On Several Composite Quadratic Lyapunov Functions for Switched Systems

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***Abstract*— Three types of composite quadratic Lyapunov funtions are used for deriving conditions of stabilization and for constructing switching laws for switched systems. The three types of functions are, the max of quadratics, the min of quadratics and the convex hull of quadratics. Directional derivatives of the Lyapunov functions are used for the characterization of convergence rate. Stability results are established with careful consideration of the existence of sliding mode and the convergence rate along the sliding mode. Dual stabilization result is established with respect to the pair of conjugate Lyapunov functions: the max of quadratics and the convex hull of quadratics. It is observed that the min of quadratics, which is nondifferentiable and nonconvex, may be a more convenient tool than the other two types of functions which are convex and/or differentiable.**

keywords: switched system, composite quadratic functions, stabilization, BMI, sliding mode.

I. Introduction

Given a family of linear systems,

$$\dot{x} = A_i x, \quad i = 1, 2, \cdots, N, \tag{1}$$

two types of switching systems can be defined. In the first case, the switch among the systems is determined by an unknown force. At each time instant, we only know that

$$\dot{x} \in \{A_i x : i = 1, 2, \cdots, N\}. \tag{2}$$

To analyze this linear differential inclusion (LDI) we assume that the switch is arbitrary and have to expect the worst situation. For this case, diverging trajectories can be produced even if all A_i's are Hurwitz (see, e.g., [3], [4]). In the second case, the switch is orchestrated by the controller/supervisor which can choose one of the systems at each time instant based on the measurement of the state or certain output. As a basic step we assume that x is available. For this case, the switching strategy can be optimized for the best performances. The system can be written as

$$\dot{x} = A_{\sigma(x)} x, \tag{3}$$

where $\sigma(x) = i$ for $x \in \Omega_i$ and $\cup_{i=1}^N \Omega_i = \mathbb{R}^n$. The design problem boils down to the construction of the sets Ω_i. With a well designed switching law, every trajectory may converge to the origin even if none of the A_i's is Hurwitz (see, e.g., [14], [20]).

To differentiate the above two cases, we simply call the system (2) the LDI and the system (3) the switched system.

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Stability and stabilization of both the LDI and the switched system have been extensively studied in recent years (see, e.g., [1], [4], [14], [15], [16] for some recent surveys). For the LDI (2), even though the vector $\dot{x}$ can be discontinuous at each time instant, the set-valued map from x to $\{A_i x : i = 1, 2, \cdots, N\}$ is continuous and linear. Because of the linearity, it was shown in [3] that asymptotic stability is equivalent to the existence of a common convex and homogeneous Lyapunov function for every member system $\dot{x} = A_i x$. Since quadratic Lyapunov functions are known to be conservative for LDIs, recent efforts have been dedicated to the construction of numerically tractable nonquadratic Lyapunov functions (e.g., [2], [7], [22]).

The situation with the switched system (3) is quite different. With a given switching strategy $\sigma(x)$, the resulting system is intrinsically nonlinear and discontinuous. There are basically two approaches to stability analysis and stabilization for switched systems: one based on detailed analysis of the vector field and the other one based on Lyapunov theory. For second order systems, some necessary and sufficient condition for stability/stabilizability have been obtained in [10], [23] through analysis on the geometric structure of the vector field. Perhaps the first constructive result by using quadratic Lyapunov functions on switched systems is the bilinear matrix inequality (BMI) obtained in [20]:

$$(\alpha A_1 + (1-\alpha) A_2)^{\mathrm{T}} P + P(\alpha A_1 + (1-\alpha) A_2) < 0,$$

for a positive definite matrix P and a real number $\alpha \in [0, 1]$. Based on this inequality, a switching law can be constructed such that the system is quadratically stable. As with LDIs, a single quadratic Lyapunov function can be too conservative and efforts have been devoted to the development of multiple Lyapunov functions (e.g., see [17], [21], [24]). In [21], two types of piecewise quadratic functions were applied and two pairs of coupled BMIs were derived as conditions for stabilizability. In [17], further conditions on stabilizability were derived as BMIs. More results based on Lyapunov theory can be found, e.g., in [11], [25].

