Available online at www.sciencedirect.com
SCIENCE DIRECT®
Systems & Control Letters 49 (2003) 389–399
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

On the tightness of a recent set invariance condition under actuator saturation☆

Tingshu Hu*, Zongli Lin

Department of Electrical and Computer Engineering, University of Virginia, P.O. Box 400743, Charlottesville, VA 22904-4743, USA

Received 20 January 2002; received in revised form 5 January 2003; accepted 26 January 2003

Abstract

A sufficient condition for an ellipsoid to be invariant was obtained recently and an LMI approach was developed to find the largest ellipsoid satisfying the condition. This condition was later shown to be necessary for the single input case. This paper is dedicated to the multi-input case. We will examine the conservatism of the condition for multi-input systems. Our investigation is conducted by studying the optimal solution to a related LMI problem. A criterion is presented to determine when the condition is not conservative and when the largest invariant ellipsoid has been obtained by using the LMI method.

Keywords: Invariant ellipsoid; Contractive invariance; Actuator saturation

1. Introduction

In this paper, we will continue to study the set invariance property for a linear system under saturated feedback

$$\dot{x} = Ax + B\,\mathrm{sat}(Fx). \tag{1}$$

This problem has been studied in our recent works [7,9]. We have restricted our attention to invariant ellipsoids since quadratic Lyapunov functions are the most popular and the results can be put into simple and compact forms, which make analysis and design easily implementable. Moreover, our method can be applied to different ellipsoids. The union of multiple invariant ellipsoids forms a new invariant set. In the literature, invariant ellipsoids have been used to estimate the domain of attraction for nonlinear systems (see e.g., [1–3,5,10,11,13] and the references therein). The problem of estimating the domain of attraction for (1) has been a focus of study in recent years.

For a matrix $F \in \mathbb{R}^{m\times n}$, denote the ith row of F as f_i and define

$$\mathscr{L}(F) := \{x \in \mathbb{R}^n \colon |f_i x| \leqslant 1, i = 1, 2, \ldots, m\}.$$

☆ This work was supported in part by the US Office of Naval Research Young Investigator Program under grant N00014-99-1-0670.
* Corresponding author.
E-mail addresses: th7f@virginia.edu, zl5y@virginia.edu (T. Hu).

0167-6911/03/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0167-6911(03)00117-8

If F is a feedback gain matrix, then $\mathscr{L}(F)$ is the region where the feedback control $u = \mathrm{sat}(Fx)$ is linear in x. We call $\mathscr{L}(F)$ the linear region of the saturated feedback sat(Fx), or simply, the linear region of saturation.

Let $P \in \mathbb{R}^{n\times n}$ be a positive-definite matrix. For a positive number ρ, denote

$$\mathscr{E}(P,\rho) = \{x \in \mathbb{R}^n\colon\ x^{\mathrm{T}}Px \leqslant \rho\}.$$

If

$$(A+BF)^{\mathrm{T}}P + P(A+BF) < 0$$

and $\mathscr{E}(P,\rho) \subset \mathscr{L}(F)$, then $\mathscr{E}(P,\rho)$ is an invariant ellipsoid inside the domain of attraction. The largest of these $\mathscr{E}(P,\rho)$'s was used as an estimate of the domain of attraction in the earlier literature (see e.g., [14]). This saturation avoidance estimation method, though simple, could be very conservative. Recent efforts have been made to extend the ellipsoid beyond the linear region $\mathscr{L}(F)$ (see, e.g., [5,10]). In particular, simple and general methods have been derived by applying the absolute stability analysis tools, such as the circle and Popov criteria, where the saturation is treated as a locally sector bounded nonlinearity.

More recently, we developed a new sufficient condition for an ellipsoid to be invariant in [9] (see also [7]). It was shown that this condition is less conservative than the existing conditions resulting from the circle criterion or the vertex analysis. The most important feature of this new condition is that it can be expressed as LMIs in terms of all varying parameters and hence can easily be used for controller synthesis. A recent discovery makes this condition even more attractive. In [8], we showed that for the single input case, this condition is also necessary, thus the largest ellipsoid obtained with the LMI approach is actually the largest one. With this new finding, we are tempted to try to understand if this condition is also necessary for the multi-input case. Our investigation identifies cases where this condition is not conservative for multiple input systems.

