider a first-order system

$$\dot{x} = \mathrm{sat}(-x) + d, \quad y = h(x) = x,$$

where $\mathrm{sat}(u) = \mathrm{sign}(u)\min\{1, |u|\}$ and d is the disturbance. If d is arbitrary, then the steady-state gain from d to x is unbounded. For instance, a constant $d > 1$ will drive x unbounded. However, if d is a sinusoidal signal $d(t) = k\sin(\beta t)$, then the steady-state gain from d to y is finite (see Angeli et al. (2001)). Here we show that the steady-state gain is bounded by $2/\beta$. We express d as

$$d = [1\ \ 0]w, \quad \dot{w} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}, \quad |w_\circ| = k.$$

Choose $p = 2$ and $W(\xi) = 2x^2 + 4xw_2/\beta + 2w_1^2/\beta^2 + 4w_2^2/\beta^2$. Then it can be verified that $x^2 \leqslant W(\xi)$ and

$$\max_{f\in F(\xi)} \langle \nabla W(\xi), f\rangle \leqslant -\frac{\mathrm{sat}(x)}{x}\left(W(\xi) - \frac{4}{\beta^2}|w|^2\right),$$

for all $x \neq 0$. Let $\eta(\xi) = \mathrm{sat}(x)/x$ and

$$\Lambda(\rho) = \left\{\xi \in \mathbb{R}^3 : W(\xi) \leqslant \rho, \frac{4|w|^2}{\beta^2} \leqslant \rho\right\}.$$

Then $\min\{\eta(\xi) : \xi \in \Lambda(\rho)\} := \overline{\eta} > 0$ and by Corollary 1, $\Lambda(\rho)$ is an invariant set for every $\rho > 0$. Moreover, for all $\xi_\circ \in \mathbb{R}^3$, the steady-state x is bounded by $\overline{\gamma}(|w_\circ|) = 2|w_\circ|/\beta = 2k/\beta$. Note that this bound is valid for all $w_\circ$. If $|w_\circ|$ is sufficiently small, then the steady-state behaves like that of a linear system hence the gain will approach $1/\sqrt{1+\beta^2}$.

3. Characterization of transient responses

Evaluation of the peak is an important problem, especially for systems that must operate under some state or output constraints. However, exact calculation of the peak of an output for a set of initial conditions is an unresolved problem even for stable linear systems. In this section, we develop a general method for evaluation of the peak and the convergence rate.

3.1. A general result

Consider the nonlinear differential inclusion (21) under the standing assumption and Assumption 3. With (23), one may obtain an estimation of the peak of the output as $\alpha_\varepsilon(|\xi_\circ|) + (1+\varepsilon)\overline{\gamma}(|w_\circ|)$. Such an estimation may be too conservative as $\alpha_\varepsilon(|\xi_\circ|)$ is a coarse estimation from the proof of Theorem 3 (see Section A.1). Besides, we may prefer a more flexible description of the set of initial states than a simple bound on $|\xi_\circ|$. For this reason, we consider a function $r : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ which measures the size of the state x. We would like to determine the peak of the output in terms of $r(x_\circ)$ and $|w_\circ|$.

Theorem 4. *Consider system* (21). *Suppose that there exist a locally Lipschitz function $W : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$, a number $\overline{\eta} \geqslant 0$, a function $\eta : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$, and class $\mathscr{K}_\infty$*-functions *$\alpha_1, \alpha_2, \alpha_3$ satisfying*

$$\eta(\xi) \geqslant \overline{\eta}, \tag{25a}$$

$$|h(x)|^p \leqslant W(\xi) \leqslant \alpha_1(r(x)) + \alpha_2(|w|^p), \tag{25b}$$

$$\max_{f\in F(\xi)} \langle \nabla W(\xi), f\rangle \leqslant -\eta(\xi)(W(\xi) - \alpha_3(|w|^p)) \quad \text{a.e.} \tag{25c}$$

Then the peak of $|y(t)|^p$ is bounded by $\max\{\alpha_1(r(x_\circ)) + \alpha_2(|w_\circ|^p), \alpha_3(|w_\circ|^p)\}$ *and the convergence rate is no less than $\overline{\eta}$. In particular, for all $t \geqslant 0$, $x_\circ \in \mathbb{R}^n$, $w_\circ \in \mathbb{R}^\ell$,*

$$|y(t)|^p \leqslant K_\circ e^{-\overline{\eta}t} + \alpha_3(|w_\circ|^p), \tag{26}$$

where $K_\circ = \max\{0, \alpha_1(r(x_\circ)) + \alpha_2(|w_\circ|^p) - \alpha_3(|w_\circ|^p)\}$.

Proof. Define $V(\xi) := \max\{0, W(\xi) - \alpha_3(|w|^p)\}$. Since $|w(t)| = |w_\circ|$ for all $t > 0$, by (25a), (25b) and (25c) we have

$$\max_{f\in F(\xi)} \langle \nabla V(\xi), f\rangle \leqslant -\eta(\xi)V(\xi) \quad \text{a.e.}\ .$$

Let $\xi(t)$ be a solution to the system with initial condition $\xi_\circ$. It can be shown (similar to Teel, Panteley, & Loria, 2002, p. 185) that for almost all $t>0$,

$$\dot{V}(\xi(t)) \leqslant -\eta(\xi(t))V(\xi(t)) \leqslant -\overline{\eta}V(\xi(t)).$$

Hence for all $t>0$,

$$V(\xi(t)) \leqslant V(\xi_\circ)\mathrm{e}^{-\overline{\eta}t}.$$

It follows from (25a), (25b) and (25c) that

$$|y(t)|^p \leqslant W(\xi(t)) \leqslant V(\xi_\circ)\mathrm{e}^{-\overline{\eta}t} + \alpha_3(|w_\circ|^p),$$

which gives (26). □

Similarly to Corollary 1, a regional result can be established if (25a), (25b) and (25c) are satisfied in a subset of the state space. Theorem 4 is stated for system (21) but can be easily adapted for linear and homogeneous systems.

