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Semi-global stabilization with guaranteed regional performance of linear systems subject to actuator saturation

Tingshu Hu[a, *, 1], Zongli Lin[a], Yacov Shamash[b]

[a] *Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22903, USA*
[b] *Department of Electrical Engineering, State University of New York, Stony Brook, NY 11794, USA*

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Abstract

For a linear system under a given saturated linear feedback, we propose feedback laws that achieve semi-global stabilization on the null controllable region while preserving the performance of the original feedback law in a fixed region. Here by semi-global stabilization on the null controllable region we mean the design of feedback laws that result in a domain of attraction that includes any a priori given compact subset of the null controllable region. Our design guarantees that the region on which the original performance is preserved would not shrink as the domain of attraction is enlarged by appropriately adjusting the feedback laws. Both continuous-time and discrete-time systems will be considered.

Keywords: Actuator saturation; Semi-global stabilization; Performance

1. Introduction

We revisit the problem of semi-globally stabilizing a linear system on its null controllable region with saturating actuators. The null controllable region, denoted as $\mathscr{C}$, is the set of states that can be steered to the origin of the state space in a finite time using saturating actuators. The problem of semi-global stabilization on the null controllable region is, for any a priori given set $\mathscr{X}$ that is in the interior of the null controllable region $\mathscr{C}$, to find a stabilizing feedback law $u = F_{\mathscr{X}}(x)$ such that the resulting domain of attraction includes $\mathscr{X}$ as a subset.

This problem has been well studied for systems that are so-called asymptotically null controllable with bounded controls (ANCBC).[2] In particular, it is established in [6,7] that, in both continuous-time and discrete-time, a linear ANCBC system is semi-globally asymptotically stabilizable on its null controllable region by saturated *linear* feedback. We note that in this case, the null controllable region is the entire state space. The key to the possibility of achieving semi-global stabilization on $\mathscr{C}$ by *linear*

* Corresponding author.
E-mail addresses: th7f@virginia.edu (T. Hu), yshamash@notes.sunysb.edu (Y. Shamash).
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[2] A continuous-time [resp. discrete-time] linear system is asymptotically null controllable with bounded controls if it is stabilizable in the usual linear sense and has all its open loop poles in the closed left-half plane [resp. the closed unit disc].

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feedback is that the open loop system is ANCBC. In general saturated linear feedback cannot achieve semi-global stabilization on $\mathscr{C}$ if the open loop system is not ANCBC, although there have been many attempts to enlarge the domain of attraction by appropriately choosing the linear feedback gains (see, for example, [3] and the references therein).

Our objective in this paper is to construct nonlinear feedback laws that semi-globally stabilize a linear system (not necessarily ANCBC) subject to actuator saturation. This problem has been addressed before. In particular, it was established in [4,5] that, in both continuous-time and discrete-time, a linear system with only two exponentially unstable modes can be semi-globally stabilized on its null controllable region by controllers that switch between two linear feedback laws. By defining these two linear feedback laws on an appropriately constructed invariant set, it is guaranteed that switching would occur at most once. In discrete-time, general systems have been considered in [1] and feedback laws were constructed that achieve semi-global stabilization on the null controllable region. More specifically, a sequence of polygons are constructed that approaches the null controllable region as the number of vertices increases. The vertices divide the polygons into cones. The state feedback laws are then constructed based on the controls that drive the vertices of a polygon to the origin according to which cone the state belongs to.

In this paper we will first consider a general linear system subject to actuator saturation,

$$x(k+1)=Ax(k)+B\sigma(u(k)), \quad x\in\mathbb{R}^n,\ u\in\mathbb{R}^m, \quad (1)$$

where σ is the standard saturation function. With a slight abuse of notation, we use the same symbol to denote both the vector saturation function and the scalar saturation function, i.e., if $v\in\mathbb{R}^m$, then $\sigma(v)=[\sigma(v_1),\sigma(v_2),\ldots,\sigma(v_m)]^{\mathrm{T}}$ and $\sigma(v_i)=\mathrm{sgn}(v_i)\min\{1,|v_i|\}$. We also assume that a feedback law $u=F_0(x)$ has been designed such that the resulting closed-loop system in the absence of the saturation function

$$x(k+1)=Ax(k)+BF_0(x(k)) \quad (2)$$

has the desired performance. We need to study the stability and performance of the actual system in the presence of actuator saturation,

$$x(k+1)=Ax(k)+B\sigma(F_0(x(k))). \quad (3)$$

Let $\mathscr{D}_0$ be an invariant set of the closed-loop system and be inside the linear region of the saturation function: $\{x\in\mathbb{R}^n\colon \|F_0(x)\|_\infty\leqslant 1\}$. For example, a linear state feedback law $u=F_0x$ could be constructed that places the closed-loop poles at certain desired locations and $\mathscr{D}_0$ can be a level set of the form $\{x\in\mathbb{R}^n\colon x^{\mathrm{T}}P_0x\leqslant 1\}$, where $P_0>0$ satisfies

$$(A+BF_0)^{\mathrm{T}}P_0(A+BF_0)-P_0<0. \quad (4)$$

Suppose that $\mathscr{D}_0$ is in the linear region, then it is an invariant set and within $\mathscr{D}_0$, the saturation function does not have an effect and hence the desired closed-loop performance is preserved.