Recently, a pair of conjugate nonquadratic Lyapunov functions have demonstrated great potential in the synthesis of LDIs and saturated linear systems [7], [8], [12], [13]. One is called the convex hull (of quadratics) function and the other is called max (of quadratics) function. When this pair of conjugate functions are applied for stability and performance analysis of LDIs, a handful of dual BMIs are derived in [7]. Numerical examples in these works demonstrate that this pair of nonquadratic Lyapunov functions can significantly enhance stability and performance analysis results.

In this paper, we would like to apply these two conjugate functions to switched systems and use their conjugate relationship to study stabilization of dual switched systems. In addition, we will study another function which is formed by taking the pointwise minimum of a family of quadratic functions. We will call this function the min (of quadratics) function. This function was actually used in [21] along with the max function for deriving two pairs of coupled matrix inequalities. Unlike the ideas behind [21] where each quadratic function (or positive definite matrix) is associated with a particular system $\dot{x} = A_i x$, we will use the max function and the min function as a single Lyapunov function. In particular, the number of positive definite matrices used to compose the Lyapunov functions needn't be the same as the number of linear systems. Moreover, since the max function and the min function are not everywhere differentiable, their directional derivatives will be characterized and used for examining the sliding behavior resulting from switch.

This paper is organized as follows. In Section II, three composite quadratic functions are briefly reviewed and their directional derivatives are derived. A switching law is defined based on a general Lypunov function with some discussion on the effect of sliding mode. Section III establishes stabilization results based on the min function. Section IV establishes a dual stabilization result for the max function and the convex hull function and Section V concludes the paper.

Notation:

- $I[k_1, k_2]$: the set of integers $\{k_1, k_1+1, k_1+2, \cdots, k_2\}$;
- Γ^J: $= \{\gamma \in \mathbb{R}^J : \Sigma_{j=1}^J \gamma_j = 1, \gamma_j \geq 0\}$;
- $\nabla V(x)$: gradient of V at x;
- $\partial V(x)$: subdifferential of V at x;
- $\dot{V}(x;\zeta)$: one-sided directional derivative at x along ζ;
- co$\{S\}$: convex hull of a set S.

II. Composite quadratic functions and switched systems

A. Three composite quadratic functions

Let J be a positive integer. Define

$$\Gamma^J := \left\{\gamma \in \mathbb{R}^J : \ \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \geq 0\right\}.$$

A vector $\gamma \in \Gamma^J$ will be used to form a convex combination of J vectors. Given J positive definite matrices $P_j = P_j^{\mathrm{T}} > 0, j \in I[1, J]$, let $Q_j = P_j^{-1}$.

In [13], three nonquadratic functions are composed from these P_j's:

$$V_{\min}(x) = \min\{x^{\mathrm{T}} P_j x : j \in I[1, J]\}, \tag{4}$$

$$V_{\max}(x) = \max\{x^{\mathrm{T}} P_j x : j \in I[1, J]\}, \tag{5}$$

$$V_c(x) = \min_{\gamma \in \Gamma^J} x^{\mathrm{T}} \left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x. \tag{6}$$

All these functions are positive definite and homogeneous of degree two. It was established that $V_{\max}$ is strictly convex, and V_c is convex and continuously differentiable with the gradient given by $\nabla V_c(x) = 2\left(\sum_{j=1}^J \gamma_j^* Q_j\right)^{-1} x$, where $\gamma^* = \arg\min_{\gamma \in \Gamma^J} x^{\mathrm{T}} \left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x$.

For a positive definite matrix P, denote

$$\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^{\mathrm{T}} P x \leq 1\}.$$

Denote the 1-level set of a general Lyaupnov function V as

$$L_V := \left\{x \in \mathbb{R}^n : \ \ V(x) \leq 1\right\}.$$

It is easy to see that $L_{V_{\min}}$ is the union of the ellipsoids $\mathcal{E}(P_j)$'s and $L_{V_{\max}}$ is the intersection of the same ellipsoids. In [13], It was shown that L_{V_c} is the convex hull of the ellipsoids $\mathcal{E}(P_j)$'s.

For simplicity, we will call $V_{\min}, V_{\max}$ and V_c the min function, the max function and the convex hull function, respectively. The convex hull function V_c was first used in [13] for stability analysis of constrained control systems. It was later used along with $V_{\max}$ in [7], [12] for various stability and performance analysis problems on saturated systems and linear differential inclusions.