3.2. Application to linear systems

Let the initial state of x be inside a set $X_\circ$, say $X_\circ = \{x \in \mathbb{R}^n : x^{\mathrm{T}}Rx \leqslant 1\}$, where $R=R^{\mathrm{T}}>0$. Assume for simplicity that $w_\circ^{\mathrm{T}}w_\circ \leqslant 1$. We take $r(x)=x^{\mathrm{T}}Rx$. Consider quadratic type Lyapunov functions $W(\xi)=\xi^{\mathrm{T}}P\xi$ and $p=2$. According to Theorem 4, if we can find a $P=P^{\mathrm{T}}>0$, and numbers $\alpha_1,\alpha_2,\alpha_3,\eta>0$ such that, for all $\xi \in \mathbb{R}^{n+\ell}$,

$$\xi^{\mathrm{T}}\begin{bmatrix} C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix}\xi \leqslant \xi^{\mathrm{T}}P\xi \leqslant \xi^{\mathrm{T}}\begin{bmatrix} \alpha_1 R & 0 \\ 0 & \alpha_2 I_\ell \end{bmatrix}\xi, \tag{27}$$

$$A_L^{\mathrm{T}}P + PA_L \leqslant -\eta\left(P - \alpha_3\begin{bmatrix} 0 & 0 \\ 0 & I_\ell \end{bmatrix}\right), \tag{28}$$

then for all $x_\circ \in X_\circ$, $|w_\circ| \leqslant 1$,

$$y(t)^{\mathrm{T}}y(t) \leqslant \max\{\alpha_1+\alpha_2, \alpha_3\} \qquad \forall t \geqslant 0. \tag{29}$$

Our objective is to compute

$$\gamma_{p*} := \inf_{\alpha_1,\alpha_2,\alpha_3,\eta>0,P} \max\{\alpha_1+\alpha_2,\alpha_3\} \tag{30}$$

s.t. (a) $\begin{bmatrix} C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix} \leqslant P$,

s.t. (b) $P \leqslant \begin{bmatrix} \alpha_1 R & 0 \\ 0 & \alpha_2 I_\ell \end{bmatrix}$,

s.t. (c) $P=P^{\mathrm{T}}>0$, (28).

It can be verified that the optimal solution can be obtained by restricting $\alpha_1+\alpha_2=\alpha_3$. In this case, all the constraints are LMIs for a fixed η and the optimal γ_{p*} can be obtained by sweeping η from 0 to ∞ and solving each resulting optimization problem. The optimal γ_{p*} then gives an upper bound for $y(t)^{\mathrm{T}}y(t)$ under the initial conditions $x_\circ \in X_\circ$, $|w_\circ| \leqslant 1$.

4. Numerical analysis of LDIs via quadratic functions

Linear differential inclusions (LDIs) have been used to describe complex nonlinear uncertain time-varying systems. This description is practical since it makes complicated systems numerically tractable. A set of LMI-based tools are developed for LDIs in Boyd et al. (1994). This section is devoted to deriving LMI-based algorithms for characterizing the output responses of LDIs.

LDIs are homogeneous of degree one systems. According to Theorem 2, it is sufficient to consider homogeneous Lyapunov functions. For practical application, we may need to restrict our attention to Lyapunov functions which are computationally tractable. Apart from quadratic functions, other types of homogeneous functions which are computationally tractable include composite quadratic functions, piecewise quadratic functions, piecewise linear functions and some polynomial functions (see, e.g., Blanchini, 1995; Chesi, Garulli, Tesi, & Vicino, 2003; Goebel, Teel, Hu, & Lin, 2004; Hu, Lin, Goebel, & Teel, 2004; Jarvis-Wloszek & Packard, 2002). These functions have been used for stability and performance analysis of LDIs and saturated linear systems. For simplicity, we would like to use quadratic functions to demonstrate the Lyapunov approach.

Let Ω be a compact convex set in $\mathbb{R}^{(n+\ell)\times(n+\ell)}$. Assume that each $A_L \in \Omega$ has the structure $A_L = \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}$ and $S+S^{\mathrm{T}}=0$. Consider the following LDI:

$$\dot{\xi} \in \{A_L\xi : A_L \in \Omega\}, \quad y = [C \ \ 0]\xi, \tag{31}$$

where $\xi = [x \ \ w]^{\mathrm{T}} \in \mathbb{R}^{(n+\ell)}$. Clearly the set-valued map $\{A_L\xi : A_L \in \Omega\}$ is homogeneous of degree one. Here we consider a polytopic LDI with

$$\Omega = \mathrm{co}\{A_{Li} : i=1,2,\ldots,N\}, \tag{32}$$

where ”co” denotes taking the convex hull of set. This LDI can be used to describe the uncertainty in the frequency of w. For example, S belongs to the set

$$\left\{\begin{bmatrix} 0 & -\phi I_m \\ \phi I_m & 0 \end{bmatrix} : \phi \in [\phi_1,\phi_2]\right\}. \tag{33}$$

We note that there is no restriction on how ϕ varies between the interval $[\phi_1,\phi_2]$. Hence this description allows nonperiodic signal w.

Consider Lyapunov functions of the type: $W(\xi)=\xi^{\mathrm{T}}\overline{P}\xi$ and $p=2$. Then the condition (19a) and (19b) can be stated as

$$\begin{bmatrix} C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix} \leqslant \gamma^2 P \tag{34}$$

$$A_L^{\mathrm{T}}P + PA_L \leqslant -2\eta\left(P - \begin{bmatrix} 0 & 0 \\ 0 & I_\ell \end{bmatrix}\right), \quad \forall A_L \in \Omega, \tag{35}$$

where $P=\overline{P}/\gamma^2$. We have replaced $\overline{P}$ with P for numerical simplicity. If there exist $P=P^{\mathrm{T}}>0$, $\gamma,\eta>0$ satisfying (34) and (35), then γ is an upper bound for the steady-state gain by Theorem 2.