Notation: In this paper, we use $\mathrm{sat}:\mathbb{R}^m \to \mathbb{R}^m$ to denote the standard saturation function of appropriate dimensions. For $u \in \mathbb{R}^m$, the ith component of sat(u) is $\mathrm{sign}(u_i)\min\{1,|u_i|\}$. The infinity norm of u is denoted as $|u|_\infty$. For an $m\times n$ matrix H, we use h_i to denote its ith row and for an $n\times m$ matrix B, we use b_i to denote its ith column.

2. A sufficient condition for set invariance

Consider the linear system subject to input saturation,

$$\dot{x} = Ax + Bu, \quad x\in\mathbb{R}^n, \quad u\in\mathbb{R}^m, \quad |u|_\infty \leqslant 1. \tag{2}$$

Under a saturated linear feedback $u = \mathrm{sat}(Fx)$, the closed-loop system is

$$\dot{x} = Ax + B\,\mathrm{sat}(Fx). \tag{3}$$

Given a positive definite matrix P, let $V(x) = x^{\mathrm{T}}Px$. The ellipsoid $\mathscr{E}(P,\rho)$ is said to be (contractively) invariant if

$$\dot{V}(x) = 2x^{\mathrm{T}}P(Ax + B\,\mathrm{sat}(Fx)) \leqslant (<)0$$

for all $x \in \mathscr{E}(P,\rho)\setminus\{0\}$. Clearly, if $\mathscr{E}(P,\rho)$ is contractively invariant, then it is inside the domain of attraction. We need more notation to present the sufficient condition for $\mathscr{E}(P,\rho)$ to be contractively invariant.

Let $\mathscr{D}$ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathscr{D}$. Suppose that each element of $\mathscr{D}$ is labeled as D_i, $i = 1,2,\ldots,2^m$. Then, $\mathscr{D} = \{D_i\colon\ i\in[1,2^m]\}$. Denote $D_i^- = I - D_i$. Clearly, D_i^- is also an element of $\mathscr{D}$ if $D_i \in \mathscr{D}$. Given two matrices $F,H \in \mathbb{R}^{m\times n}$,

$$\{D_iF + D_i^-H\colon\ i\in[1,2^m]\}$$

is the set of matrices formed by choosing some rows from F and the rest from H.

Theorem 1 (Hu et al. [9] and Hu and Lin [7]). *Given an ellipsoid $\mathscr{E}(P,\rho)$, if there exists an $H \in \mathbf{R}^{m\times n}$ such that*

$$(A + B(D_iF + D_i^- H))^{\mathrm{T}} P + P(A + B(D_iF + D_i^- H)) < 0, \quad \forall i \in [1, 2^m] \tag{4}$$

and $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$, then $\mathscr{E}(P,\rho)$ is contractively invariant under the feedback $u = \mathrm{sat}(Fx)$.

This theorem was originally obtained in [9] and an intuitive explanation was later provided in [7]. It is actually motivated by the simple fact: for two vectors $u, v \in \mathbb{R}^m$, if $|v|_\infty \leqslant 1$, then

$$\mathrm{sat}(u) \in \mathrm{co}\{D_i u + D_i^- v\colon\ i \in [1, 2^m]\},$$

where "co" stands for the convex hull. In the context of Theorem 1, under the condition $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$, we have $|Hx|_\infty \leqslant 1$ for all $x \in \mathscr{E}(P,\rho)$. Hence

$$\mathrm{sat}(Fx) \in \mathrm{co}\{D_iFx + D_i^- Hx\colon\ i \in [1, 2^m]\} \quad \forall x \in \mathscr{E}(P,\rho)$$

and the theorem readily follows from Lyapunov stability analysis.

It is clear that the condition

$$(A + BF)^{\mathrm{T}} P + P(A + BF) < 0 \tag{5}$$

which corresponds to (4) for $D_i = I$, is necessary for the contractive invariance of $\mathscr{E}(P,\rho)$ for any $\rho > 0$. Hence we assume throughout the paper that (5) is satisfied.

For the single input case ($m = 1$), we have shown in [8] that the condition in Theorem 1 is also necessary.

Theorem 2. *Assume that $m = 1$. Given an ellipsoid $\mathscr{E}(P,\rho)$, suppose that (5) is satisfied. Then $\mathscr{E}(P,\rho)$ is contractively invariant under the feedback $u = \mathrm{sat}(Fx)$ if and only if there exists an $H \in \mathbb{R}^{1\times n}$ such that*

$$(A + BH)^{\mathrm{T}} P + P(A + BH) < 0 \tag{6}$$

and $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$.

Here we note that when $m = 1$, there are only two inequalities in (4), namely, (5) and (6).

