ELSEVIER

Systems & Control Letters 40 (2000) 59–69

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

On enlarging the basin of attraction for linear systems under saturated linear feedback ☆

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Received 9 May 1999; received in revised form 9 September 1999; accepted 18 December 1999

Abstract

We consider the problem of enlarging the basin of attraction for a linear system under saturated linear feedback. An LMI-based approach to this problem is developed. For discrete-time system, this approach is enhanced by the lifting technique, which leads to further enlargement of the basin of attraction. The low convergence rate inherent with the large invariant set (hence, the large basin of attraction) is prevented by the construction of a sequence of invariant ellipsoids nested within the large one obtained. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Basin of attraction; Invariant set; Saturation; Lifting technique

1. Introduction

The notion of invariant set has played an important role in the analysis and design of dynamical systems (see, e.g., [2,3,7,15] and the references therein). For a stable linear system, a simple and popular type of invariant set is the level set $\Omega(P) = \{x: x^{\mathrm{T}}Px \leqslant 1\}$, associated with the Lyapunov function $V(x) = x^{\mathrm{T}}Px$. Another type of well-studied invariant set is polyhedra (see, e.g., [1,7]).

For linear systems under saturated stabilizing linear state feedback, both the problem of estimating the basin of attraction (the largest invariant set) for a specific feedback gain matrix and that of searching for an appropriate feedback gain matrix to result in a large basin of attraction are of paramount importance and have attracted a great deal of attention from the control research community. Although these problems are still far from being completely solved, recent literature shows that they have been examined extensively from various aspects (see, e.g., [1,6,9,11] and the recent survey paper [2]). In particular, in [9], we investigated continuous-time linear systems under saturated stabilizing linear feedback. We showed that, if the system is of second order and has both open-loop poles in the open right half-plane, the boundary of the basin of attraction is the unique unstable limit cycle of the closed-loop system and can be easily obtained from its time-reversed system. Moreover, a family of gain matrices can be designed to obtain a basin of attraction that is arbitrarily close to the null controllable region, the largest possible basin of attraction under any bounded

☆ This work was supported in part by the US Office of Naval Research Young Investigator Program under grant N00014-99-1-0670.

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PII: S0167-6911(00)00002-5

controls. Though complete for the second-order continuous-time systems, these results cannot be extended in an obvious way to either general higher-order systems or discrete-time systems, which do not always have time-reversed systems.

For general linear systems under saturated linear stabilizing feedback, usually only an estimate of the basin of attraction can be obtained. A simple way of estimating the basin of attraction is the largest linear invariant ellipsoid associated with a quadratic Lyapunov function. By linear invariant ellipsoid, we mean an invariant ellipsoid that is completely within the linear region of the saturation function. This estimate, though often conservative, can be improved by an appropriate choice of the Lyapunov function and the feedback gain matrix. For example, in the case that the open-loop system is not exponentially unstable, the linear invariant ellipsoid can be made large enough to cover any a priori given (arbitrarily large) bounded set [11].

The objective of this paper is to present a systematic approach to the design of the feedback laws that result in large basin of attraction for general linear systems, both in continuous- and in discrete-time. More specifically, we will present an LMI-based approach to maximizing the linear invariant ellipsoid. Given a reference set X_R, the maximization is in the sense that the linear invariant ellipsoid contains the set αX_R with α being maximized. In the case that the open-loop system is not exponentially unstable, our design results in linear invariant ellipsoid that includes any a priori given (arbitrarily large) bounded set as a subset. This is the so-called semi-global stabilization [11–13]. For discrete-time systems, we show that this approach can be enhanced by the lifting technique, which leads to further enlargement of the basin of attraction. Finally, the low convergence rate inherent with the large basin of attraction can be prevented by constructing a sequence of invariant ellipsoids nested within the large one obtained and optimizing the convergence rate of the piecewise linear controller of Wredenhagen and Belanger [15].

The remainder of this paper is organized as follows. In Section 2, we present an LMI approach to the maximization of the linear invariant ellipsoid for both continuous- and discrete-time systems. In Section 3, we show how the lifting technique can be used to further enlarge the basin of attraction for discrete-time systems. In Section 4, we show how the closed-loop system convergence rate can be increased by switching the feedback gains between nested sequence of linear invariant ellipsoids. Two examples are included in Section 5 to demonstrate the effectiveness of the proposed design techniques. Concluding remarks are made in Section 6.