For polytopic LDIs, (35) is satisfied if and only if

$$A_{Li}^{\mathrm{T}} P + P A_{Li} \leqslant -2\eta \left(P - \begin{bmatrix} 0 & 0 \\ 0 & I_\ell \end{bmatrix} \right),$$
$$\forall i = 1, 2, \cdots, N. \tag{36}$$

Therefore, the bound on the steady-state gain can be sharpened by solving

$$\inf_{\gamma, \eta > 0, P > 0} \gamma, \quad \text{s.t. } (34),\ (36) \tag{37}$$

For a fixed $\eta > 0$, this is a standard "gevp" (generalized eigenvalue problem) in LMI (see Boyd et al. (1994)). For structured (or norm-bounded) LDIs, similar optimization problem can be derived. To estimate the peak and the convergence rate of the output response with initial state inside a given set, we can also develop LMI-based methods from Theorem 4.

Example 2. In this example, we apply the numerical analysis method to a fluid pump for the estimation of the rotor's displacement under sinusoidal disturbances resulting from fluid and unbalance. The rotor is suspended by one active magnetic bearing (AMB) and two passive magnetic bearings (PMB). Due to the nonlinearity and uncertainty of the PMBs, the open-loop system is described as the following LDI with four vertices:

$$\dot{x} \in \mathrm{co}\{A_i x + Bu + E_1 d_1 + E_2 d_2 : i = 1, 2, 3, 4\},$$

where $x \in \mathbb{R}^4$ and $u, d_1, d_2 \in \mathbb{R}$. The states x_1 and x_2 are the displacements of the rotor (unit: meter) at two locations and x_3 and x_4 are the velocities; u is the control input (unit: Newton), the force generated by the AMB; d_1 and d_2 are normalized disturbances of the form $d_1(t) = \sin(\beta_\circ t + \theta_1)$, $d_2(t) = \sin(4\beta_\circ t + \theta_2)$, where $\beta_\circ$ is the rotational frequency of the rotor. The disturbance d_2 has a frequency four times that of d_1 since the rotor has four blades on its surface. The gap between the rotor and the stator is very small and it is required that the displacements x_1 and x_2 be restricted within 0.5 mm. Here are the parameters for the open-loop system:

$$A_i = \begin{bmatrix} 0 & I \\ H_i * 10^5 & 0 \end{bmatrix}, \quad i = 1, 2, 3, 4, \text{ where}$$
$$H_1 = \begin{bmatrix} 3.22 & 3.90 \\ -0.04 & -2.56 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 3.22 & 3.64 \\ -0.04 & -2.18 \end{bmatrix},$$
$$H_3 = \begin{bmatrix} 2.73 & 3.58 \\ -0.04 & -2.55 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 2.73 & 3.32 \\ -0.04 & -2.17 \end{bmatrix},$$
$$B = [0\ \ 0\ \ 192.35\ \ -59.68]^{\mathrm{T}},$$
$$E_1 = [0\ \ 0\ \ -21.18\ \ 25.15]^{\mathrm{T}},$$
$$E_2 = [0\ \ 0\ \ -163.0\ \ 145.9]^{\mathrm{T}}.$$

The rotational frequency is $\beta_\circ = 314.1593$ rad/s (3000 rpm). An LQR controller is designed based on (A_1, B) by taking $Q = \mathrm{diag}[5000\ 5000\ 20\ 20]$ and $R = 1$. The resulting controller is

$$u = Kx = 10^3 \times [-5.48\ \ -1.03\ \ -0.01\ \ -0.007]x.$$

We would like to estimate the maximal displacement x_1 and x_2 at steady-state for the closed-loop system. First, we need to describe the disturbance in the standard form. The total disturbance $E_1 d_1 + E_2 d_2$ can be described as

$$E_1 d_1 + E_2 d_2 = Ew, \quad \dot{w} = Sw, \quad |w_\circ| \leqslant 1, \text{ where}$$
$$E = \left[\frac{1}{\alpha} E_1 [1\ \ 0]\ \ \frac{1}{\sqrt{1-\alpha^2}} E_2 [1\ \ 0] \right],$$
$$S = \mathrm{diag}\left\{ \begin{bmatrix} 0 & \beta_\circ \\ -\beta_\circ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4\beta_\circ \\ -4\beta_\circ & 0 \end{bmatrix} \right\}.$$

Here $\alpha \in (0, 1)$ is used to reduce the conservatism in the estimation due to the transformation into the standard form We estimate the magnitudes of x_1 and x_2 by solving the optimization problem (37) with $C = [1\ 0\ 0\ 0]$ and $C = [0\ 1\ 0\ 0]$, respectively. For the LDI system with four vertices, let γ_{1*} and γ_{2*} be the minimal values of the solutions of (37) for all possible α and η corresponding to $y = x_1$ and $y = x_2$, respectively. Then $\gamma_{1*} = 2.8308 \times 10^{-4}$ m and $\gamma_{2*} = 4.5178 \times 10^{-4}$ m. If we ignore the frequency information, then by using the method in Boyd et al. (1994) (p. 82–84), the bound of x_1 is 18.9789×10^{-4} m and the bound of x_2 is 30.1974×10^{-4} m, significantly greater than γ_{1*} and γ_{2*}.

5. An observation on frequency response vs L_2 gain

For linear systems, we know that the peak of the frequency response equals the L_2 gain. For LDIs, it may be expected that the peak of the frequency response is no greater than the L_2 gain. If this is the case, then the peak of the frequency response can be suppressed indirectly by minimizing the L_2 gain, which can be easily addressed by solving LMIs. However, the following example demonstrates that the L_2 gain of an LDI system could be less than the peak of the frequency response. This means that the frequency analysis has to be performed separately from the L_2 gain analysis to ensure that the output is below a desirable value.

Example 3. Consider the LDI:

$$\dot{x} \in \mathrm{co}\{A_1 x + B_1 u, A_2 x + B_2 u\}, \tag{38}$$

$$A_1 = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.3 & -2.5 \\ 2.5 & -0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}.$$

The output is $y = Cx = [1\ \ 0]x$. The L_2 gains of the two linear systems (C, A_1, B_1) and (C, A_2, B_2) are both 0.8333. An upper bound for the L_2 gain of the LDI is computed as $\gamma_\infty^u = 0.9906$ (with the algorithm in Boyd et al. (1994)).