The min function $V_{\min}$ is not considered in these papers since it is not convex and it can be shown that V_c always yield better results than $V_{\min}$ for LDIs. As will be shown in this paper, when it comes to the synthesis of switched systems, it may be more convenient to use $V_{\min}$.

B. The directional derivatives for $V_{\max}$ and $V_{\min}$

Clearly $V_{\max}$ and $V_{\min}$ are not everywhere differentiable. The directional derivatives of $V_{\max}$ and $V_{\min}$ will be crucial for the characterization of the behavior at nondifferentiable points and they can be explicitly obtained.

Let $V_{\max}$ and $V_{\min}$ be constructed from J positive definite matrices as in (4) and (5). For a given x, define

$$I_{\max}(x) := \{j \in I[1, J] : x^{\mathrm{T}} P_j x = V_{\max}(x)\},$$
$$I_{\min}(x) := \{j \in I[1, J] : x^{\mathrm{T}} P_j x = V_{\min}(x)\}.$$

Let

$$\Phi_j = \{x \in \mathbb{R}^n : x^{\mathrm{T}} P_j x < x^{\mathrm{T}} P_k x \ \ \forall k \neq j\}, \tag{7}$$

$$\Psi_j = \{x \in \mathbb{R}^n : x^{\mathrm{T}} P_j x > x^{\mathrm{T}} P_k x \ \ \forall k \neq j\}. \tag{8}$$

Then for every $x \in \Phi_j$, $V_{\min}(x) = x^{\mathrm{T}} P_j x$ and $V_{\min}$ is differentiable at x. For every $x \in \Psi_j$, $V_{\max}(x) = x^{\mathrm{T}} P_j x$ and $V_{\max}$ is differentiable at x.

Given a vector $\zeta \in \mathbb{R}^n$, the one-sided directional derivative of $V_{\max}$, and that of $V_{\min}$, at x along ζ are defined as

$$\dot{V}_{\max}(x;\zeta) := \lim_{t \to 0, t>0} \frac{V_{\max}(x + t\zeta) - V_{\max}(x)}{t},$$
$$\dot{V}_{\min}(x;\zeta) := \lim_{t \to 0, t>0} \frac{V_{\min}(x + t\zeta) - V_{\min}(x)}{t}.$$

For a nonlinear vector field $\dot{x} = h(x)$, $\dot{V}_{\max}(x; h(x))$ measures the (forward) time derivative of $V_{\max}$ at x.

Following the definition of [18] (page 215), a subgradient of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 is a vector $v \in \mathbb{R}^n$ such that

$$f(x) - f(x_0) \geq v^{\mathrm{T}}(x - x_0) \ \ \forall\, x \in \mathbb{R}^n, \tag{9}$$

and the subdifferential, denoted as $\partial f(x_0)$, is the set of all subgradient at x_0. The function $f(x)$ is differentiable at x_0

if and only if $\partial f(x_0)$ has only one vector. In this case we have $\partial f(x_0) = \nabla f(x_0)$. We use $\partial V_{\max}(x)$ to denote the subdifferential of $V_{\max}$ at x.

Lemma 1: Consider $x_0 \in \mathbb{R}^n$. We have

1) $\partial V_{\max}(x_0) = \text{co}\{2P_j x_0 : j \in I_{\max}(x_0)\}$.
2) For a vector $\zeta \in \mathbb{R}^n$, the directional derivative of $V_{\max}$ at x_0 along ζ is

$$\dot{V}_{\max}(x_0; \zeta) = \max\{2x_0^{\mathrm{T}} P_j \zeta : j \in I_{\max}(x_0)\}. \quad (10)$$

Combining items 1 and 2 in Lemma 1, we have

$$\dot{V}_{\max}(x; \zeta) = \max_{\xi \in \partial V_{\max}(x)} \{\xi^{\mathrm{T}} \zeta\}.$$

This equality relates the directional derivative of $V_{\max}$ with its subdifferential. At x where $V_{\max}$ is differentiable, $I_{\max}(x)$ contains only one integer and $\partial V_{\max}(x)$ is single valued. In this case, we have $\dot{V}_{\max}(x; \zeta)) = (\nabla V_{\max}(x))^{\mathrm{T}} \zeta = 2x^{\mathrm{T}} P_{I_{\max}(x)} \zeta$.