Remark 1. Here we note that there is some resemblence between Theorem 2 and certain results in [4]. Using this paper's notation, [4] implies that if $F \in \mathbb{R}^{m\times n}$ can be decomposed as $F = H - kBB^{\mathrm{T}}P$ for some $k > 0$ and H satisfying (6) and $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$, then $\mathscr{E}(P,\rho)$ is contractively invariant under the feedback $u = \mathrm{sat}(Fx)$. Theorem 2, although applies only to single input systems, imposes no relation between F and H, and the condition is both necessary and sufficient.

3. Tightness of the set invariance condition

In this section, we will study the necessity of the condition in Theorem 1 for multi-input case. First, let us consider the largest ellipsoid that satisfies the condition. Let $P > 0$ be given, define

$$\rho^* := \sup_H \rho \tag{7}$$

$$\text{s.t.} \quad \text{(a)}\ (A + B(D_iF + D_i^- H))^{\mathrm{T}} P + P(A + B(D_iF + D_i^- H)) < 0, \quad i \in [1, 2^m],$$

$$\text{(b)}\ \rho h_j P^{-1} h_j^{\mathrm{T}} \leqslant 1, \quad j \in [1, m].$$

Recall from [6,7] that constraint (b) is equivalent to $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$. Consider a closely related optimization problem

$$\rho_1^* := \sup_H \rho \tag{8}$$

$$\text{s.t.} \quad \text{(a) } (A + B(D_iF + D_i^- H))^{\mathrm{T}} P + P(A + B(D_iF + D_i^- H)) \leqslant 0, \quad i \in [1, 2^m],$$

$$\text{(b) } \rho h_j P^{-1} h_j^{\mathrm{T}} \leqslant 1, \quad j \in [1, m].$$

The only difference between (7) and (8) is the "<" in (7a) and "$\leqslant$" in (8a). This means that the feasible domain of (8) is the closure of that of (7). Under the assumption of (5), the feasible domain of (8) has an interior point. It follows that $\rho^* = \rho_1^*$.

From Theorem 1, we know that if $\rho < \rho^*$, then $\mathscr{E}(P,\rho)$ is contractively invariant. If we can conclude that $\rho \geqslant \rho^*$ implies that $\mathscr{E}(P,\rho)$ is not contractively invariant, then the condition in Theorem 1 is not conservative. However, this is not always the case.

Define

$$\rho_c := \sup\{\rho > 0 \text{: } \mathscr{E}(P,\rho) \text{ is contractively invariant}\}.$$

It is clear that $\rho_c \geqslant \rho^*$. We will show that $\rho_c = \rho^*$ is conditional.

Also, let $(\rho^*, \bar{H})$ be an optimal solution to (8). There must be a j such that $\rho^* \bar{h}_j P^{-1} \bar{h}_j^{\mathrm{T}} = 1$ and there must be an i such that

$$\lambda_{\max}((A + B(D_iF + D_i^- \bar{H}))^{\mathrm{T}} P + P(A + B(D_iF + D_i^- \bar{H}))) = 0. \tag{9}$$

Theorem 3. *Let $(\rho^*, \bar{H})$ be an optimal solution to (8). Suppose that*

(1) *there is only one j such that $\rho^* \bar{h}_j P^{-1} \bar{h}_j^{\mathrm{T}} = 1$ (i.e., the boundary of $\mathscr{E}(P,\rho^*)$ only touches one pair of the hyperplanes $\bar{h}_j x = \pm 1$);*
(2) *there is only one i satisfying (9), the matrix in (9) has a single eigenvalue at 0 and the only nonzero element in D_i^- is the jth diagonal one ($D_i^- \bar{H}$ chooses only $\bar{h}_j$).*

Let $x_0 = \rho^* P^{-1} \bar{h}_j^{\mathrm{T}}$, then x_0 is the unique intersection of $\mathscr{E}(P,\rho^*)$ with $\bar{h}_j x = 1$. If
(3) *$|f_k x_0| \leqslant 1$ for all $k \neq j$,*
then $\rho^* = \rho_c$.

Proof. To simplify the proof, we would like to make some special assumptions. First, we assume that $\rho^* = 1$. Otherwise we can scale the matrix P to make it so. We also assume that $j = 1$. Otherwise we can permute the columns of the B matrix.