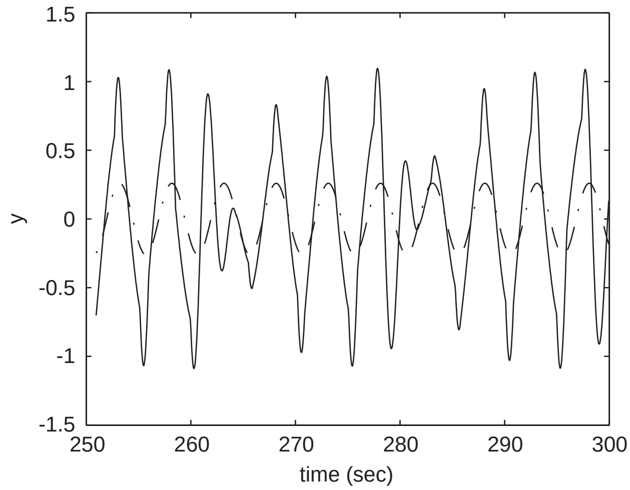

Fig. 1. Two steady-state responses.

Now assume that $u=\sin(\phi t+\theta)$. By using our method at $\phi=1.26$, an upper bound for the steady-state output y_{ss} is computed as 1.2169. An actual magnitude of y_{ss}, for a specific phase θ and a specific switching strategy, is detected as 1.0965. This shows that the steady-state gain at this frequency is no less than 1.0965. Hence the peak of the frequency response is greater than $1.0965>0.9906=\gamma_\infty^u$. This never happens with linear systems (see, e.g., Zhou, Doyle, & Glover (1996)). The switching strategy is chosen such that

$$\langle \nabla W, f\rangle,$$
$$f\in \mathrm{co}\left\{\begin{bmatrix} A_1 & B_1\Gamma\\ 0 & S\end{bmatrix}\begin{bmatrix} x\\ w\end{bmatrix}, \begin{bmatrix} A_2 & B_2\Gamma\\ 0 & S\end{bmatrix}\begin{bmatrix} x\\ w\end{bmatrix}\right\}$$

takes the maximal value, where

$$W=\begin{bmatrix} x\\ w\end{bmatrix}^{\mathrm{T}} P\begin{bmatrix} x\\ w\end{bmatrix},$$
$$P=\begin{bmatrix} 7.86 & -0.25 & 1.39 & -0.80\\ -0.25 & 7.884 & -2.02 & -0.03\\ 1.39 & -2.02 & 4.10 & 1.28\\ -0.80 & -0.03 & 1.28 & 2.67\end{bmatrix}$$
$$S=\begin{bmatrix} 0 & 1.26\\ -1.26 & 0\end{bmatrix},\qquad \Gamma=[1\ \ 0].$$

Actually, under this particular switching strategy, we detected two steady-state responses of the output, corresponding to different initial phase θ (or different $|w_\circ|$). These two steady-state responses are plotted in Fig. 1, where the response plotted with solid line corresponds to the initial condition of $x_\circ=0$ and $w_\circ=(1,0)$, and the response plotted with dash-dotted line corresponds to $x_\circ=0$ and $w_\circ=(0,1)$ The response (solid) has a peak larger than 1 but the energy over the time interval is only 0.556 of the energy of the disturbance $u=\sin(1.26t+\pi/2)$ over the same interval. From the figure, we see that the highest peaks of the output are produced in a period about four or five times the period of the input. And these peaks are much sharper than those of a sinusoidal signal. This explains the low energy of the output even with a high peak.

6. Conclusions

This paper pursues a Lyapunov approach to the evaluation of quantitative measures of input–output characteristics for systems with oscillatory inputs. A prominent feature of the Lyapunov approach is its capability of handling nonlinear and uncertain systems such as linear and nonlinear differential inclusions. Currently it has been shown that the Lyapunov approach is numerically tractable through quadratic Lyapunov functions for LDIs. Guided by the numerical result based on quadratic Lyapunov functions for a second order LDI, an interesting phenomenon is observed that the peak of the frequency response (the maximal steady-state gain over all the frequencies) can be strictly larger than the L_2 gain. The Lyapunov approach is potentially numerically tractable through homogeneous Lyapunov functions for homogeneous differential inclusions. As has been justified in this work, it is sufficient to characterize the output properties with homogeneous Lyapunov functions for homogeneous systems. With the advancement in the construction of numerically tractable homogeneous Lyapunov functions, it is expected that the newly proposed Lyapunov approach will undergo further significant development and find wide applications in practical systems which are driven by oscillatory inputs.

Appendix A. Proofs of Theorems 1–3

To save space, the theorems are proved in the general framework for nonlinear differential inclusions (Theorem 3) with specific explanation to linear systems and homogeneous systems. Section A.1 shows that (a) implies (b) and Section A.2 shows that (b) implies (a).

A.1. *Characterization of output responses:* (*a*) *to* (*b*)

Let W, $p>1$, $\eta>0$, $\varepsilon>0$ and $\overline{\gamma}\in\mathscr{K}$ satisfy (22a) and (22b) where $\overline{\gamma}$ is locally Lipschitz. In the linear case, $p=2$ and $W(\xi)=\xi^T P\xi$ with $P=P^{\mathrm{T}}>0$. In both the linear and homogeneous cases, $\overline{\gamma}(s)=\overline{\gamma}\cdot s$ and $\varepsilon=\gamma/\overline{\gamma}-1$. Define

$$V(\xi):=\max\{0, W(\xi)-(1+\varepsilon)^p\overline{\gamma}(|w|)^p\}$$

and note that (using $a^p\geqslant(a-b)^p+b^p$ whenever $a\geqslant b$)

$$(\max\{0,|h(x)|-(1+\varepsilon)\overline{\gamma}(|w|)\})^p \leqslant \max\{0,|h(x)|^p-(1+\varepsilon)^p\overline{\gamma}(|w|)^p\}\leqslant V(\xi). \quad \text{(A.1)}$$