The objective of this paper is to construct feedback laws that semi-globally stabilize the system (1) on its null controllable region and in the mean time preserve the desired closed-loop performance in the region $\mathscr{D}_0$. The structure of our feedback laws is completely different from that of [1]. Instead of resorting to the cones of the polygons which are not invariant sets, we design our controller by combining a sequence of feedback laws $u=F_i(x),\ i=0,1,\ldots,M$, in a way that the union of the invariant sets corresponding to each of the feedback laws is also an invariant set, which is shown to be in the domain of attraction. By appropriately selecting this sequence of feedback laws, the union of the invariant sets can then be made large enough to enclose any subset in the interior of the null controllable region. This idea was made feasible by the use of the lifting technique, which was used in [2] to provide an alternative proof of the results of [7] mentioned earlier. We will also extend the above results to continuous-time systems.

This paper is organized as follows. In Section 2 we propose a method for expanding the domain of attraction by switching between a finite sequence of feedback laws. This switching design is then used in Section 3 to show that the domain of attraction can be enlarged to include any subset in the interior of the null controllable region. Section 4 extends the results of Section 3 to continuous-time systems. An example is given in Section 5 to illustrate our design results. Finally, a brief concluding remark is made in Section 6.

2. Expansion of the domain of attraction

Let $u=F_i(x),\ i=0,1,\ldots,M$, be a finite sequence of stabilizing feedback laws. Among these feedback laws, $u=F_0(x)$ can be viewed as the feedback law that was originally designed to guarantee certain desired closed-loop performance in a given region and the remaining feedback laws have been introduced for the purpose of enlarging the domain of attraction while preserving the regional performance of the original feedback law $u = F_0(x)$.

For each $i = 0, 1, \ldots, M$, let $\mathscr{D}_i$ be an invariant set inside the domain of attraction of the equilibrium $x=0$ of the closed-loop system under feedback law $u = F_i(x)$,

$$x(k+1) = Ax(k) + B\sigma(F_i(x)). \tag{5}$$

Denote

$$\Omega_i = \bigcup_{j=0}^{i} \mathscr{D}_j, \quad i = 0, 1, \ldots, M.$$

Then, $\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_M$.

Theorem 1. *For each $i = 0, 1, \ldots, M$, Ω_i is an invariant set inside the domain of attraction of $x = 0$ of the closed-loop system*

$$x(k+1) = Ax(k) + B\sigma(G_i(x(k))), \tag{6}$$

where

$$G_i(x) := \begin{cases} F_0(x), & \text{if } x \in \Omega_0, \\ F_1(x), & \text{if } x \in \Omega_1 \backslash \Omega_0, \\ \vdots & \vdots \\ F_i(x), & \text{if } x \in \Omega_i \backslash \Omega_{i-1}. \end{cases} \tag{7}$$

Here we note that, for each $i=1, 2, \ldots, M$, $\Omega_i \backslash \Omega_{i-1} = \mathscr{D}_i \backslash \bigcup_{j=0}^{i-1} \mathscr{D}_j$.

Proof. We prove the theorem by induction. The statement is trivially true for $i = 0$. Suppose it is true for $i \geqslant 0$, we need to show that it is also true for $i + 1$. Let us write $G_{i+1}(x)$ as

$$G_{i+1}(x) = \begin{cases} G_i(x), & \text{if } x \in \Omega_i, \\ F_{i+1}(x), & \text{if } x \in \Omega_{i+1} \backslash \Omega_i. \end{cases} \tag{8}$$

If $x(0) \in \Omega_i$, then under the feedback $u = G_i(x)$, $x(k) \in \Omega_i$ for all k and $\lim_{k\to\infty} x(k) = 0$. If $x(0) \in \Omega_{i+1} \backslash \Omega_i = \mathscr{D}_{i+1} \backslash \Omega_i$, since $\mathscr{D}_{i+1}$ is inside the domain of attraction under the feedback $u = F_{i+1}(x)$ and Ω_i is a neighborhood of the origin, $x(k)$ will enter Ω_i at some $k_1 < \infty$. After that, the control is switched to $u = G_i(x)$ and by the foregoing argument, we also have $\lim_{k\to\infty} x(k) = 0$. This shows that Ω_{i+1} is inside the domain of attraction.