Next, we assume some special forms of the matrices P and H. Suppose that we have a state transformation, $x \to z = Tx$. Then the invariance of $\mathscr{E}(P,\rho)$ for the x system is equivalent to the invariance of $\mathscr{E}(\bar{P},\rho)$ for the z system

$$\dot{z} = \hat{A}z + \hat{B}\,\mathrm{sat}(\hat{F}z)$$

with

$$\hat{P} = (T^{-1})^{\mathrm{T}} P T^{-1}, \quad \hat{A} = TAT^{-1}, \quad \hat{B} = TB, \quad \hat{F} = FT^{-1}.$$

Also let $\hat{\bar{H}} = \bar{H}T^{-1}$. With the above transformation, the three conditions (1)–(3) in Theorem 3 remain unchanged. In view of the above arguments, we can assume that $P = I$ and $\bar{h}_1 = [1 \ 0 \ \cdots \ 0]$. Otherwise, we can use a unitary transformation ($T^{\mathrm{T}}T = I$) to make it so, noting that $\rho^* \bar{h}_1 P^{-1} \bar{h}_1^{\mathrm{T}} = 1$ and $\rho^* = 1$.

In summary, we assume that $\rho^*=1, P=I$ and

$$\bar{h}_1=[1 \quad 0 \quad \cdots \quad 0].$$

In this case, we have $x_0=[1\ 0\cdots\ 0]^{\mathrm{T}}$, $x_0^{\mathrm{T}}Px_0=1=\rho^*$ and $\bar{h}_1x_0=1$, i.e., x_0 is the unique intersection of the ellipsoid $\mathscr{E}(P,\rho^*)$ with the hyperplane $\bar{h}_1x=1$.

Denote

$$Q(h_1)=\left(A+B\begin{bmatrix}h_1\\ f_2\\ \vdots\\ f_m\end{bmatrix}\right)^{\mathrm{T}}P+P\left(A+B\begin{bmatrix}h_1\\ f_2\\ \vdots\\ f_m\end{bmatrix}\right)$$

$$=\left(A+\sum_{j=2}^{m}b_jf_j\right)^{\mathrm{T}}P+P\left(A+\sum_{j=2}^{m}b_jf_j\right)+(b_1h_1)^{\mathrm{T}}P+Pb_1h_1.$$

Under condition (2), we have $\lambda_{\max}\ (Q(\bar{h}_1))=0$. Let the unit eigenvector of $Q(\bar{h}_1)$ corresponding to the zero eigenvalue be v, i.e., $v^{\mathrm{T}}v=1$ and $Q(\bar{h}_1)v=0$. In the remaining part of the proof, we will first show that $Q(\bar{h}_1)x_0=0$, and under condition (3), we will further have $\dot{V}(x_0)=0$. This leads to $\rho_c\leqslant\rho^*$ and hence $\rho_c=\rho^*$.

Step 1: $Q(\bar{h}_1)x_0=0$.

Under condition (1), the only equality in (8b) is

$$\rho^*\bar{h}_1P^{-1}\bar{h}_1^{\mathrm{T}}=1$$

and all the others have strict "<". Under condition (2), we have

$$(A+B(D_iF+D_i^-\bar{H}))^{\mathrm{T}}P+P(A+B(D_iF+D_i^-\bar{H}))<0$$

for all $i\in[1,2^m]$, except

$$Q(\bar{h}_1)\leqslant 0.$$

Hence, there is a neighborhood of $\bar{H}$, $\mathscr{N}(\bar{H})$, where all the conditions in (8a) and (8b) are satisfied except $\rho^*h_1P^{-1}h_1^{\mathrm{T}}\leqslant 1$ and $Q(h_1)\leqslant 0$. Let us restrict $H\in\mathscr{N}(\bar{H})$, we must also have

$$\rho^*=\sup_{H\in\mathscr{N}(\bar{H})}\rho \quad \text{s.t.} \quad (8a),\ (8b) \tag{10}$$

with an optimizer $\bar{H}$. Since for any $H\in\mathscr{N}(\bar{H})$, all the conditions in (8a) and (8b) are satisfied except $\rho^*h_1P^{-1}h_1^{\mathrm{T}}\leqslant 1$ and $Q(h_1)\leqslant 0$, the pair $(\rho^*,\bar{H})$ must also be the optimal solution to

$$\sup_{H\in\mathscr{N}(\bar{H})}\rho \tag{11}$$

$$\text{s.t.}\quad \rho h_1P^{-1}h_1\leqslant 1, \tag{12}$$

$$Q(h_1)\leqslant 0. \tag{13}$$

The optimality of the solution means that if we scale down h_1 from $\bar{h}_1$ to $k\bar{h}_1,\ k<1$, condition (13) must be violated, otherwise a ρ greater than ρ^* would be allowed for condition (12). Observing the special form of $\bar{h}_1$, we have