Since $|w(t)|=|w_\circ|$ for all $t\geqslant 0$, we have

$$\max_{f\in F(\xi)}\langle\nabla V(\xi), f\rangle\leqslant -p\eta V(\xi)\quad \text{a.e. .} \quad \text{(A.2)}$$

Let $\xi(t)$ be a solution to the system with initial condition $\xi_\circ$. It can be shown (similarly to Teel & Praly (2000), p. 185) that for almost all $t > 0$,

$$\dot{V}(\xi(t)) \leqslant -p\eta V(\xi(t)). \tag{A.3}$$

Now integrate (A.3) to get $V(\xi(t)) \leqslant V(\xi_\circ)\mathrm{e}^{-p\eta t}$ and

$$\max\{0, |y(t)| - (1+\varepsilon)\overline{\gamma}(|w(t)|)\} \leqslant V(\xi(t))^{1/p} \leqslant V(\xi_\circ)^{1/p}\mathrm{e}^{-\eta t}. \tag{A.4}$$

Since $|w(t)| = |w_\circ|$ for all $t \geqslant 0$, if we add $(1+\varepsilon)\overline{\gamma}(|w_\circ|)$ to both sides of (A.4), we obtain

$$|y(t)| \leqslant V(\xi_\circ)^{1/p}\mathrm{e}^{-\eta t} + (1+\varepsilon)\overline{\gamma}(|w_\circ|). \tag{A.5}$$

The result follows with $\overline{\eta} = \eta$ and α_ε satisfying

$$V(\xi)^{1/p} \leqslant \alpha_\varepsilon(|\xi|). \tag{A.6}$$

In the case where W is homogeneous of degree p and $\overline{\gamma}(s) = \overline{\gamma} \cdot s$ so that V is homogeneous of degree p, $V(\xi)^{1/p}$ is homogeneous of degree one and so we can take $\alpha_\varepsilon(s) = K_\varepsilon \cdot s$ for some $K_\varepsilon > 0$.

A.2. Construction of Lyapunov functions: (b) *to* (a)

The following can be established:

Lemma 1. *For $\rho \in (0,1)$, $a, b \geqslant 0$ and $p > 1$,*

$$a^p \leqslant \rho^{-1}(\max\{0, a-b\})^p + \left(\frac{1}{1-\rho^{1/p}}\right)^p b^p.$$

A.2.1. Under a temporary assumption on Lyapunov functions

Temporary assumption: The bound on the output responses implies the existence of a function V that is zero at zero, and a positive real number η such that

$$(\max\{0, |h(x)| - (1+\varepsilon)\overline{\gamma}(|w|)\})^p \leqslant V(\xi), \tag{A.7}$$

$$\max_{f\in F(\xi)} \langle \nabla V(\xi), f\rangle \leqslant -p\eta V(\xi). \tag{A.8}$$

In the case of linear systems, we suppose V is quadratic and positive when $w = 0$ and $x \neq 0$. Moreover, $p = 2$. In the homogeneous case, we suppose V is continuously differentiable and homogeneous of degree p. In the nonlinear case, we suppose that V is locally Lipschitz and hence that (A.8) holds almost everywhere. In the linear and homogeneous cases, $\overline{\gamma}(s) = \overline{\gamma} \cdot s$ and $\varepsilon = -1 + \gamma/\overline{\gamma}$.

Let $\rho \in (0,1)$ and define the family of functions

$$W_\rho(\xi) = \rho^{-1}V(\xi) + \left(\frac{1+\varepsilon}{1-\rho^{1/p}}\right)^p \overline{\gamma}(|w|)^p. \tag{A.9}$$

Note that $W_\rho(0) = 0$. Applying Lemma 1 with $a = |h(x)|$ and $b = (1+\varepsilon)\overline{\gamma}(|w|)$, we have

$$|h(x)|^p \leqslant W_\rho(\xi). \tag{A.10}$$

Since $|w(t)| = |w_\circ|$ for all $t \geqslant 0$,

$$\max_{f\in F(\xi)} \langle \nabla W_\rho(\xi), f\rangle \leqslant -p\eta\left(W_\rho(\xi) - \left(\frac{1+\varepsilon}{1-\rho^{1/p}}\right)^p \overline{\gamma}(|w|)^p\right).$$

Observing that $\varepsilon > 0$ and $\rho > 0$ can be taken to be arbitrarily small, we obtain the result.

The theorems will be established once we remove the temporary assumption in the next section.

A.2.2. Removing the temporary assumption

Linear systems

Let Π be the solution to $A\Pi - \Pi S = E$. Then

$$\dot{x} + \Pi\dot{w} = A(x + \Pi w).$$

It follows from Assumption 1 that there exists $P_\circ = P_\circ^{\mathrm{T}} > 0$ and $\eta > 0$ such that, with $V_\circ(\xi) = (x+\Pi w)^T P_\circ(x+\Pi w)$ and $p = 2$, the relation (A.8) holds. By Claim 1, we have $\overline{\gamma} \geqslant \|C\Pi\|$. So we can write

$$\begin{aligned} V_\circ(\xi) &= (x+\Pi w)^{\mathrm{T}} P_\circ(x+\Pi w) \\ &\geqslant \lambda_{\min}(P_\circ)|x+\Pi w|^2 \\ &\geqslant \lambda_{\min}(P_\circ)\|C\|^{-2}|Cx + C\Pi w|^2 \\ &\geqslant \lambda_{\min}(P_\circ)\|C\|^{-2}\max\{0, |Cx| - (1+\varepsilon)\overline{\gamma}|w|\}^2. \end{aligned}$$

So the temporary assumption is satisfied with $V(\xi) = \|C\|^2 V_\circ(\xi)/\lambda_{\min}(P_\circ)$.