2. Maximizing the linear invariant ellipsoid

Consider the system

$$x(k+1) = Ax(k) + B\sigma(u(k)), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \tag{1}$$

where (A, B) is stabilizable. In this paper, we use $\sigma(\cdot)$ to denote a standard saturation function of appropriate dimensions. For example, in the above system, $\sigma : \mathbb{R}^m \to \mathbb{R}^m$, and $\sigma(u) = [\sigma(u_1), \sigma(u_2), \ldots, \sigma(u_m)]^T$, where $\sigma(u_i) = \operatorname{sgn}(u_i)\min\{1, |u_i|\}$. The closed-loop system under the feedback law $u = Fx$ is given by

$$x(k+1) = Ax(k) + B\sigma(Fx(k)). \tag{2}$$

Let f_i be the ith row of F. Denote the linear region of system (2) as

$$L(F) := \{x \in \mathbb{R}^n : |f_i x| \leqslant 1, i = 1, 2, \ldots, m\}.$$

Let F be such that $A + BF$ has all its eigenvalues inside the unit circle, then there exists $P > 0$ such that

$$(A + BF)^T P (A + BF) - P < 0.$$

Denote the Lyapunov level set as $\Omega(P) := \{x \in \mathbb{R}^n : x^T P x \leqslant 1\}$. If $\Omega(P) \subset L(F)$, then $\Omega(P)$ is an invariant set and we call it a linear invariant ellipsoid.

Our design objective is to choose F and P such that $\Omega(P) \subset L(F)$ is maximized in some sense. In the literature, e.g., [4], the largeness of a set is usually measured by its volume. Here we will take its shape

into consideration and will maximize $\Omega(P)$ with respect to some reference set. To make the problem well formulated, we introduce a reference set X_R. Let X_R be a bounded convex set, denote

$$\alpha X_R = \{\alpha x : x \in X_R\}.$$

The linear invariant ellipsoid $\Omega(P)$ is said to be maximized over F and P if α is maximized such that $\alpha X_R \subset \Omega(P) \subset L(F)$. In practice, the shape of X_R can be determined by the partial knowledge of the initial states. It can also be chosen according to the shape of the null controllable region, as identified in [9,8]. In this paper, we will consider two types of X_R:

- The polygon: $X_R = \mathrm{co}\{x_1, x_2, \ldots, x_l\}$ is the convex hull of a given set of states $x_1, x_2, \ldots, x_l \in \mathbb{R}^n$;
- The ellipsoid: $X_R = \{x \in \mathbb{R}^n : x^T R x \leqslant 1\},\ R > 0$.

Given system (1) and X_R, the optimization problem can be described as follows:

$$\begin{array}{ll} \sup\limits_{P>0,F} & \alpha \\ \text{s.t.} & \text{(a) } \alpha X_R \subset \Omega(P), \\ & \text{(b) } \Omega(P) \subset L(F), \\ & \text{(c) } (A+BF)^T P (A+BF) - P < 0. \end{array} \tag{3}$$

We also define the supremum of α as α^*.

If X_R is a polygon, then constraint (a) is equivalent to

$$\alpha^2 x_i^T P x_i \leqslant 1, \quad i = 1, 2, \ldots, l. \tag{4}$$

If X_R is an ellipsoid, then constraint (a) is equivalent to

$$\alpha^2 P \leqslant R. \tag{5}$$

On the other hand, constraint (b) is equivalent to

$$\min\{x^T P x : f_i x = 1\} \geqslant 1, \quad i = 1, 2, \ldots, m. \tag{6}$$

To see this, note that $\Omega(P) \subset L(F)$ if and only if all the hyperplanes $f_i x = \pm 1,\ i = 1, 2, \ldots, m$, lie completely outside of $\Omega(P) = \{x \in \mathbb{R}^n : x^T P x \leqslant 1\}$, i.e., at each point x on the hyperplanes $f_i x = \pm 1$, we have $x^T P x \geqslant 1$.

The left-hand side of (6) is a convex optimization problem and has a unique minimum. By using the Lagrange multiplier method, we obtain

$$\min\{x^T P x : f_i x = 1\} = (f_i P^{-1} f_i^T)^{-1}.$$

Consequently, constraint (b) is equivalent to

$$f_i P^{-1} f_i^T \leqslant 1, \quad i = 1, 2, \ldots, m. \tag{7}$$

Thus, if X_R is a polygon, then (3) can be rewritten as follows:

$$\begin{array}{ll} \sup\limits_{P>0,F} & \alpha \\ \text{s.t.} & \text{(a) } \alpha^2 x_i^T P x_i \leqslant 1, \quad i = 1, 2, \ldots, l, \\ & \text{(b) } f_i P^{-1} f_i^T \leqslant 1, \quad i = 1, 2, \ldots, m, \\ & \text{(c) } (A+BF)^T P (A+BF) - P < 0. \end{array} \tag{8}$$

If X_R is an ellipsoid, we just need to replace (a) with (5).