It is also easy to see that Ω_{i+1} is an invariant set under $u = G_{i+1}(x)$. □

From (7), we see that if $x \in \Omega_0 = \mathscr{D}_0$, then $u = F_0(x)$ is in effect and hence the pre-designed performance is guaranteed on $\mathscr{D}_0$.

For later use in Section 4, it can be verified in a similar way that Theorem 1 is also true for a continuous-time system $\dot{x}(t) = f(x, u, t)$, in particular,

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)), \tag{9}$$

with a set of stabilizing feedback laws $u = F_i(x)$, $i = 0, 1, \ldots, M$. In the context of continuous-time systems, the existence and uniqueness of the solution of the closed-loop system equation is guaranteed by the fact that Ω_i's are invariant sets and nested to each other. In other words, a trajectory starting from a set Ω_i will remain in it. Once it enters a smaller set Ω_j, $j < i$, it will again remain in it.

3. A semi-global stabilization strategy

In this section, we utilize the lifting technique to design a sequence of ellipsoids that cover any prescribed compact subset of the null controllable region. Each ellipsoid is invariant and in the domain of attraction for the lifted closed-loop system under an appropriately chosen linear feedback. This, by Theorem 1, would achieve semi-global stabilization for the lifted system, and hence for the original system.

The null controllable region of (1) at step K, denoted as $\mathscr{C}(K)$, is the set of states that can be steered to the origin in K steps [5]. We see that $x_0 \in \mathscr{C}(K)$ if and only if there exists a control $u(\cdot)$, $\|u(k)\|_\infty \leqslant 1$, $k = 0, 1, \ldots, K-1$, such that

$$A^K x_0 + \sum_{i=0}^{K-1} A^{K-i-1} B u(i) = 0. \tag{10}$$

The null controllable region, denoted as $\mathscr{C}$, is the set of states that can be steered to the origin in a finite number of steps. Clearly, $\mathscr{C} = \bigcup_{K\geqslant 0} \mathscr{C}(K)$ and it can be shown by standard analysis that any compact subset of $\mathscr{C}$ is a subset of $\mathscr{C}(K)$ for some K. For simplicity, we assume that the pair (A, B) is controllable and A is nonsingular. Then there is an integer $n_0 \leqslant n$ such that, for all $K \geqslant n_0$, $\mathscr{C}(K)$ contains the origin in its interior and is bounded.

For a positive integer L, the lifted system of (1) with step L is given by

$$x_L(k+1) = A_L x_L(k) + B_L \sigma(u_L(k)), \tag{11}$$

where

$$x_L(k) = x(kL), \quad u_L(k) = \begin{bmatrix} u(kL) \\ u(kL+1) \\ u(kL+L-1) \end{bmatrix},$$

$$A_L = A^L, B_L = [A^{L-1}B A^{L-2}B \quad \cdots \quad AB \quad B]. \quad (12)$$

We have more flexibility in the design of a system by using the lifting technique because it allows us to see further the effect of a control law and to consider the combined effect of the control action at several steps.

For a feedback matrix $F \in \mathbb{R}^{mL\times n}$, denote the unsaturated region (linear region) of the closed-loop system

$$x_L(k+1) = A_L x_L(k) + B_L \sigma(F x_L(k)) \quad (13)$$

as

$$\mathscr{L}(F) := \{x \in \mathbb{R}^n\colon |f_j x| \leqslant 1, j = 1, 2, \ldots, mL\},$$

where f_j is the jth row of F. If $L \geqslant n_0$, then there exists an F such that $A_L + B_L F = 0$. For such an F, there is a corresponding $\mathscr{L}(F)$ and for all $x_{L0} = x_0 \in \mathscr{L}(F)$, $A_L x_0 + B_L \sigma(F x_0) = (A_L + B_L F)x_0 = 0$. Hence $\mathscr{L}(F)$ is an invariant set of the lifted system (13) and is inside the domain of attraction.

For a positive definite matrix $P \in \mathbb{R}^{n\times n}$, denote

$$\mathscr{E}(P) = \{x \in \mathbb{R}^n\colon x^{\mathrm{T}} P x \leqslant 1\}.$$

Suppose that $\mathscr{E}(P) \subset \mathscr{L}(F)$, then under the feedback law $u_L = F x_L$, $\mathscr{E}(P)$ is also an invariant set inside the domain of attraction. Here we are interested in the ellipsoids because they can be generalized to the Lyapunov level sets for the case $A_L + B_L F \neq 0$. We will show that any compact subset of the null controllable region can be covered by the union of a finite set of such ellipsoids.