Nonlinear systems

For general nonlinear systems, the temporary assumption follows from the proof in Teel and Praly (2000). Indeed, the bound (23) corresponds to $\mathscr{KL}$ stability with respect to the two measures

$$\omega_1(\xi) = \max\{0, |h(x)| - (1+\varepsilon)\overline{\gamma}(|w|)\}, \quad \omega_2(\xi) = \alpha_\varepsilon(|\xi|).$$

Since the $\mathscr{KL}$ bound is already exponential, the differential inclusion is locally Lipschitz, $\overline{\gamma}$ is locally Lipschitz and the desired function only needs to be locally Lipschitz, we can take Eq. (131) in Teel and Praly (2000)

$$V(\xi) = \sup_{t\geqslant 0, \phi\in\mathscr{S}(\xi)} \omega_1(\phi(t,\xi))^p \mathrm{e}^{\eta t}, \tag{A.11}$$

where $\phi(t,\xi)$ is a solution of the differential inclusion under initial condition ξ, and $\mathscr{S}(\xi)$ is the set of solutions.

Homogeneous systems

Like for the general nonlinear case, the bound (20) corresponds to $\mathscr{KL}$ stability with respect to the two measures

$$\omega_1(\xi) = \max\{0, |h(x)| - (1+\varepsilon)\overline{\gamma}|w|\}, \quad \omega_2(\xi) = K|\xi|.$$

Since the $\mathscr{KL}$ bound is already exponential and the differential inclusion is globally Lipschitz we can get a preliminary function V_1 (defined as in (A.11) above with $p=1$) that is globally Lipschitz and homogeneous of degree one and in turn, following the proof of Teel and Praly (2000) (especially Eqs. (179)–(182)) the existence of a smooth function V satisfying

$$\kappa_1(V_1(\xi)) \leqslant V(\xi) \leqslant \kappa_2(V_1(\xi)), \tag{A.12}$$

$$\alpha_1(\omega_1(\xi)) \leqslant V(\xi) \leqslant \alpha_2(\omega_2(\xi)), \tag{A.13}$$

$$\max_{f\in F(\xi)} \langle \nabla V(\xi), f\rangle \leqslant -\mu V(\xi). \tag{A.14}$$

for some $\mu > 0$ and $\alpha_1, \alpha_2, \kappa_1, \kappa_2 \in \mathscr{K}_\infty$. Using the global Lipschitz continuity of V_1, V can be constructed to satisfy the additional condition

$$|\nabla V(\xi)| \leqslant \sigma(V(\xi)), \tag{A.15}$$

where $\sigma \in \mathscr{K}_\infty$.[1] These observations allow us to remove the temporary assumption via the following theorem which is based on the main result in Rosier (1992).

Theorem 5. *Assume that there exist a smooth function V, class-$\mathscr{K}_\infty$ functions $\alpha_1, \alpha_2, \kappa_1, \kappa_2$, homogeneous of degree one functions $V_1, \omega_1, \omega_2 : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$, and a number $\mu > 0$, satisfying* (A.12)–(A.14). *Define $\mathscr{A} := \{\xi \in \mathbb{R}^{n+\ell} : V(\xi) = 0\}$. Then for each positive number k there exist a continuous function $\overline{V} : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geqslant 0}$ that is homogeneous of degree k and smooth on $\mathbb{R}^{n+\ell}\backslash\mathscr{A}$, and there exist positive numbers $\overline{\alpha}_1$, $\overline{\alpha}_2$ $\overline{\kappa}_1$, $\overline{\kappa}_2$,$\overline{\mu}$ such that*

$$\overline{\kappa}_1 \cdot V_1(\xi)^k \leqslant \overline{V}(\xi) \leqslant \overline{\kappa}_2 \cdot V_1(\xi)^k \tag{A.16}$$

$$\overline{\alpha}_1 \cdot \omega_1(\xi)^k \leqslant \overline{V}(\xi) \leqslant \overline{\alpha}_2 \cdot \omega_2(\xi)^k \tag{A.17}$$

and, for all ξ where $\overline{V}$ is continuously differentiable,

$$\max_{f\in F(\xi)} \langle \nabla \overline{V}(\xi), f\rangle \leqslant -\overline{\mu}\,\overline{V}(\xi). \tag{A.18}$$

Moreover, suppose there exist $M > 0$, $\sigma \in \mathscr{K}_\infty$ such that

$$|\nabla V(\xi)| \leqslant M + \sigma(V(\xi)). \tag{A.19}$$

Then $\overline{V}$ is C^1 on $\mathbb{R}^n$.

Proof. From the definition of $\mathscr{A}$, the continuity of V and the upper bound in (A.12) or (A.13), it follows that $\mathscr{A}$ is a closed set and $0 \in \mathscr{A}$. For $\xi \in \mathbb{R}^n$, define $|\xi|_{\mathscr{A}} := \min\{|\xi - \zeta| : \zeta \in \mathscr{A}\}$.

Definition of $\overline{V}$ and homogeneity: Guided by Rosier (1992) we let $\rho : \mathbb{R} \to [0, 1]$ be a smooth, nondecreasing function that is zero on $(-\infty, \tau_1]$ and one on $[\tau_2, \infty)$ where $0 < \tau_1 < \tau_2$ and that has a strictly positive derivative on the interval $[\kappa_1(1), \kappa_2(2)]$, i.e., there exists $\overline{\rho} > 0$ such that

$$\rho'(s) \geqslant \overline{\rho} \quad \forall s \in [\kappa_1(1), \kappa_2(2)]. \tag{A.20}$$

Clearly this requires $\tau_1 < \kappa_1(1)$ and $\tau_2 > \kappa_2(2)$. Then we define, for each $\xi \in \mathbb{R}^n$,

$$\overline{V}(\xi) = \int_\circ^\infty \frac{1}{t^{k+1}} \rho(V(t\xi))\,\mathrm{d}t. \tag{A.21}$$

The function $\overline{V}$ is well-defined due to the way ρ is defined together with the observation that $0 \in \mathscr{A}$, the continuity of V, and the lower bound on $V(t\xi)$ for $\xi \in \mathbb{R}^n\backslash\mathscr{A}$ and t sufficiently large, via (A.12). As in Rosier (1992), a simple change of variables shows that $\overline{V}$ is homogeneous. In particular, $\overline{V}(\lambda\xi) = \lambda^k\overline{V}(\xi)$.