Constraints (a)–(c) are nonlinear and convex. The standard tool to transform such constraints into LMI is Schur complements: Suppose $Q > 0$, then the LMI

$$\begin{bmatrix} R & S \\ S^T & Q \end{bmatrix} \geqslant 0$$

if and only if $R-SQ^{-1}S^{\mathrm{T}}\geqslant 0$. Let $\gamma=1/\alpha^2$, $Q=P^{-1}$, $Y=FP^{-1}$, then we can transform (8) into the following LMI problem:

$$\begin{aligned}
&\inf_{Q,Y} \quad \gamma\\
&\text{s.t.} \quad \text{(a)} \begin{bmatrix} \gamma & x_i^{\mathrm{T}} \\ x_i & Q \end{bmatrix} \geqslant 0, \quad i=1,2,\ldots,l,\\
&\qquad \text{(b)} \begin{bmatrix} 1 & y_i \\ y_i^{\mathrm{T}} & Q \end{bmatrix} \geqslant 0, \quad i=1,2,\ldots,m,\\
&\qquad \text{(c)} \begin{bmatrix} Q & QA^{\mathrm{T}}+Y^{\mathrm{T}}B^{\mathrm{T}} \\ AQ+BY & Q \end{bmatrix} > 0,
\end{aligned} \tag{9}$$

where we have used y_i to denote the ith row of Y. For the case where X_{R} is an ellipsoid, we can simply replace (a) in (9) with $\gamma Q\geqslant R^{-1}$. We will denote the infimum of γ in the above optimization problem as $\gamma^*=1/(\alpha^*)^2$.

Remark 1. When $\gamma=\gamma^*$, there may not exist Q and Y that satisfy (a)–(c) in (9). In this case, we can choose $\gamma=\gamma^*+\varepsilon$ with ε arbitrarily small and solve for feasible solutions satisfying the constraints. For example, suppose A has all its eigenvalues on or inside the unit circle, then $\gamma^*=0$ [11] and no $Q>0$ satisfies (a) or (5). By taking γ arbitrarily small, we can make the set $\alpha X_{\mathrm{R}}\subset\Omega(P)\subset L(F)$ arbitrarily large, i.e., semi-global stabilization [12,13] can be achieved.

Remark 2. The above optimization method can be easily adapted to the continuous-time system by replacing (c) in (8) with $(A+BF)^{\mathrm{T}}P+P(A+BF)<0$ and (c) in (9) with $QA^{\mathrm{T}}+AQ+Y^{\mathrm{T}}B^{\mathrm{T}}+BY<0$.

3. Further enlargement of basin of attraction via lifting technique

The lifting technique has been used to improve the robust performance of discrete-time systems in [10] and to design semi-global stabilizing controller in [5]. Here we will show that it can also be efficiently used to enlarge the basin of attraction. Let $N\geqslant 1$ be a positive integer. Denoting

$$\bar{A}=A^N, \quad \bar{B}=[\,A^{N-1}B\ A^{N-2}B\ \cdots B\,]$$

and

$$\bar{x}(k)=x(kN), \quad \bar{u}(k)=\begin{bmatrix} u(kN) \\ u(kN+1) \\ \vdots \\ u(kN+N-1) \end{bmatrix},$$

we obtain the lifted N-step system

$$\bar{x}(k+1)=\bar{A}\bar{x}(k)+\bar{B}\sigma(\bar{u}(k)), \quad \bar{x}\in\mathbb{R}^n, \ \bar{u}\in\mathbb{R}^{Nm}. \tag{10}$$