For the non-convex function $V_{\min}$, we have,

Lemma 2: Consider $x_0 \in \mathbb{R}^n$. For a vector $\zeta \in \mathbb{R}^n$, the directional derivative of $V_{\min}$ at x_0 along ζ is

$$\dot{V}_{\min}(x_0; \zeta) = \min\{2x_0^{\mathrm{T}} P_j \zeta : j \in I_{\min}(x_0)\}. \quad (11)$$

C. Switching laws via the directional derivatives and some discussions on sliding motion

Consider a switched system constructed from N linear systems,

$$\dot{x} = A_i x, \qquad i = 1, 2, \cdots, N. \quad (12)$$

Let V be a positive definite function with directional derivative $\dot{V}(x; \zeta)$ well defined for all $x \neq 0$ and $\zeta \in \mathbb{R}^n$. Denote

$$\Omega_i = \{x \in \mathbb{R}^n : \dot{V}(x; A_i x) \leq \dot{V}(x; A_k x) \ \ \forall k \in I[1, N]\}.$$

To make V decrease with the fastest convergence rate, we may construct a switching law as follows,

$$\sigma(x) = i, \quad \text{if } \ x \in \Omega_i. \quad (13)$$

Equivalently, we can also write

$$\sigma(x) = \arg \min_{i \in I[1,N]} \dot{V}(x; A_i x).$$

The resulting switched system is

$$\dot{x} = A_{\sigma(x)} x. \quad (14)$$

If $\sigma(x)$ is multi-valued, say, $\sigma(x) = \{i_1, i_2, \cdots, i_K\}$, then $\dot{x}$ could be any one of these $A_{i_k} x$'s. Suppose that there exists an $\eta < 0$ such that

$$\min\{\dot{V}(x; A_i x) : i \in I[1, N]\} \leq \eta V(x) \qquad \forall x \neq 0. \quad (15)$$

It seems that the resulting system (14) should be stable with the convergence rate η. This is easy to see if there is no sliding motion, where $\dot{x} = A_i x$ for some $i \in \sigma(x)$ at each time instant. In this case $\dot{V} \leq \eta V$ for all t and we have $V(x(t)) \leq V(x(0))e^{\eta t}$ and the convergence rate is ensured.

When a sliding motion occurs, $x(t)$ stays on a switching surface and the effective $\dot{x}$ may not equal to any of $A_i x$. Instead $\dot{x}$ is a convex combination of those $A_i x$'s which are involved in the vicinity of x. In particular, let $i_k, k \in I[1, K]$ be integers such that there exist a sequence of x_ℓ satisfying $\lim_{\ell \to \infty} x_\ell = x$ and $\sigma(x_\ell) = i_k$ for all ℓ. Then $\dot{x} = \Sigma_{k=1}^{K} \alpha_k A_{i_k} x$ for certain $\alpha \in \Gamma^K$ such that $\dot{x}$ is tangential to the sliding surface (see [6]).

As remarked in [19], a sliding motion is an approximation of the solution of a real system, by letting the nonidealities, such as time-delay, hysteresis, and other types (e.g., sampling time), go to zero. Thus it is expected that when a switching law is implemented by a computer and an nonideal actuator, the solution of the real system would be closely described by the sliding motion. Because of the practical importance and because it is often unavoidable in a switched system, we need to pay particular attention to sliding motion, especially, when a switching rule is based on a nondifferentiable Lyapunov function such as $V_{\min}$ and $V_{\max}$.

III. Stabilization via the min function

A sliding mode is relatively easier to analyze if it is within the set of points where the Lyapunov function is differentiable than the other case.

Proposition 1: Consider the N linear systems given by (12). Let $V_{\min}(x) = \min\{x^{\mathrm{T}} P_j x : j \in I[1, J]\}$ with $P_j = P_j^{\mathrm{T}} > 0, j \in I[1, J]$. Denote

$$\Omega_i = \{x \in \mathbb{R}^n : \dot{V}_{\min}(x; A_i x) \leq \dot{V}_{\min}(x; A_k x), k \in I[1, N]\}. \quad (16)$$

Let the switching law be constructed as

$$\sigma(x) = i \quad \text{for} \quad x \in \Omega_i. \quad (17)$$

Then there exist no sliding mode on the set of x where $V_{\min}$ is not differentiable.