Lemma 1. *Given an integer $L \geqslant n_0$ and a positive number $\beta < 1$, there exists a family of $F_i \in \mathbb{R}^{mL\times n}$, $i = 1, 2, \ldots, M$, with corresponding positive definite matrices P_i's, such that $A_L + B_L F_i = 0$,*

$$\mathscr{E}(P_i) \subset \mathscr{L}(F_i), \quad i = 1, 2, \ldots, M,$$

and

$$\beta\mathscr{C}(L) \subset \bigcup_{i=1}^{M} \mathscr{E}(P_i),$$

where $\beta\mathscr{C}(L) = \{\beta x\colon x \in \mathscr{C}(L)\}$.

Proof. Let $\partial(\beta\mathscr{C}(L))$ be the boundary of $\beta\mathscr{C}(L)$. Firstly, we show that, there exists an $\varepsilon > 0$ such that, for any $x_1 \in \partial(\beta\mathscr{C}(L))$, there exist an $F \in \mathbb{R}^{mL\times n}$ and $P > 0$ that satisfy

$$A_L + B_L F = 0 \quad \text{and} \quad \mathscr{B}(x_1, \varepsilon) \subset \mathscr{E}(P) \subset \mathscr{L}(F),$$

where $\mathscr{B}(x_1, \varepsilon) = \{x \in \mathbb{R}^n\colon \|x - x_1\|_2 \leqslant \varepsilon\}$.

Let e_ℓ be the unit vector in $\mathbb{R}^n$ whose ℓth element is 1 and other elements are zeros. For simplicity, assume $x_1 = \gamma e_1$, otherwise we can use a unitary transformation $x \to Vx$, $V^{\mathrm{T}} V = I$, to satisfy this. Note that a unitary transformation is equivalent to rotating the state space and does not change the shapes of $\mathscr{B}(x_1, \varepsilon)$, $\mathscr{E}(P)$ and $\mathscr{C}(L)$.

Since $x_1 = \gamma e_1 \in \beta\mathscr{C}(L)$, it follows from (10) and (12) that there exists a u_{L1}, $\|u_{L1}\|_\infty \leqslant \beta$, such that

$$A_L \gamma e_1 + B_L u_{L1} = 0. \quad (14)$$

Define

$$\mu = \frac{\max\{\|x\|_2\colon x \in \partial\mathscr{C}(L)\}}{\min\{\|x\|_2\colon x \in \partial\mathscr{C}(L)\}}.$$

Since $L \geqslant n_0$, $\mathscr{C}(L)$ includes the origin in its interior and $\mu < \infty$. It follows that $\gamma e_\ell \in \mu\beta\mathscr{C}(L)$ for all $\ell \geqslant 2$. Therefore, for each $\ell \geqslant 2$, there exists a $u_{L\ell}$, $\|u_{L\ell}\| \leqslant \mu\beta$, such that

$$A_L \gamma e_\ell + B_L u_{L\ell} = 0. \quad (15)$$

Let $F = \{f_{j\ell}\}$ be chosen as

$$F = \frac{1}{\gamma}[u_{L1} \quad u_{L2} \quad \cdots \quad u_{Ln}],$$

then $|f_{j1}| \leqslant \beta/\gamma$ and $|f_{j\ell}| \leqslant \mu\beta/\gamma$ for $\ell = 2, \ldots, n$ and $j = 1, 2, \ldots, mL$. From (14) and (15), we have

$$(A_L + B_L F)e_\ell = A_L e_\ell + \frac{1}{\gamma} B_L u_{L\ell} = 0, \quad \ell = 1, 2, \ldots, n.$$

This shows that $A_L + B_L F = 0$.

Let

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 I_{n-1} \end{bmatrix},$$

where

$$p_1 = \frac{1}{\gamma^2}\left(\frac{2\beta}{\beta+1}\right)^2,$$

$$p_2 = (n-1)\left(\frac{\beta\mu}{\gamma}\right)^2\left(1 - \frac{(\beta+1)^2}{4}\right)^{-1}.$$

Let $\gamma_{\min} = \min\{\|x\|\colon x \in \partial(\beta\mathscr{C}(L))\}$ and

$$\varepsilon = \left(1 - \frac{2\beta}{\beta+1}\right)\gamma_{\min}\left\{\max\left(\frac{2\beta}{\beta+1}, \frac{2\sqrt{n-1)}\beta\mu}{\sqrt{4-(\beta+1)^2}}\right)\right\}^{-1}.$$

Then $\|P^{1/2}\|_2\varepsilon \leqslant 1-2\beta/(\beta+1)$. Note that ε is independent of γ and a particular x_1.