Let $\bar{u}(k)=\bar{F}\bar{x}(k)$, $\bar{F}\in\mathbb{R}^{Nm\times n}$ be a stabilizing feedback. The closed-loop system is

$$\bar{x}(k+1)=\bar{A}\bar{x}(k)+\bar{B}\sigma(\bar{F}\bar{x}(k)). \tag{11}$$

Similar to the one-step case, the problem of maximizing the linear invariant ellipsoid can be described as

$$\begin{aligned}
&\sup_{P>0,\bar{F}} \quad \alpha\\
&\text{s.t.} \quad \text{(a) } \alpha X_{\mathrm{R}}\subset\Omega(P),\\
&\qquad \text{(b) } \Omega(P)\subset L(\bar{F}) \quad (\text{or } \bar{f}_i P^{-1}\bar{f}_i^{\mathrm{T}}\leqslant 1, \ i=1,2,\ldots,Nm),\\
&\qquad \text{(c) } (\bar{A}+\bar{B}\bar{F})^{\mathrm{T}}P(\bar{A}+\bar{B}\bar{F})-P<0
\end{aligned} \tag{12}$$

which can be solved by the LMI approach proposed in the previous section. Denoting the supremum of α as $\alpha^*(N)$, we have the following theorem that justifies the use of lifting technique.

Theorem 1. *For any integers* $p, N \geqslant 1$, $\alpha^*(p) \leqslant \alpha^*(pN)$.

Proof. *Case* 1: $p = 1$. Denote the set of feasible (α, P) satisfying constraints (a)–(c) as

$$\Phi(N) = \{(\alpha, P): \exists \bar{F} \text{ s.t. (a), (b) and (c) are satisfied}\}.$$

It suffices to show that $\Phi(1) \subset \Phi(N)$.

Suppose that $(\alpha, P) \in \Phi(1)$, then there exists an $F \in \mathbb{R}^{m\times n}$ such that

$$f_i P^{-1} f_i^{\mathrm{T}} \leqslant 1, \quad i = 1, 2, \ldots, m \tag{13}$$

and

$$(A + BF)^{\mathrm{T}} P (A + BF) - P < 0 \tag{14}$$

which is equivalent to

$$\begin{bmatrix} P & (A + BF)^{\mathrm{T}} \\ A + BF & P^{-1} \end{bmatrix} > 0$$

and to

$$(A + BF) P^{-1} (A + BF)^{\mathrm{T}} - P^{-1} < 0. \tag{15}$$

Let

$$\bar{F} = \begin{bmatrix} F \\ F(A + BF) \\ \vdots \\ F(A + BF)^{N-1} \end{bmatrix},$$

then

$$\bar{A} + \bar{B}\bar{F} = A^N + A^{N-1}BF + A^{N-2}BF(A + BF) + \cdots + BF(A + BF)^{N-1} = (A + BF)^N.$$

It then follows from (14) that

$$(\bar{A} + \bar{B}\bar{F})^{\mathrm{T}} P (\bar{A} + \bar{B}\bar{F}) = ((A + BF)^{\mathrm{T}})^N P (A + BF)^N < ((A + BF)^{\mathrm{T}})^{N-1} P (A + BF)^{N-1} < \cdots < P$$

which shows that P and $\bar{F}$ satisfy constraint (c).

Since $\bar{f}_j = f_i (A + BF)^q$ for some $i \leqslant m$, $q \leqslant N - 1$, we have

$$\bar{f}_j P^{-1} \bar{f}_j^{\mathrm{T}} = f_i (A + BF)^q P^{-1} ((A + BF)^{\mathrm{T}})^q f_i^{\mathrm{T}}.$$

It follows from (13) and (15) that

$$\bar{f}_j P^{-1} \bar{f}_j^{\mathrm{T}} \leqslant f_i (A + BF)^{q-1} P^{-1} ((A + BF)^{\mathrm{T}})^{q-1} f_i^{\mathrm{T}} \leqslant \cdots \leqslant f_i P^{-1} f_i^{\mathrm{T}} \leqslant 1$$

which shows that P and $\bar{F}$ also satisfy constraint (b). Hence $(\alpha, P) \in \Phi(N)$.

Case 2: $p > 1$. Let

$$\hat{A} = A^p, \quad \hat{B} = [A^{p-1}B \;\; A^{p-2}B \;\; \cdots B]$$

and

$$\bar{A} = A^{pN}, \quad \bar{B} = [A^{pN-1}B \;\; A^{pN-2}B \;\; \cdots B],$$

then

$$\bar{A} = \hat{A}^N, \quad \bar{B} = [\hat{A}^{N-1}\hat{B} \;\; \hat{A}^{N-2}\hat{B} \;\; \cdots \hat{B}].$$

Suppose we first lift system (1) with step p to get $\hat{x}(k) = x(kp)$,

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\sigma(\hat{u}(k)),$$

then lift the above system with step N to get $\bar{x}(k) = \hat{x}(kN) = x(kNp)$,

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\sigma(\bar{u}(k)).$$

Applying the result in Case 1, we immediately have $\alpha^*(p) \leqslant \alpha^*(pN)$. □

Remark 3. The equality $\alpha^*(p) = \alpha^*(pN)$ with $N > 1$ can occur in some special cases. For example, let $A = a > 1$, $B = 1$, and $X_R = [-1, 1]$. It can be verified that $\alpha^*(N) = 1/(a-1)$ for all $N \geqslant 1$.