Note that Proposition 1 does not eliminate the possibility of a piece of trajectory staying within the set of nondifferentiable points. Such a piece of trajectory must overlap with the trajectory of $\dot{x} = A_i x$ for a certain i. Hence it is not a sliding motion. Proposition 1 makes it easy for us to derive a condition on stabilizability by combining the conditions on each subset Φ_j, where $V_{\min}$ is differentiable.

Proposition 2: Suppose that there exist real matrices $P_j = P_j^{\mathrm{T}} > 0$, $j \in I[1, J]$ and real numbers η, $\alpha_{ij} \in [0, 1], \beta_{jk} \geq 0, i \in I[1, N], j, k \in I[1, J]$, such that $\Sigma_{i=1}^{N} \alpha_{ij} = 1$ for each j, and

$$\left(\Sigma_{i=1}^{N} \alpha_{ij} A_i\right)^{\mathrm{T}} P_j + P_j \left(\Sigma_{i=1}^{N} \alpha_{ij} A_i\right) \leq \sum_{k=1}^{J} \beta_{jk} (P_j - P_k) + \eta P_j, \quad (18)$$

for all $j \in I[1, J]$. Let $V_{\min}$ be composed from these P_j's. Under the switching law (17), we have $V_{\min}(x(t)) \leq V_{\min}(x(0))e^{\eta t}$ for every solution $x(\cdot)$ (including sliding motion).

The stability condition (18) in Proposition 2 includes a few earlier conditions in [20], [21] as special cases. The relationship among these conditions is addressed as follows. Here we consider the case where the number of linear systems is 2, i.e., $N = 2$.

Case 1. When $J = 1$, $V_{\min}$ reduces to a quadratic function and (18) reduces to a single matrix inequality

$$(\alpha A_1 + (1-\alpha)A_2)^{\mathrm{T}} P + P(\alpha A_1 + (1-\alpha)A_2) < \eta_0 P. \quad (19)$$

For $\eta_0 = 0$, this corresponds to the stability condition in [20]. Let η_0^* be the minimal η_0 satisfying the inequality.

Case 2. When $J = 2$ and $\alpha_{11} = \alpha_{22} = 1, \alpha_{12} = \alpha_{21} = 0$, (18) reduces to the following matrix inequalities

$$\begin{array}{l} A_1^{\mathrm{T}} P_1 + P_1 A_1 < \beta_1(P_1 - P_2) + \eta_1 P_1 \\ A_2^{\mathrm{T}} P_2 + P_2 A_2 < \beta_2(P_2 - P_1) + \eta_1 P_2 \end{array}, \quad (20)$$

where $\beta_1, \beta_2 \geq 0$. If $\eta_1 = 0$, we have the stability condition obtained in [21].

Case 3. For the general case where $V_{\min}$ is composed from J positive definite matrices, we use η_J to replace η in (18) and let η_J^* be the minimal η_J satisfying the inequalities.

Claim 1: $\eta_0^* \geq \eta_1^*$.

Based on the stability condition in Proposition 2, an optimization problem can be formulated for the optimization of the convergence rate as follows

$$\begin{array}{ll} \inf\limits_{P_j, \alpha_{ij}, \beta_{jk}} \eta & (21) \\ \text{s.t. } 1)(18) & \\ \quad 2)\ P_j = P_j^{\mathrm{T}} > 0, \alpha_{ij} \in [0, 1], \beta_{jk} \geq 0, \Sigma_{i=1}^N \alpha_{ij} = 1. & \end{array}$$

The matrix inequality involves bilinear terms as the product of a scalar variable and a matrix variable. They are similar to those bilinear matrix inequalities we have encountered in [7], [12] resulting from using V_c and $V_{\max}$ on LDIs. Extensive numerical experience shows that a combination of the path-following method in [9] and the direct iteration works very well on this type of BMI problems. We developed a two-step iterative algorithm. The first step uses the path-following method to update all the parameters at the same time. The second step fixes β_{jk}'s and α_{ij}'s and solves the resulting "gevp" problem which includes only P_j's as variables. This procedure is repeated until the improvement is within a small range.