Upper and lower bounds on $\overline{V}$: First we note that for each positive integer j, each $\omega > 0$, and each function γ of the form $\gamma = \rho \circ \alpha$ where $\alpha \in \mathscr{K}_\infty$, the change of variable $r = t\omega^{1/j}$ can be used to show that

$$\int_\circ^\infty \frac{1}{t^{k+1}} \gamma(t^j\omega)\,\mathrm{d}t = \omega^{k/j} \int_\circ^\infty \frac{1}{r^{k+1}} \gamma(r^j)\,\mathrm{d}r. \tag{A.22}$$

For $i = 1, 2$, we define the positive real numbers

$$\overline{\kappa}_i = \int_\circ^\infty \frac{1}{r^{k+1}}\, \rho\circ\kappa_i(r)\,\mathrm{d}r,$$
$$\overline{\alpha}_i = \int_\circ^\infty \frac{1}{r^{k+1}}\, \rho\circ\alpha_i(r)\,\mathrm{d}r.$$

It then follows from (A.12), (A.13), (A.21) and (A.22) that

$$\overline{\kappa}_1 \cdot V_1(\xi)^k \leqslant \overline{V}(\xi) \leqslant \overline{\kappa}_2 \cdot V_1(\xi)^k, \tag{A.23}$$

$$\overline{\alpha}_1 \cdot \omega_1(\xi)^k \leqslant \overline{V}(\xi) \leqslant \overline{\alpha}_2 \cdot \omega_2(\xi)^k. \tag{A.24}$$

The bound (A.23) together with (A.12) and the definition of $\mathscr{A}$ imply that $\overline{V}(\xi) = 0$ if and only if $\xi \in \mathscr{A}$.

Smoothness of $\overline{V}$ on $\mathbb{R}^n\backslash\mathscr{A}$: We follow the argument in Rosier (1992). Suppose $\zeta \in \mathbb{R}^n\backslash\mathscr{A}$. Define

$$\underline{\theta} := \inf_{\{\xi : |\xi-\zeta| \leqslant 0.5|\zeta|_{\mathscr{A}}\}} V_1(\xi),$$
$$\overline{\theta} := \sup_{\{\xi : |\xi-\zeta| \leqslant 0.5|\zeta|_{\mathscr{A}}\}} V_1(\xi).$$

According to (A.12) with $\lambda=1$, $\underline{\theta} > 0$ and $\overline{\theta} < \infty$. Moreover, for each $\zeta_\circ$ on the boundary of $\mathscr{A}$,

$$\lim_{\zeta\to\zeta_\circ, \zeta\in\mathbb{R}^n\backslash\mathscr{A}} \underline{\theta} = \lim_{\zeta\to\zeta_\circ, \zeta\in\mathbb{R}^n\backslash\mathscr{A}} \overline{\theta} = 0. \tag{A.25}$$

Now define

$$\ell := \frac{\kappa_2^{-1}(\tau_1)}{\overline{\theta}}, \quad L := \frac{\kappa_1^{-1}(\tau_2)}{\underline{\theta}}. \tag{A.26}$$

Then, according to (A.12) and the definition of ρ, for all ξ such that $|\xi - \zeta| \leqslant 0.5|\zeta|_{\mathscr{A}}$ we have

$$\overline{V}(\xi) = \int_\ell^L \frac{1}{t^{k+1}}\, \rho(V(t\xi))\,\mathrm{d}t + \frac{1}{kL^k}. \tag{A.27}$$

[1] Let the Lipschitz constant of V_1 be K. In the construction of V_2 satisfying (179) and (180) in Teel and Praly (2000) by Lemma 16, further requirement can be made that V_2 have a Lipschitz constant $K + 1$. This is achieved by extending Lemma 16 to allow two sets of (F, α, ν). The original set ensures (179) and (180) while the second set $(\tilde{F}, \tilde{\alpha}, \tilde{\nu})$ ensures $|\nabla V_2(\xi)| \leqslant K + 1$ with $\tilde{\alpha} = K$, $\tilde{\nu} = 1$ and $\tilde{F}$ being the unit ball. Notice that $|\nabla V_1(\xi)| = \max_{|f|\leqslant 1}\langle \nabla V_1(\xi), f\rangle \leqslant K$. The function V_2 can be constructed to satisfy $|\nabla V_2(\xi)| = \max_{|f|\leqslant 1}\langle \nabla V_2(\xi), f\rangle \leqslant K+1$. When applying Lemma 17 with $V(\xi) = (\rho(V_2(\xi)))^2$, we have $|\nabla V(\xi)| \leqslant 2(K+1)\rho(\rho^{-1}(V^{1/2}(\xi)))\rho'(\rho^{-1}(V^{1/2}(\xi))) =: \sigma(V(\xi))$.

So the smoothness on $\mathbb{R}^n \backslash \mathscr{A}$ follows from the smoothness of ρ and V.

Continuity of $\overline{V}$ *on* $\mathbb{R}^n$: It follows from (A.25), (A.27), and the fact that $\overline{V}(\zeta) = 0$ if and only if $\zeta \in \mathscr{A}$ that, for each $\zeta \in \mathscr{A}$, $\lim_{\xi \to \zeta} \overline{V}(\xi) = 0 = \overline{V}(\zeta)$, i.e., $\overline{V}$ is continuous on $\mathscr{A}$. Combined with the smoothness of $\overline{V}$ on $\mathbb{R}^n \backslash \mathscr{A}$, $\overline{V}$ is continuous on $\mathbb{R}^n$.