From the above theorem, we see that

$$\alpha^*(1) \leqslant \alpha^*(2) \leqslant \alpha^*(4) \leqslant \alpha^*(8) \cdots,$$

$$\alpha^*(1) \leqslant \alpha^*(3) \leqslant \alpha^*(6) \leqslant \alpha^*(12) \cdots.$$

But $\alpha^*(N_1) \leqslant \alpha^*(N_2)$ does not necessarily hold for all $N_1 < N_2$. It should also be noted that, because of lifting, the resulting $\Omega(P)$ is not necessarily invariant for the original system at each step (see Fig. 2).

4. Performance improvement

Inherent with the achieved large basin of attraction is however the low convergence rate. To improve the convergence performance, we can use the idea of piecewise linear control [15] to design a set of nested ellipsoids $\Omega(P_M) \subset \Omega(P_{M-1}) \subset \cdots \subset \Omega(P_1) \subset \Omega(P_0)$, such that when the state enters an inner ellipsoid, the controller is switched to another feedback which makes this ellipsoid invariant with an increased convergence rate. Here we would like to explore the possibility of further increasing the overall convergence rate by maximizing the convergence rate in each of the nested ellipsoids. The nested invariant sets can be simply chosen by setting

$$P_i = \beta_i P_0, \quad 1 < \beta_1 < \beta_2 < \cdots < \beta_M.$$

The convergence rate inside $\Omega(P)$ under a feedback $u = Fx$ can be measured by a positive number $c < 1$ such that

$$(A + BF)^{\mathrm{T}} P (A + BF) - cP \leqslant 0. \tag{16}$$

We note that such F and c always exist for any $P = P_i$ since P_0, F_0 and $c = 1$ satisfy (16). Smaller c indicates faster convergence rate. Now let $P = \beta P_0$ be fixed, we need to design F such that c is minimized. The problem can be stated as follows. For a given β,

$$\begin{aligned} \min_{F} \quad & c \\ \text{s.t.} \quad & \text{(a) } f_i P_0^{-1} f_i^{\mathrm{T}} \leqslant \beta, \quad i = 1, 2, \ldots, m, \quad (\Omega(\beta P_0) \subset L(F)), \\ & \text{(b) } (A + BF)^{\mathrm{T}} P_0 (A + BF) - cP_0 \leqslant 0. \end{aligned} \tag{17}$$

We denote the minimum of c as $c^*(\beta)$. For the lifted N-step controller design, we can replace A, B and F with $\bar{A}, \bar{B}$ and $\bar{F}$, respectively. As in the previous sections, this optimization problem can also be put into the LMI framework. We note here that other performance criteria can also be formulated into a similar optimization problem (see, e.g., [14]).

Proposition 1. *$c^*(\beta)$ is decreased as β is increased. If B has full row rank, then there exists a β_0 such that $c^*(\beta) = 0$ for all $\beta > \beta_0$.*

Table 1
The increase of $\alpha^*(N)$

N	1	2	4	8	16	32
$\alpha^*(N)$	1.0650	1.0930	1.1896	1.4017	1.5164	1.5426

Proof. Constraint (b) in (17) is equivalent to

$$P_0^{-(1/2)}(A+BF)^{\mathrm{T}}P_0(A+BF)P_0^{-(1/2)} \leqslant cI.$$

Hence

$$\begin{aligned} c^*(\beta) = \ & \min \quad \lambda_{\max}[P_0^{-(1/2)}(A+BF)^{\mathrm{T}}P_0(A+BF)P_0^{-(1/2)}] \\ & \text{s.t.} \quad f_i P_0^{-1} f_i^{\mathrm{T}} \leqslant \beta, \quad i = 1,2,\ldots,m. \end{aligned}$$

As β is increased, the constraint $f_i P_0^{-1} f_i^{\mathrm{T}} \leqslant \beta$ becomes less restrictive, hence $c^*(\beta)$ will decrease.