We also have

$$f_jP^{-1}f_j^{\mathrm{T}} = \frac{1}{p_1}f_{j1}^2 + \frac{1}{p_2}\sum_{\ell=2}^{n} f_{j\ell}^2$$
$$\leqslant \frac{1}{p_1}\left(\frac{\beta}{\gamma}\right)^2 + \frac{n-1}{p_2}\left(\frac{\beta\mu}{\gamma}\right)^2 = 1, \quad (16)$$

which implies that $\mathscr{E}(P) \subset \mathscr{L}(F)$. To see this, we verify that, for any $x \in \mathscr{E}(P)$,

$$|f_jx| = |f_jP^{-(1/2)}P^{1/2}x|$$
$$\leqslant (f_jP^{-1}f_j^{\mathrm{T}})^{1/2}(x^{\mathrm{T}}Px)^{1/2} \leqslant 1.$$

For $x \in \mathscr{B}(x_1,\varepsilon)$, we have

$$\|P^{1/2}x\|_2 \leqslant \|P^{1/2}x_1\|_2 + \|P^{1/2}(x-x_1)\|_2$$
$$\leqslant \frac{2\beta}{\beta+1} + \|P^{1/2}\|_2\varepsilon \leqslant 1.$$

This shows that $x^{\mathrm{T}}Px \leqslant 1$ and hence $\mathscr{B}(x_1,\varepsilon) \subset \mathscr{E}(P) \subset \mathscr{L}(F)$.

Because $\partial(\beta\mathscr{C}(L))$ is a compact set, there exists a finite set of $x_i \in \partial(\beta\mathscr{C}(L))$, $i=1,2,\ldots,M$, such that $\partial(\beta\mathscr{C}(L)) \subset \bigcup_{i=1}^{M}\mathscr{B}(x_i,\varepsilon)$. By the foregoing proof, we know that for each $x_i \in \partial(\beta\mathscr{C}(L))$, there exist an F_i and P_i such that $A_L + B_LF_i = 0$ and

$$\mathscr{B}(x_i,\varepsilon) \subset \mathscr{E}(P_i) \subset \mathscr{L}(F_i).$$

Hence,

$$\partial(\beta\mathscr{C}(L)) \subset \bigcup_{i=1}^{M}\mathscr{E}(P_i).$$

It then follows that

$$\beta\mathscr{C}(L) \subset \bigcup_{i=1}^{M}\mathscr{E}(P_i).$$

To see this, for any $x \in \beta\mathscr{C}(L)$, let y be an intersection point of $\partial(\beta\mathscr{C}(L))$ with the straight line passing through the origin and x. Hence, $y \in \mathscr{E}(P_{i_0})$ for some i_0. Since $\mathscr{E}(P_{i_0})$ is convex and contains the origin, $x \in \mathscr{E}(P_{i_0})$. □

Remark 1. We would like to point out that, the family of F_i's may contain repeated members with different P_i's. This is the case, for example, when the system has a single input ($m=1$) and the lifting step L is the same as n, the dimension of the state space. In this case, we have only a unique $F_i = -B_L^{-1}A_L$ with $\mathscr{C}(L) \subset \mathscr{L}(F_i)$.

Lemma 1 shows that $\beta\mathscr{C}(L)$ can be covered by a finite number of ellipsoids and within each ellipsoid there is a corresponding linear feedback law such that the state of (11) will be steered to the origin the next step, or equivalently, the state of (1) will be steered to the origin in L steps. Because β can be made arbitrarily close to 1 and L can be made arbitrarily large, any compact subset of $\mathscr{C}$ can be covered by a family of such ellipsoids. It should be noted that as β gets closer to 1, ε will decrease and we need more ellipsoids to cover $\beta\mathscr{C}(L)$, although the determination of these ellipsoids could be technically involved for higher order systems. Also, in the above development, we need to lift the system by L steps to cover $\beta\mathscr{C}(L)$. Actually, the lifting step can be reduced if we replace the dead-beat condition $A_L + B_LF = 0$ with a less restrictive one:

$$(A_L + B_LF)^{\mathrm{T}}P(A_L + B_LF) - cP \leqslant 0,$$

where $c \in (0,1)$ specifies the requirement of the convergence rate. A direct consequence of Lemma 1 is