C^1 *on* $\mathbb{R}^n$ *under* (A.19): We first show that for each $\zeta_\circ$ on the boundary of $\mathscr{A}$,

$$\lim_{\zeta \to \zeta_\circ, \zeta \in \mathbb{R}^n \backslash \mathscr{A}} \nabla \overline{V}(\zeta) = 0. \tag{A.28}$$

According to (A.27), we have

$$|\nabla \overline{V}(\zeta)| \leqslant \int_\ell^L \frac{1}{t^k} |\rho'(V(t\zeta)) \nabla V(t\zeta)| \, \mathrm{d}t.$$

We claim that $\rho'(V(t\zeta))\sigma(V(t\zeta))$ is uniformly bounded. Because $\rho'(s) = 0$ for all $s \geqslant \tau_2$, since $\rho(s) = 1$ for all $s \geqslant \tau_2$. Thus

$$\begin{aligned} &|\rho'(V(t\zeta)\sigma(V(t\zeta))| \\ &\quad \leqslant \sup_{s \in [0,\infty)} |\rho'(s)\sigma(s)|, \\ &\quad = \sup_{s \in [0,\tau 2]} |\rho'(s)\sigma(s)| \leqslant \sigma(\tau_2) \sup_{s \in [0,\tau 2]} |\rho'(s)|, \\ &\quad \leqslant \sigma(\tau_2) N \quad \text{for some } N > 0, \end{aligned}$$

and (A.28) follows from that $k > p = 1$ and $\ell, L \to \infty$ as $\zeta \to \zeta_\circ, \zeta \in \mathbb{R}^n \backslash \mathscr{A}$ (see (A.25) and (A.26)). In view of (A.28), it is sufficient to establish that for each $\zeta_\circ$ on the boundary of $\mathscr{A}$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each $\tau \in (0, \delta]$ and each unit vector u, we have

$$\overline{V}(\zeta_\circ + \tau u) = |\overline{V}(\zeta_\circ + \tau u) - \overline{V}(\zeta_\circ)| \leqslant \tau\varepsilon. \tag{A.29}$$

We only need to consider values of τ and u such that $\zeta_\circ + \tau u \in \mathbb{R}^n \backslash \mathscr{A}$ and $\tau \in (0, \delta]$. We define $\overline{\chi} := \max\{\chi \in [0, 1] : \zeta_\circ + \chi\tau u \in \mathscr{A}\}$. Since $\mathscr{A}$ is closed, $\zeta_\circ \in \mathscr{A}$, and $\zeta_\circ + \tau u \in \mathbb{R}^n \backslash \mathscr{A}$, the value $\overline{\chi}$ is well-defined and $\overline{\chi} < 1$. We note that $\overline{V}(\zeta_\circ + \overline{\chi}\tau u) = 0$ and $\zeta_\circ + \chi\tau u \in \mathbb{R}^n \backslash \mathscr{A}$ for all $\chi \in (\overline{\chi}, 1]$. Using $\overline{V}(\zeta_\circ + \overline{\chi}\tau u) = 0$ and the mean value theorem, for any $\chi \in (\overline{\chi}, 1]$ we can write, for some $s \in [0, 1]$

$$\begin{aligned} &\overline{V}(\zeta_\circ + \tau u) \\ &\quad = \overline{V}(\zeta_\circ + \tau u) - \overline{V}(\zeta_\circ + \chi\tau u) \\ &\qquad + \overline{V}(\zeta_\circ + \chi\tau u) - \overline{V}(\zeta_\circ + \overline{\chi}\tau u), \\ &\quad = \langle \nabla \overline{V}(\zeta_\circ + (s + (1-s)\chi)\tau u), (1-\chi)\tau u \rangle \\ &\qquad + \overline{V}(\zeta_\circ + \chi\tau u) - \overline{V}(\zeta_\circ + \overline{\chi}\tau u), \\ &\quad \leqslant h |\nabla \overline{V}(\zeta_\circ + (s + (1-s)\chi)\tau u)| \\ &\qquad + \overline{V}(\zeta_\circ + \chi\tau u) - \overline{V}(\zeta_\circ + \overline{\chi}\tau u), \\ &\quad \leqslant h\varepsilon + \overline{V}(\zeta_\circ + \chi\tau u) - \overline{V}(\zeta_\circ + \overline{\chi}\tau u). \end{aligned}$$

The result then follows by letting $\chi \to \overline{\chi}$ and using the continuity of $\overline{V}$.

Bounding $\max_{f \in F(\xi)} \langle \nabla \overline{V}(\xi), f \rangle$: We have

$$\begin{aligned} &\max_{f \in F(\xi)} \langle \nabla \overline{V}(\xi), f \rangle \\ &\quad = \max_{f \in F(\xi)} \int_\circ^\infty \frac{1}{t^{k+1}} \rho'(V(t\xi)) t \langle \nabla V(t\xi), f \rangle \, \mathrm{d}t, \\ &\quad \leqslant \int_\circ^\infty \frac{1}{t^{k+1}} \rho'(V(t\xi)) \max_{f \in F(t\xi)} \langle \nabla V(t\xi), f \rangle \, \mathrm{d}t, \\ &\quad \leqslant -\mu \int_\circ^\infty \frac{1}{t^{k+1}} \rho'(V(t\xi)) V(t\xi) \, \mathrm{d}t, \\ &\quad \leqslant -\mu \int_\circ^\infty \frac{1}{t^{k+1}} \rho'(V(t\xi)) \kappa_1(t V_1(\xi)) \, \mathrm{d}t. \end{aligned}$$

Now we make the change of variable $r = tV_1(\xi)$ and get

$$\begin{aligned} &\max_{f \in F(\xi)} \langle \nabla \overline{V}(\xi), f \rangle \\ &\quad \leqslant -\mu V_1(\xi)^k \int_\circ^\infty \frac{1}{r^{k+1}} \rho'(\phi(r, \xi)) \kappa_1(r) \, \mathrm{d}r, \end{aligned}$$

where, using (A.12),

$$\phi(r, \xi) = V(r/V_1(\xi)\xi) \in [\kappa_1(r), \kappa_2(r)]. \tag{A.30}$$

It follows from (A.20) and (A.23) that

$$\max_{f \in F(\xi)} \langle \nabla \overline{V}(\xi), f \rangle \leqslant -\frac{\mu\overline{\rho}\kappa_1(1)}{2^{k+1}} \frac{\overline{V}(\xi)}{\overline{\kappa}_2}. \qquad \square$$

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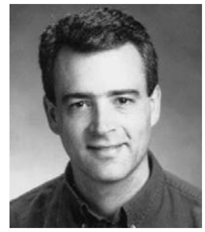

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