If B has full row rank, then there exists F_1 such that $A + BF_1 = 0$. Let the ith row of F_1 be f_{1i}. Let $\beta_0 = \max\{f_{1i}P_0^{-1}f_{1i}^{\mathrm{T}} : i = 1,2,\ldots,m\}$, then for all $\beta > \beta_0$, $c^*(\beta) = 0$. □

Usually, for system (1), B does not have full row rank. For the lifted system (10), if (A,B) is controllable, then $\bar{B}$ will have full row rank when $N \geqslant n$. We will see in the examples that the lifting design method is efficient not only in enlarging the linear invariant ellipsoid, but also in increasing the convergence rate.

Now, let $1 < \beta_1 < \beta_2 < \cdots < \beta_M$ be a sequence of numbers. Denote the optimal solution of (17) corresponding to β_i as c_i^* and F_i^*. A switching feedback law can be designed as

$$u(k) = \begin{cases} F_0 x(k) & \text{if } x(k) \in \Omega(P_0)\backslash\Omega(\beta_1 P_0), \\ F_1^* x(k) & \text{if } x(k) \in \Omega(\beta_1 P_0)\backslash\Omega(\beta_2 P_0), \\ & \vdots \\ F_M^* x(k) & \text{if } x(k) \in \Omega(\beta_M P_0). \end{cases}$$

In the set $\Omega(\beta_i P_0) \setminus \Omega(\beta_{i+1}P_0)$, the convergence rate is c_i^*. As the state enters the inner set $\Omega(\beta_{i+1}P_0)\backslash \Omega(\beta_{i+2}P_0)$, the convergence rate is increased to c_{i+1}^*.

5. Examples

Example 1. Consider a second-order system in the form of (1) with

$$A = \begin{bmatrix} 0.9510 & 0.5408 \\ -0.2704 & 1.7622, \end{bmatrix}, \quad B = \begin{bmatrix} 0.0980 \\ 0.5408 \end{bmatrix}.$$

A has two unstable eigenvalues $\{1.2214,\ 1.4918\}$. The reference set $X_{\mathrm{R}} = \{x \in \mathbb{R}^2 : x^{\mathrm{T}}Rx \leqslant 1\}$, where

$$R = \begin{bmatrix} 1.2862 & -1.0310 \\ -1.0310 & 4.7138 \end{bmatrix},$$

is chosen according to the shape of the null controllable region. Table 1 shows the computational result for $\alpha^*(N), N = 1,2,4,8,16,32$.

Fig. 1 shows the effectiveness of the lifting design. The innermost curve is the boundary of $\alpha^*(1)X_{\mathrm{R}}$. For $N = 2,4,8,16,32$, the set $\alpha^*(N)X_{\mathrm{R}}$ grows bigger. The outermost curve is the boundary of the null controllable region obtained by the method proposed in [8].

We see that the increase from $\alpha^*(16)$ to $\alpha^*(32)$ is small so we take $N = 16$ as the lifting step. Now we design for the 16-step lifted system a set of nested invariant ellipsoids to accelerate the convergence rate. The optimal P_0 corresponding to $\alpha^*(16)$ is

$$P_0 = \begin{bmatrix} 0.5593 & -0.4483 \\ -0.4483 & 2.0497 \end{bmatrix} = 0.4348R = \frac{1}{(\alpha^*(16))^2}R.$$

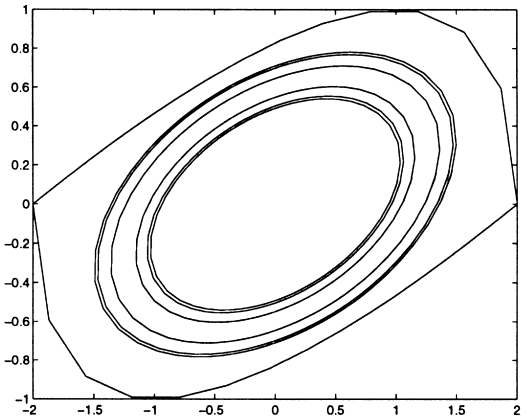

Fig. 1. The sets $\alpha^*(N)X_R$.

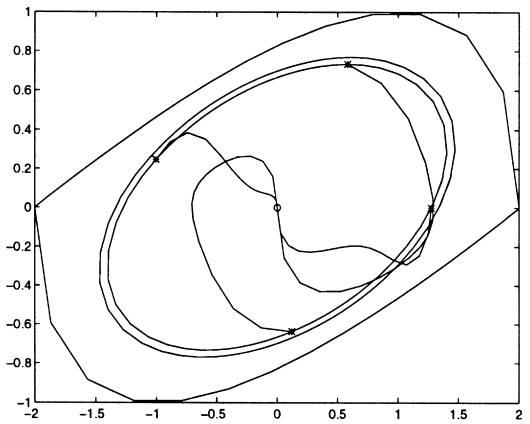

Fig. 2. The deadbeat control.