$$v^{\mathrm{T}} \left. \frac{\partial Q(h_1)}{\partial h_{11}} v \right|_{h_1=\bar{h}_1} < 0 \tag{14}$$

which means that a decrease of h_{11} from 1 would increase the largest eigenvalue of $Q(h_1)$. This relation is obtained by using eigenvalue perturbation theory (e.g., see [12]). Recalling that $P=I$, we can rewrite (14) as

$$v^{\mathrm{T}} \begin{bmatrix} 2b_{11} & b_{12} & \cdots & b_{1n} \\ b_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ b_{1n} & 0 & \cdots & 0 \end{bmatrix} v = 2v_1 \sum_{i=1}^{n} b_{1i} v_i < 0. \tag{15}$$

Also, the optimality of ρ^* means that if we change other elements of h_1 over the sphere surface $\rho^* h_1 P^{-1} h_1^{\mathrm{T}} = h_1 h_1^{\mathrm{T}} = 1$, we will have $\lambda_{\max}(Q(h_1)) \geqslant 0$, otherwise a ρ greater than ρ^* would be allowed. All the h_1 in the surface $h_1 h_1^{\mathrm{T}} = 1$ and in a neighborhood of $\bar{h}_1$ can be expressed as

$$h_1 = [\sqrt{1-(d_2^2+d_3^2+\cdots+d_n^2)}\ d_2\ \cdots\ d_n], \quad d_2^2+d_3^2+\cdots+d_n^2 < 1.$$

Since $\lambda_{\max}(Q(h_1))$ has a local minimum at $h_1 = \bar{h}_1$, by eigenvalue perturbation theory, we must have

$$v^{\mathrm{T}} \left. \frac{\partial Q(h_1)}{\partial d_j} v \right|_{h_1=\bar{h}_1} = 0, \quad j \in [2,n]. \tag{16}$$

With the special form of P, we can rewrite (16) as

$$v^{\mathrm{T}} \begin{bmatrix} 0 & \cdots & b_{11} & \cdots & 0 \\ \vdots & & \vdots & & \\ b_{11} & \cdots & 2b_{1j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \\ 0 & \cdots & b_{1n} & \cdots & 0 \end{bmatrix} v = 2v_j \sum_{i=1}^{n} b_{1i} v_i = 2v_j \sum_{i=1}^{n} b_{1i} v_i = 0. \tag{17}$$

Relations (15) and (17) jointly show that $v_1 \neq 0$ and $v_j = 0, j \in [2,n]$, and hence v is aligned with x_0. Therefore

$$Q(\bar{h}_1) x_0 = 0.$$

Step 2: $\dot{V}(x_0) = 0$.

From (15) and $v_j = 0,\ j \in [2,n]$, we have $b_{11} < 0$. From condition (3) of the theorem, $|f_k x_0| \leqslant 1$ for all $k \in [2,m]$. It follows that

$$\begin{aligned} \dot{V}(x_0) &= x_0^{\mathrm{T}} \left(A^{\mathrm{T}} P + PA + \sum_{k=2}^{m} (f_k^{\mathrm{T}} b_k^{\mathrm{T}} P + P b_k f_k) \right) x_0 + 2x_0^{\mathrm{T}} P b_1 \mathrm{sat}(f_1 x_0) \\ &= x_0^{\mathrm{T}} \left(A^{\mathrm{T}} P + PA + \sum_{k=2}^{m} (f_k^{\mathrm{T}} b_k^{\mathrm{T}} P + P b_k f_k) \right) x_0 + 2b_{11} \mathrm{sat}(f_1 x_0). \end{aligned}$$

Since $Q(\bar{h}_1)x_0 = 0$, we have

$$\begin{aligned}0 &= x_0^{\mathrm{T}} Q(\bar{h}_1)x_0\\ &= x_0^{\mathrm{T}}\left(A^{\mathrm{T}}P + PA + \sum_{k=2}^{m}(f_k^{\mathrm{T}}b_k^{\mathrm{T}}P + Pb_kf_k) + \bar{h}_1^{\mathrm{T}}b_1^{\mathrm{T}}P + Pb_1\bar{h}_1\right)x_0\\ &= x_0^{\mathrm{T}}\left(A^{\mathrm{T}}P + PA + \sum_{k=2}^{m}(f_k^{\mathrm{T}}b_k^{\mathrm{T}}P + Pb_kf_k)\right)x_0 + 2b_{11},\end{aligned} \tag{18}$$

noting that $\bar{h}_1x_0 = 1$.