Theorem 2. *Given any compact subset X_0 of $\mathscr{C}$ and a number $c \in (0,1)$, there exist an $L \geqslant 1$ and a family of $F_i \in \mathbb{R}^{mL\times n}$, $i=1,2,\ldots,M$, with corresponding positive definite matrices P_i's, such that*

$$(A_L + B_LF_i)^{\mathrm{T}}P_i(A_L + B_LF_i) - cP_i \leqslant 0, \quad (17)$$

$$\mathscr{E}(P_i) \subset \mathscr{L}(F_i), \quad i=1,2,\ldots,M, \quad (18)$$

and

$$X_0 \subset \bigcup_{i=1}^{M}\mathscr{E}(P_i). \quad (19)$$

Because of (17) and (18), $\mathscr{E}(P_i)$ is an invariant set inside the domain of attraction for the closed-loop system

$$x_L(k+1) = A_Lx_L(k) + B_L\sigma(F_ix_L(k)).$$

By Theorem 1, we can use a switching controller to make $\bigcup_{i=1}^{M}\mathscr{E}(P_i)$ inside the domain of attraction. Once the state enters the region $\mathscr{E}(P_0)$, the controller switches to the feedback law

$$u_L(k) = \bar{F}_0(x_L(k)) = \begin{bmatrix} F_0(x_L(k)) \\ F_0(x(kL+1)) \\ \vdots \\ F_0(x(kL+L-1)) \end{bmatrix}, \quad (20)$$

where the variables $x(kL+i)$, $i=1,2,\dots,L-1$, can be recursively computed from the state $x_L(k)$ as follows:

$$x(kL+1)=Ax_L(k)+BF_0(x_L(k)),$$

$$\begin{aligned}x(kL+2)&=Ax(kL+1)+BF_0(x(kL+1)\\&=A(Ax_L(k)+BF_0(x_L(k)))\\&\quad+BF_0(Ax_L(k)+BF_0(x_L(k)))\end{aligned}$$

$$\vdots$$

$$x(kL+i+1)=Ax(kL+i)+BF_0(x(kL+i)).$$

Since feedback law (20) corresponds to $u=F_0(x)$ in the original time index, under which $\mathscr{E}(P_0)$ is an invariant set, $\mathscr{E}(P_0)$ is also an invariant set under feedback law (20) in the lifted time index and the desired performance in this region is preserved.

We also observe that, due to the switching and lifting that are involved in the construction of feedback laws, our final semi-globally stabilizing feedback laws, when implemented in the original system (1), are nonlinear and periodic in time.

4. Continuous-time systems

In this section, we consider the continuous-time counterpart of the system (1)

$$\dot{x}(t)=Ax(t)+B\sigma(u(t)),\quad x\in R^n,\ u\in\mathbb{R}^m. \tag{21}$$

The null controllable region at time T, denoted as $\mathscr{C}(T)$, is the set of states that can be steered to the origin in time T by a measurable control input u. The null controllable region, denoted as $\mathscr{C}$, is $\bigcup_{T\geqslant 0}\mathscr{C}(T)$.

Let $h>0$ be the lifting period. We are now interested in controlling the state of (21) at times kh, $k=1,2,\dots$. Denote $x_h(k)=x(kh)$ and $u_h(k,\tau)=u(kh+\tau)$. Let $A_h=\mathrm{e}^{Ah}$; then the lifted system is

$$x_h(k+1)=A_hx_h(k)+\int_0^h \mathrm{e}^{A(h-\tau)}B\sigma(u_h(k,\tau))\,\mathrm{d}\tau. \tag{22}$$

Denote the set of $m\times n$ dimensional measurable functions defined on $[0,h)$ as $\mathscr{F}^{m\times n}$. With a matrix function $F\in\mathscr{F}^{m\times n}$, let the feedback control be $u_h(k,\tau)=F(\tau)x_h(k)$. Then the closed-loop system is

$$x_h(k+1)=A_hx_h(k)+\int_0^h \mathrm{e}^{A(h-\tau)}B\sigma(F(\tau)x_h(k))\,\mathrm{d}\tau. \tag{23}$$

The unsaturated region of the feedback law is then given by,

$$\mathscr{L}(F):=\{x\in\mathbb{R}^n:\ |f_j(\tau)x|\leqslant 1,\quad j=1,2,\dots,m,\ \tau\in[0,h)\},$$

where $f_j\in\mathscr{F}^{1\times n}$ is the jth row of F. If $x_h(k)\in\mathscr{L}(F)$, then $\sigma(F(\tau)x_h(k))=F(\tau)x_h(k)$ and

$$x_h(k+1)=\left(A_h+\int_0^h \mathrm{e}^{A(h-\tau)}BF(\tau)\,\mathrm{d}\tau\right)x_h(k). \tag{24}$$

The feedback $u_h(k,\tau)=F(\tau)x_h(k)$ is stabilizing if there exists $P>0$ such that

$$\left(A_h+\int_0^h \mathrm{e}^{A(h-\tau)}BF(\tau)\,\mathrm{d}\tau\right)^{\mathrm{T}}\times P\left(A_h+\int_0^h \mathrm{e}^{A(h-\tau)}BF(\tau)\,\mathrm{d}\tau\right)-P\leqslant 0.$$

Note that P can be scaled such that $\mathscr{E}(P)\subset\mathscr{L}(F)$. In this case, $\mathscr{E}(P)$ is an invariant set inside the domain of attraction for the system (23). Since for all $x_h(k)\in\mathscr{E}(P)$, the control is linear in $x_h(k)$, so, when $x_h(k)$ tends to the origin, the control $u_h(k,\tau)=F(\tau)x_h(k)$ will gets smaller and hence the state of the original system (21) between $t=kh$ and $t=(k+1)h$ will stay close to $x_h(k)$. Similar to the discrete-time case, we have the following lemma.