So $\Omega(P_0) = \alpha^*(16)X_R$. The optimal feedback is

$$\bar{F}_0^T = \begin{bmatrix} 0.3504 & 0.4636 & 0.6129 & 0.7324 & 0.7279 & 0.6374 & 0.5467 & 0.4777 \\ -1.4294 & -1.3917 & -1.2360 & -0.8490 & -0.2872 & 0.1679 & 0.4481 & 0.6167 \end{bmatrix}$$
$$\begin{bmatrix} 0.4279 & 0.3918 & 0.3653 & 0.3454 & 0.3302 & 0.3185 & 0.3094 & 0.3021 \\ 0.7225 & 0.7924 & 0.8406 & 0.8750 & 0.9003 & 0.9193 & 0.9337 & 0.9447 \end{bmatrix}.$$

The eigenvalues of $\bar{A} + \bar{B}\bar{F}$ are $0.2758 \pm j0.8814$, which indicates a low convergence rate.

We take $\beta = 1.04, 1.08, 1.1$, and get the corresponding $c^*(\beta)$ as $0.2650, 0.005, 0$. This shows that the convergence rate is accelerated. The fact that $c^*(1.1) = 0$ implies that all the states in $\Omega(1.1P_0)$ can be steered to the origin in 16 steps (counted for the original unlifted system) by a linear feedback controller. The deadbeat feedback matrix is

$$\bar{F}_0^T = \begin{bmatrix} 0.3115 & 0.4671 & 0.6665 & 0.7789 & 0.7236 & 0.6235 & 0.5433 & 0.4864 \\ -1.4990 & -1.4655 & -1.2419 & -0.6839 & -0.0779 & 0.3118 & 0.5342 & 0.6662 \end{bmatrix}$$
$$\begin{bmatrix} 0.4461 & 0.4170 & 0.3953 & 0.3788 & 0.3659 & 0.3555 & 0.3470 & 0.3398 \\ 0.7494 & 0.8047 & 0.8428 & 0.8697 & 0.8887 & 0.9020 & 0.9110 & 0.9164 \end{bmatrix}.$$

Fig. 2 illustrates this design result, where the innermost ellipsoid is $\Omega(1.1P_0)$ and the larger ellipsoid is $\Omega(P_0) = \alpha^*(16)X_R$. The outermost curve is the boundary of the null controllable region. The initial states

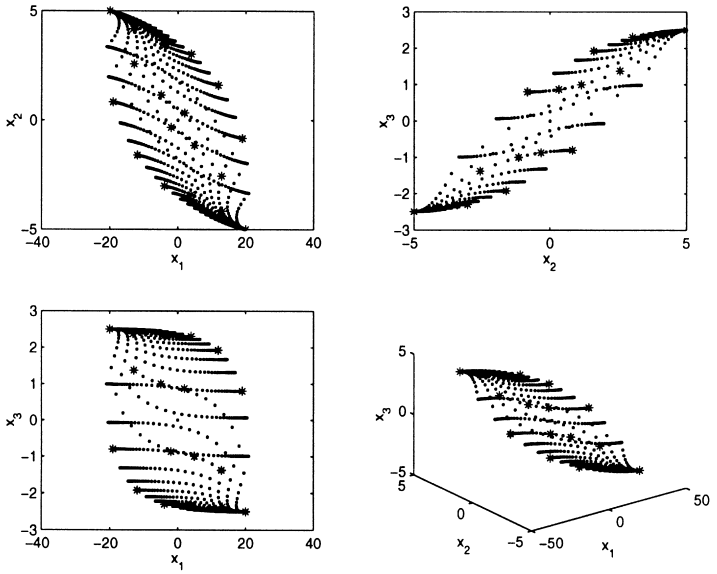

Fig. 3. The vertices of X_R.