On the other hand, from (5)

$$x_0^{\mathrm{T}}((A+BF)^{\mathrm{T}}P + P(A+BF))x_0 = x_0^{\mathrm{T}}\left(A^{\mathrm{T}}P + PA + \sum_{k=2}^{m}(f_k^{\mathrm{T}}b_k^{\mathrm{T}}P + Pb_kf_k)\right)x_0 + 2b_{11}f_1x_0 < 0. \tag{19}$$

By comparing (18) with (19), we know that

$$2b_{11}f_1x_0 < 2b_{11}.$$

Recalling that $b_{11} < 0$, we obtain $f_1x_0 > 1$. Thus $\mathrm{sat}(f_1x_0) = 1$ and

$$\dot{V}(x_0) = x_0^{\mathrm{T}}Q(\bar{h}_1)x_0 = 0.$$

Since $x_0 \in \mathscr{E}(P, \rho^*)$, this implies that $\rho_c \leqslant \rho^*$. Observing that $\rho_c \geqslant \rho^*$, we finally have $\rho_c = \rho^*$. □

Corollary 1. *If the system has only one input, i.e.,* $m = 1$, *then* $\rho_c = \rho^*$.

Proof. In this case, (8b) has only one equality and for an optimal solution, we must have $\rho^*HP^{-1}H^{\mathrm{T}} = 1$. Hence condition (1) in Theorem 3 is satisfied. As to condition (2), there are two inequalities involved

$$(A+BF)^{\mathrm{T}}P + P(A+BF) \leqslant 0 \quad \text{and} \quad (A+BH)^{\mathrm{T}}P + P(A+BH) \leqslant 0.$$

For the first one, we have the strict "$<$" by assumption and for the second one, we must have

$$\lambda_{\max}((A+BH)^{\mathrm{T}}P + P(A+BH)) = 0.$$

Hence condition (2) is also satisfied. Since $m = 1$, condition (3) vanishes (or is satisfied automatically). □

For systems with multiple inputs, computational experience shows that conditions (1) and (2) of Theorem 3 are generally true. This can be explained as follows. It is easy to see that (1) is generally true. Assume that it is $\rho^*\bar{h}_1P^{-1}\bar{h}_1^{\mathrm{T}} = 1$. For condition (2), there is also generally only one D_i such that

$$\lambda_{\max}((A + B(D_iF + D_i^-\bar{H}))^{\mathrm{T}}P + P(A + B(D_iF + D_i^-\bar{H}))) = 0. \tag{20}$$

We would like to show that this D_i^- should be the matrix whose only nonzero element is at $(1,1)$. First, D_i^- must chooses $\bar{h}_1$. Otherwise, (8a) would be true for all h_1 in a neighborhood of $\bar{h}_1$, allowing a greater ρ^*. Suppose that D_i^- also chooses some other h_j, say h_2, then we would have the term

$$(b_2\bar{h}_2)^{\mathrm{T}}P + Pb_2\bar{h}_2$$

in the matrix in (20). Since $\rho^*\bar{h}_2P^{-1}\bar{h}_2^{\mathrm{T}} < 1$, we can let h_2 vary in a neighborhood of $\bar{h}_2$ without violating other conditions except (20). Generally, there would be certain direction Δh_2 such that the additional term

$$(b_2\,\Delta h_2)^{\mathrm{T}}P + Pb_2\,\Delta h_2,$$

will cause

$$\lambda_{\max}((A+B(D_iF+D_i^-H))^{\mathrm{T}}P+P(A+B(D_iF+D_i^-H)))<0.$$

This would also allow a greater ρ^*.

However, condition (3) in Theorem 3 is not generally satisfied. In that case, we may have $\rho_c>\rho^*$. This will be illustrated in an example.

4. An example

Consider a two-input system with

$$A=\begin{bmatrix}0.6 & -0.8\\ 0.8 & 0.6\end{bmatrix}.$$

The input matrix B is generated randomly with normal distribution. F is a feedback matrix such that $A+BF$ has eigenvalues $-1\pm \mathrm{j}0.6$, and P is the solution to

$$(A+BF)^{\mathrm{T}}P+P(A+BF)=-I.$$

Computational results show that conditions (1) and (2) are generally satisfied (97 out of 100). Condition (3) is often satisfied but not always (88 out of 100).

The following are two sets of parameters B generated randomly and the corresponding optimization results.