Example 1: Consider a third-order switched system with

$$A_1 = \begin{bmatrix} -3 & -6 & 3 \\ 2 & 2 & -3 \\ 1 & 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 & 3 \\ -1 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}.$$

Both A_1 and A_2 are neutrally stable. There exist no convex combination of A_1 and A_2 that is Hurwitz.

We turn to use $V_{\min}$ composed from two quadratic functions and minimize η_1 subject to (20). The minimal η_1 is quickly determined through the path-following algorithm as $\eta_1^* = -0.3375$. The same η_1^* is obtained by sweeping β_1 and β_2 in (20) from 0 to ∞.

If we use $V_{\min}$ composed from more quadratic functions, the convergence rate can be further increased. With $J = 3$, a local minimum η^* for the BMI problem (21) is found to be $\eta_3^* = -0.3836$. With $J = 4$, a local minimum η^* is found to be $\eta_4^* = -0.4656$. This shows that $V_{\min}$ with three or more quadratic functions can work better than those with two quadratic functions even for a system that switches between two A_i's. To demonstrate that the switching feedback law corresponding to η_4^* is stabilizing with the guaranteed convergence rate, we run a set of simulations from different initial conditions for the system with A_1 and A_2 replaced with $\bar{A}_1 = A_1 - \eta_4^* I/2$ and $\bar{A}_2 = A_2 - \eta_4^* I/2$. It can be seen that $\bar{A}_1$ and $\bar{A}_2$ satisfy the matrix inequalities with the same parameters except that $\eta = 0$, which means that a guaranteed convergence rate is 0. A typical result is plotted in Fig. 1 which shows $V_{\min}$ as a function of time and the dotted line identifies the index j such that $x^{\mathrm{T}} P_j x = V_{\min}$. In Fig. 2, the convergence rate $\dot{V}/V$ is plotted as a function of time. The maximum value of $\dot{V}/V$ is -3.773×10^{-4}, slightly below 0 (not exactly 0 because of sampling rate). This shows that the convergence rate is closely reflected from the matrix equations.

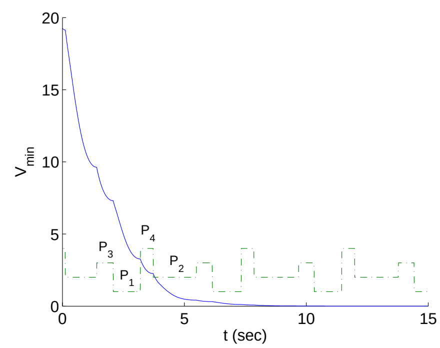

Fig. 1. $V_{\min}$ as a function of time.

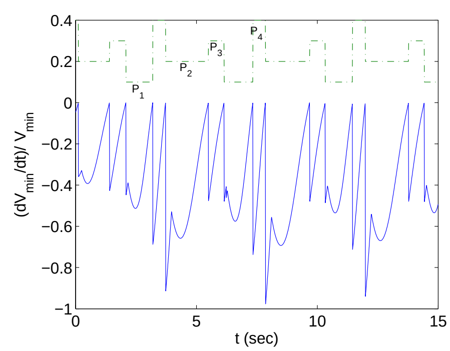

Fig. 2. The convergence rate as a function of time.

IV. Stabilization via Max Function and Convex Hull Function

As established in [8], $V_{\max}$ and V_c are conjugate types. In particular, for

$$V_{\max}(x) = \frac{1}{2} \max\{x^{\mathrm{T}} P_j x : \ j \in I[1, J]\}, \quad (22)$$

its conjugate function is

$$V_c(\xi) = \frac{1}{2} \min_{\gamma \in \Gamma^J} \xi^{\mathrm{T}} \left(\sum_{j=1}^{J} \gamma_j P_j \right)^{-1} \xi. \quad (23)$$

Because of the conjugate relationship, we have

$$\xi \in \partial V_{\max}(x), V_{\max}(x) = \frac{1}{2} \Leftrightarrow x \in \partial V_c(\xi), V_c(\xi) = \frac{1}{2}. \quad (24)$$

Since V_c is continuously differentiable, $\partial V_c(\xi)$ is single valued and equals its gradient. At a point x where $V_{\max}$ is not differentiable, $\partial V_{\max}(x)$ is a convex set given by Lemma 1. By (24), for all $\xi \in \partial V_{\max}(x)$, their gradient $\nabla V_c(\xi) = \partial V_c(\xi) = x$.