Lemma 2. *Given $h>0$ and a positive number $\beta<1$, there exists a family of $F_i\in\mathscr{F}^{m\times n}$, $i=1,2,\dots,M$, with corresponding positive definite matrices P_i's, such that*

$$A_h+\int_0^h \mathrm{e}^{A(h-\tau)}BF_i(\tau)\,\mathrm{d}\tau=0,$$

$$\mathscr{E}(P_i)\subset\mathscr{L}(F_i),\quad i=1,2,\dots,M,$$

and

$$\beta\mathscr{C}(h)\subset\bigcup_{i=1}^M\mathscr{E}(P_i).$$

Proof. The idea of the proof is the same as that of Lemma 1. Here we just show how to construct ε, F and P for a given $x_1\in\partial(\beta\mathscr{C}(h))$. We also assume that $x_1=\gamma e_1$. Since $\gamma e_1\in\partial(\beta\mathscr{C}(h))$, there exists a $u_1\in\mathscr{F}^{m\times 1}$, $\|u_1(\tau)\|_\infty\leqslant\beta$ for all $\tau\in[0,h)$, such that

$$A_h\gamma e_1+\int_0^h \mathrm{e}^{A(h-\tau)}Bu_1(\tau)\,\mathrm{d}\tau=0,$$

and for $\ell \geqslant 2$, there exists a $u_\ell \in \mathscr{F}^{m\times 1}$, $\|u_\ell(\tau)\|_\infty \leqslant \beta\mu$ for all $\tau \in [0, h)$, such that

$$A_h\gamma e_\ell + \int_0^h \mathrm{e}^{A(h-\tau)}Bu_\ell(\tau)\,\mathrm{d}\tau = 0.$$

Let $F = 1/\gamma[u_1\ u_2\ \cdots\ u_n]$, and P, ε be the same as those in the proof of Lemma 1, the remaining part of the proof will be the same as that of Lemma 1 except that (16) is replaced with

$$f_j(\tau)P^{-1}f_j^{\mathrm{T}}(\tau) \leqslant 1, \quad \forall\tau \in [0, h),\ j = 1, 2, \ldots, m. \quad \Box$$

The following is the counterpart of Theorem 2 for the discrete-time system (1).

Theorem 3. *Given any compact subset X_0 of $\mathscr{C}$ and a number $c \in (0, 1)$, there exist an $h > 0$ and a family of $F_i \in \mathscr{F}^{m\times n}$, $i = 1, 2, \ldots, M$, with corresponding positive definite matrices P_i's, such that*

$$\left(A_h + \int_0^h \mathrm{e}^{A(h-\tau)}BF_i(\tau)\,\mathrm{d}\tau\right)^{\mathrm{T}}$$

$$\times P_i\left(A_h + \int_0^h \mathrm{e}^{A(h-\tau)}BF_i(\tau)\,\mathrm{d}\tau\right) - cP_i \leqslant 0,$$

$$\mathscr{E}(P_i) \subset \mathscr{L}(F_i), \quad i = 1, 2, \ldots, M,$$

and

$$X_0 \subset \bigcup_{i=1}^{M} \mathscr{E}(P_i).$$

Again, by Theorem 1, we can use a switching controller to make $\bigcup_{i=0}^{M} \mathscr{E}(P_i)$ inside the domain of attraction and hence semi-global stabilization can be achieved. Moreover, once the state enters the region $\mathscr{E}(P_0)$, the controller switches to the feedback law $u = F_0(x)$ and hence the desired performance in this region is preserved.