Case 1:

$$B=\begin{bmatrix}0.8030 & 0.9455\\ 0.0839 & 0.9159\end{bmatrix}.$$

The pole assignment feedback matrix F and the P matrix are

$$F=\begin{bmatrix}-1.2031 & 1.7926\\ -0.4441 & -2.1447\end{bmatrix},\qquad P=\begin{bmatrix}0.5366 & -0.2676\\ -0.2676 & 0.7179\end{bmatrix}.$$

The optimal solution to (8) is $\rho^*=0.4050$ and

$$\bar{H}=\begin{bmatrix}-0.6633 & 1.1973\\ -0.0359 & -1.1828\end{bmatrix}.$$

We also have

$$x_0=\begin{bmatrix}-0.4420\\ -0.8320\end{bmatrix}.$$

All the conditions in Theorem 3 are satisfied. This is illustrated in Fig. 1, where the four solid lines are $h_1x=\pm 1$ and $h_2x=\pm 1$, and the four dotted lines are $f_1x=\pm 1$ and $f_2x=\pm 1$. The ellipsoid only intersects $h_2x=\pm 1$. This shows that condition (1) is satisfied. condition (2) is verified by checking the eigenvalues of the matrices in (8a). We also see that x_0 is between the two lines $f_1x=\pm 1$. This means that condition (3) is satisfied. According to Theorem 3, $\mathscr{E}(P,\rho^*)$ is the largest invariant ellipsoid. This is verified in Fig. 2, where $\dot{V}(x)$ along the boundary of the ellipsoid $\mathscr{E}(P,\rho^*)$ is plotted. We see that the maximal value reaches 0.

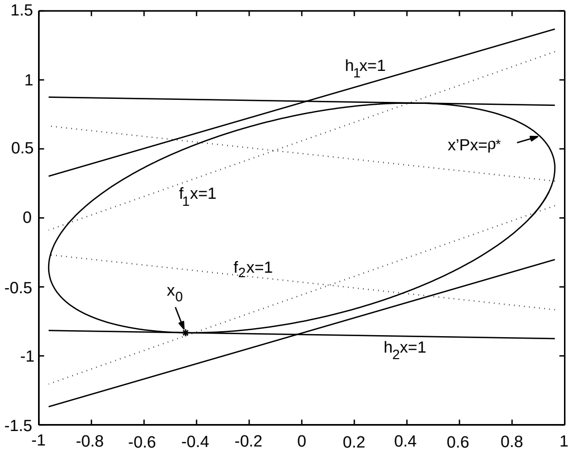

Fig. 1. Illustration of the conditions for Case 1.

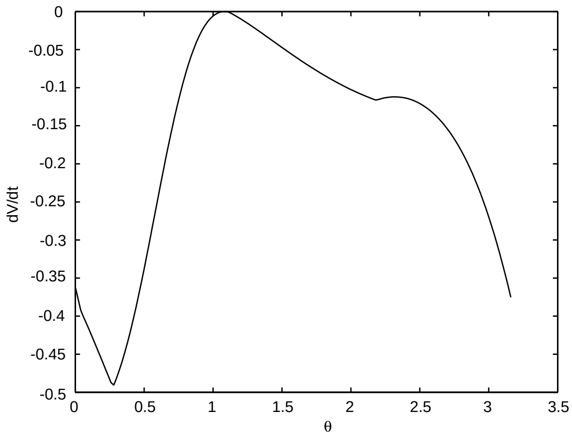

Fig. 2. The derivative $\dot{V}(x)$ along $\partial\mathscr{E}(P,\rho)$: Case 1.

Case 2: We have

$$B = \begin{bmatrix} 0.8828 & -0.1455 \\ 0.2842 & -0.0896 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -2.6921 & -9.1511 \\ -0.9778 & -18.2487 \end{bmatrix}, \qquad P = \begin{bmatrix} 0.2773 & -0.3815 \\ -0.3815 & 7.8606 \end{bmatrix}.$$

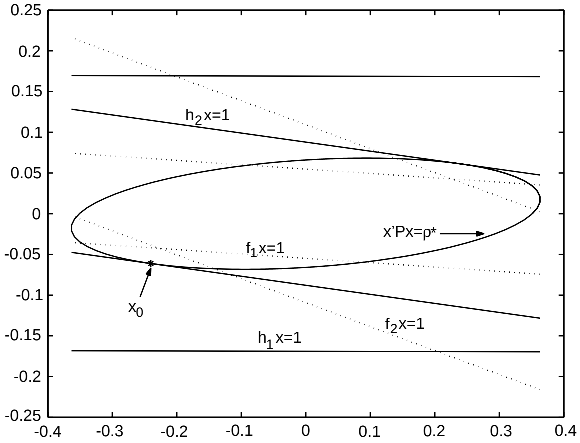

Fig. 3. Illustration of the conditions for Case 2.