Based on the equivalence relation (24), a set of dual relationships are easily established in [8] for dual linear differential inclusions. For dual state-dependent switched systems, the dual relationship is complicated by the discontinuity of switch and possible sliding motion.

Given $A_1, A_2, \cdots, A_N$ and a pair of conjugate functions $V_{\max}(x)$ and $V_c(\xi)$. Consider the dual switched systems

$$\dot{x} = A_{\sigma_1(x)} x, \quad (25)$$

where $\sigma_1(x) = \arg\min_{i \in I[1,N]} \dot{V}_{\max}(x; A_i x)$, and

$$\dot{\xi} = A^{\mathrm{T}}_{\sigma_2(\xi)} \xi, \quad (26)$$

where $\sigma_2(\xi) = \arg\min_{i \in I[1,N]} \dot{V}_c(\xi; A_i^{\mathrm{T}} \xi)$.

Proposition 3: Let $\eta \in \mathbb{R}$ be given. The following two statements are equivalent:

1) For every $x \neq 0$ there exist $\lambda \in \Gamma^N$ such that

$$\dot{V}_{\max}\left(x; \left(\Sigma_{i=1}^N \lambda_i A_i\right) x\right) \leq \eta V_{\max}(x). \quad (27)$$

2) For every $\xi \neq 0$ there exist $\lambda \in \Gamma^N$ such that

$$\dot{V}_c\left(\xi; \left(\Sigma_{i=1}^N \lambda_i A_i^{\mathrm{T}}\right) \xi\right) \leq \eta V_c(\xi). \quad (28)$$

Conditon 2) ensures that $V_c(\xi(t)) \leq V_c(\xi(0)) e^{\eta t}$ for every solution of (26) (including sliding motion). In case $N = 2$, condition 1) ensures that $V_{\max}(x(t)) \leq V_{\max}(x(0)) e^{\eta t}$ for all solutions (including sliding motion) for (25).

Since $V_{\max}$ is piecewise quadratic and V_c is not, it seems to be easier to establish a matrix condition for (27) via the $\mathcal{S}$ procedure. Because of the dual relationship in Proposition 3, we can establish a matrix condition for V_c by using $V_{\max}$ on the dual systems A_i^{T}. For example, consider the case $N = 2$ and $J = 2$. Using S procedure, a set of matrix inequality conditions can be established to ensure (27): there exist real numbers $\alpha_{ij} \in [0,1], \beta_{jk} \geq 0, i \in I[1,2], j, k \in I[1,2]$ such that $\Sigma_{i=1}^2 \alpha_{ij} = 1$ and

$$\begin{aligned}(\Sigma_{i=1}^2 \alpha_{ij} A_i)^{\mathrm{T}} P_j + P_j (\Sigma_{i=1}^2 \alpha_{ij} A_i) \\ \leq \sum_{k=1}^{J} \beta_{jk} (P_k - P_j) + \eta P_j \quad j = 1, 2, \end{aligned} \quad (29)$$

and there exist $a_i \in [0,1]$, $\Sigma_{i=1}^2 a_i = 1$, and $b_j \in \mathbb{R}, d_j \in [0,1], j = 1,2$ such that

$$\begin{aligned}(\Sigma_{i=1}^2 a_i A_i)^{\mathrm{T}} P_j + P_j (\Sigma_{i=1}^2 a_i A_i) \\ \leq b_j (P_1 - P_2) + \eta (d_j P_1 + (1 - d_j) P_2), \ j = 1, 2. \end{aligned} \quad (30)$$

The condition (29) ensures (27) for x where $x^{\mathrm{T}} P_1 x \neq x^{\mathrm{T}} P_2 x$ and (30) ensures (27) for x where $x^{\mathrm{T}} P_1 x = x^{\mathrm{T}} P_2 x$.

The following example demonstrates the dual relationship in Proposition 3.