5. Example

Consider the system (1) with

$$A = \begin{bmatrix} 0.8876 & -0.5555 \\ 0.5555 & 1.5542 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1124 \\ 0.5555 \end{bmatrix}.$$

The matrix A is exponentially unstable with a pair of eigenvalues $1.2209 \pm j0.4444$. The LQR controller corresponding to the cost function $J = \sum(x(k)^{\mathrm{T}}Qx(k) + u(k)^{\mathrm{T}}Ru(k))$, with $Q = I$, $R = 1$ is

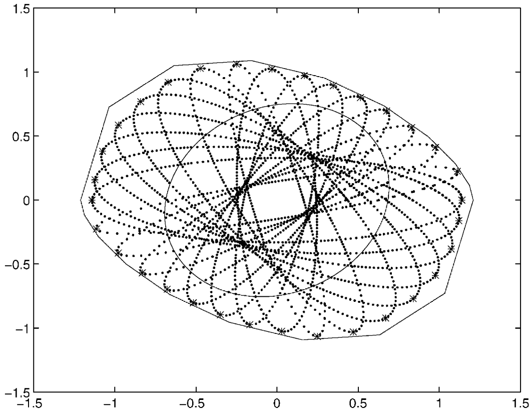

Fig. 1. The union of the invariant ellipsoids.

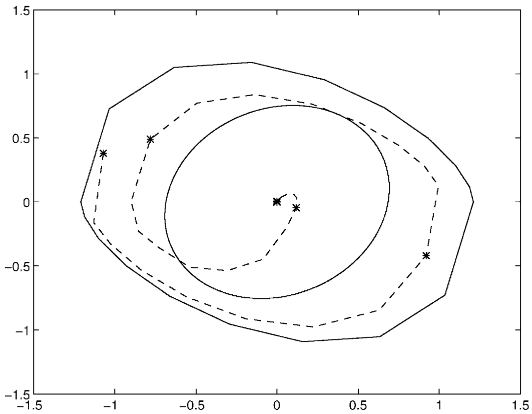

Fig. 2. A trajectory under the multiple switching control.

$u = F_0(x) = [\,-0.2630\ \ -2.1501\,]x$. Let $\mathscr{D}_0$ be obtained as

$$\mathscr{E}(P_0), \quad P_0 = \begin{bmatrix} 2.1367 & -0.2761 \\ -0.2761 & 1.7968 \end{bmatrix},$$

see the ellipsoid enclosed by the solid curve in Fig. 1.

To enlarge the domain of attraction, we take a lifting step of 8 and obtain 16 invariant ellipsoids with corresponding feedback controllers, see the ellipsoids enclosed by the dotted curves in Fig. 1. Each invariant ellipsoid is optimal with respect to certain x_i in the sense that it contains αx_i with $|\alpha|$ maximized, see the points marked with '*'. This is computed by using the LMI method [3]. The outermost curve in Fig. 1 is the boundary of the null controllable region $\mathscr{C}$. We see that the union of the ellipsoids covers a large portion of $\mathscr{C}$.

Figs. 2–4 show some simulation results of the closed-loop system under the multiple switching

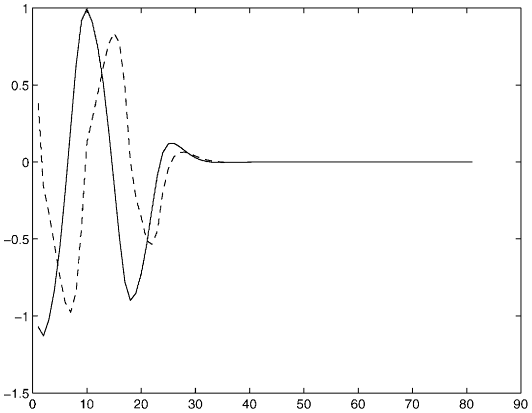

Fig. 3. Simulation: x_1, '-'; x_2, '–'.

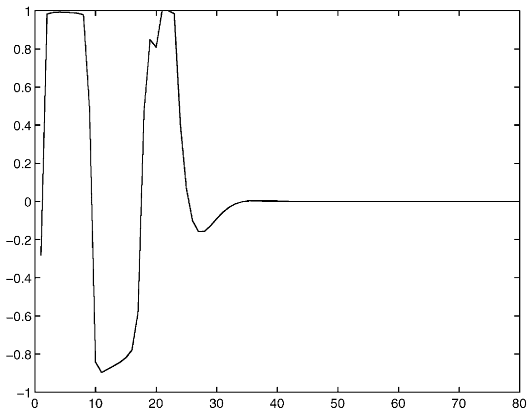

Fig. 4. Simulation: the control.

controls. The initial state is very close to the boundary of $\mathscr{C}$. In Fig. 2 the dashed trajectory is that of the unlifted system (1) under the switched control, and the trajectory of the lifted system is marked with '$*$'. Figs. 3 and 4 are the state and control of the original unlifted system.

6. Conclusions

In this paper, we have proposed a control design method for linear systems that are subject to actuator saturation. This design method applies to general (possibly exponentially unstable) systems in either continuous-time or discrete-time. The resulting feedback laws expand the domain of attraction achieved by an a priori designed feedback law to include any bounded set in the interior of the null controllable region, while preserving the desired performance of the original feedback law in a fixed region.

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