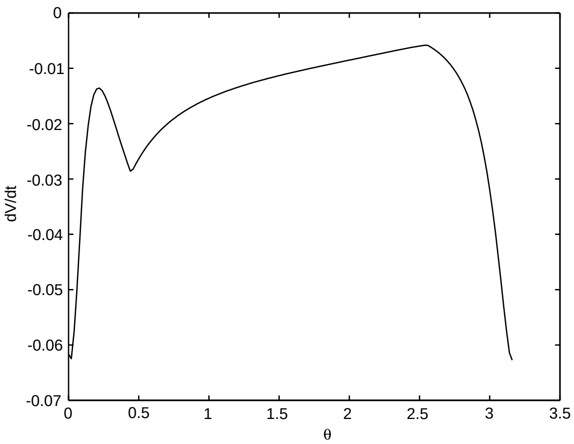

Fig. 4. The derivative $\dot{V}(x)$ along $\partial\mathscr{E}(P,\rho^*)$: Case 2.

The optimal solution to (8) is $\rho^* = 0.0342$, and

$$\bar{H} = \begin{bmatrix} -1.2666 & -11.3745 \\ 0.0110 & 5.9181 \end{bmatrix}.$$

Here we have

$$x_0 = \begin{bmatrix} -0.2403 \\ -0.0612 \end{bmatrix}.$$

It is verified that conditions (1) and (2) are satisfied, but condition (3) is not, as we can see in Fig. 3, where $|f_i x_0| > 1,\ i = 1, 2$. In this case, it is likely that $\mathscr{E}(P, \rho^*)$ is not the largest invariant ellipsoid. As can be seen from Fig. 4, the maximal value of $\dot{V}(x)$ along the ellipsoid is strictly less than 0. This means the largest invariant ellipsoid is strictly larger than $\mathscr{E}(P, \rho^*)$.

5. Conclusions

We investigated the necessity of a recent condition for set invariance by studying the optimal solution of a related LMI problem. We developed criterion for checking if the largest invariant ellipsoid has been obtained by solving the LMI problem. Examples show that the condition may be conservative under certain circumstances.

References

[1] F. Blanchini, Set invariance in control—a survey, Automatica 35 (11) (1999) 1747–1767.

[2] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in Systems and Control Theory, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1994.

[3] E.J. Davison, E.M. Kurak, A computational method for determining quadratic Lyapunov functions for non-linear systems, Automatica 7 (1971) 627–636.

[4] P.O. Gutman, P. Hagander, A new design of constrained controllers for linear systems, IEEE Trans. Automat. Control 30 (1985) 22–33.

[5] H. Hindi, S. Boyd, Analysis of linear systems with saturating using convex optimization, Proceedings of the 37th IEEE CDC, Orlando, FL, 1998, pp. 903–908.

[6] T. Hu, Z. Lin, On enlarging the basin of attraction for linear systems under saturated linear feedback, Systems and Control Lett. 40 (1) (2000) 59–69.

[7] T. Hu, Z. Lin, Control Systems with Actuator Saturation: Analysis and Design, Birkhäuser, Boston, 2001.

[8] T. Hu, Z. Lin, Exact characterization of invariant ellipsoids for linear systems with saturating actuators, IEEE Trans. Automat. Control 47 (1) (2002) 164–169.

[9] T. Hu, Z. Lin, B.M. Chen, An analysis and design method for linear systems subject to actuator saturation and disturbance, Automatica 38 (2) (2002) 351–359.

[10] H. Khalil, Nonlinear Systems, Prentice-Hall, Upper Saddle River, NJ, 1996.

[11] K.A. Loparo, G.L. Blankenship, Estimating the domain of attraction of nonlinear feedback systems, IEEE Trans. Automat. Contr. 23 (4) (1978) 602–607.

[12] G.W. Stewart, J. Sun, Matrix Perturbation Theory, Academic Press, Boston, 1990.

[13] S. Weissenberger, Application of results from the absolute stability to the computation of finite stability domains, IEEE Trans. Automat. Control AC-13 (1968) 124–125.

[14] G.F. Wredenhagen, P.R. Belanger, Piecewise-linear LQ control for systems with input constraints, Automatica 30 (1994) 403–416.