Table 2
The increase of $\alpha^*(N)$

N	1	2	4	8	16	32
$\alpha^*(N)$	0.4274	0.4382	0.4593	0.4868	0.5564	0.6041

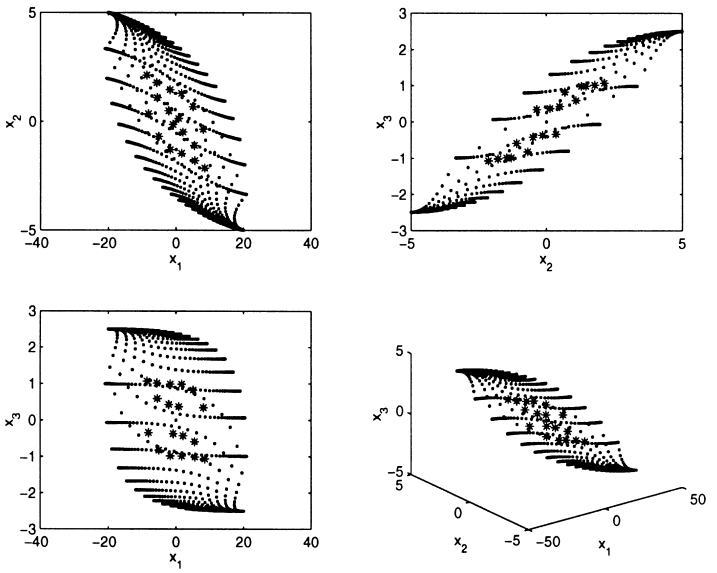

Fig. 4. $\alpha^*(1)X_R$ and the null controllable region.

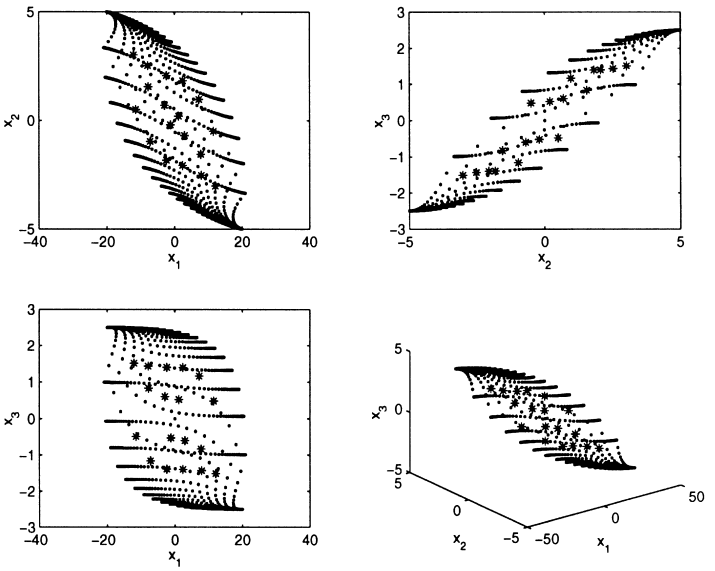

Fig. 5. $\alpha^*(32)X_R$ and the null controllable region.

on the boundary of $\Omega(1.1P_0)$ are marked with "*". They are all driven to the origin by the linear feedback control in 16 steps. Fig. 2 also shows some trajectories of the unlifted system under this 16-step control law.

Example 2. Consider a third-order system in the form of (1) with

$$A = \begin{bmatrix} 1.1972 & 1.0775 & 0 \\ 0 & 1.1972 & 0 \\ 0 & 0 & 1.4333 \end{bmatrix}, \quad B = \begin{bmatrix} 1.4431 \\ 0.9861 \\ 1.0833 \end{bmatrix}.$$

All of the eigenvalues of A are unstable. For the purpose of comparison, we choose 18 points on the boundary of the null controllable region as the vertices of X_R (see Fig. 3), where the vertices of X_R are marked with '*' and the vertices of the null controllable region are marked with '.'. Table 2 shows the computational result for $\alpha^*(N)$, $N = 1, 2, 4, 8, 16, 32$.

We also see that $\alpha^*(N)$ increases significantly as N is increased. (See Fig. 4 for the vertices of $\alpha^*(1)X_R$ and Fig. 5 for the vertices of $\alpha^*(32)X_R$, both in comparison with the null controllable region.)

6. Conclusions

We have proposed an LMI-based approach to the maximization of the linear invariant ellipsoid for linear systems under saturated linear feedback. The proposed approach applies to both continuous- and discrete-time systems. For discrete-time systems, we also showed that the lifting technique can be used to further enlarge the basin of attraction. Finally, the low convergence rate inherent with the large basin of attraction is increased by switching feedback laws between a sequence of nested invariant ellipsoids. Two examples are worked out to demonstrate the effectiveness of the proposed design techniques.

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