Example 2: Consider a second-order switched system with

$$A_1 = \begin{bmatrix} 0 & 0 \\ -0.5 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

A_1 has one eigenvalue at 0 and A_2 has a pair of complex eigenvalues on the imaginary axis. Let $V_{\max}$ be constructed from

$$P_1 = \begin{bmatrix} 0.45 & 0.6 \\ 0.6 & 1.1 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0.7 & 0.7 \\ 0.7 & 0.9 \end{bmatrix}.$$

The inequality (27) is satisfied for every differentiable x with $\eta = -0.5263$, as can be seen in Fig. 3, where the directional derivatives $\dot{V}_{\max}(x; A_1 x)$ and $\dot{V}_{\max}(x; A_2 x)$ are plotted along the boundary of the 1-level set $L_{V_{\max}}$ from $\angle x = 0$ to $\angle x = 2\pi$. The solid curve corresponds to $\dot{V}_{\max}(x; A_1 x)$ and the dash-dotted corresponds to $\dot{V}_{\max}(x; A_2 x)$. Both of these curves are discontinuous. At one pair of nondifferentiable points, we have $\min\{\dot{V}_{\max}(x; A_i x) : i = 1, 2\} \leq \eta V_{\max}(x)$. At another pair of nondifferentiable points, $\dot{V}_{\max}(x; A_i x) > 0$ for both $i = 1, 2$, but there exist a convex combination of $A_1 x$ and $A_2 x$, such that $\dot{V}_{\max}(x; (\lambda A_1 + (1 - \lambda) A_2 x) < -0.25 V_{\max}(x)$. This is also illustrated in Fig. 4, where the boundary of $L_{V_{\max}}$ is plotted in thick solid curve and the directions of $A_1 x$ and $A_2 x$ are indicated by dashed line segments pointing from the boundary. At the lower-right (or upper-left) vertex of the level set, $A_1 x$ and $A_2 x$ both point outward of the level set but a convex combination of $A_1 x$ and $A_2 x$ points inward of it. By Proposition 3, the system under the corresponding switching law is stable and has a guaranteed convergence rate $\eta_1 = -0.25$ even if sliding motion occurs, as confirmed by a trajectory in Fig. 4. This trajectory starts at a point marked with "$*$" and enters a sliding mode on the line where $x^{\mathrm{T}} P_1 x = x^{\mathrm{T}} P_2 x$.

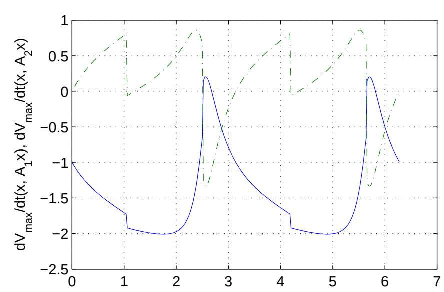

Fig. 3. Directional derivatives along the boundary of $L_{V_{\max}}$

By Proposition 3, we can use P_1 and P_2 to construct a function $V_c(\xi) = \min\{\xi^{\mathrm{T}}(\gamma P_1 + (1 - \gamma) P_2)^{-1} \xi : \gamma \in [0,1]\}$ so that the dual switched system with A_1^{T} and A_2^{T} is stable and $V_c(\xi(t))$ has a convergence rate $\eta_1 = -0.25$. This is confirmed by Figs. 5 and 6. The directional derivatives $\dot{V}_c(\xi; A_1^{\mathrm{T}} \xi)$ and $\dot{V}_c(\xi; A_2^{\mathrm{T}} \xi)$ are plotted in Fig. 5 along the boundary of the 1-level set L_{V_c} from $\angle \xi = 0$ to $\angle \xi = 2\pi$, where the solid curve corresponds to $\dot{V}_c(\xi; A_1^{\mathrm{T}} \xi)$ and the dash-dotted curve corresponds to $\dot{V}_c(\xi; A_2^{\mathrm{T}} \xi)$. Both

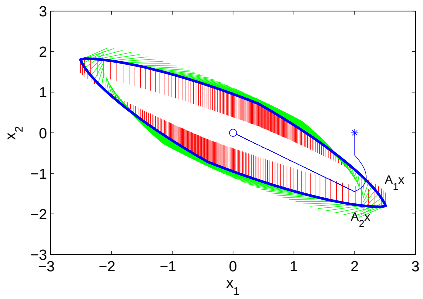

Fig. 4. Boundary of the level set $L_{V_{\max}}$ and a converging trajectory

of these curves are continuous and it can be seen that $\min\{\dot{V}_c(\xi; A_i^{\mathrm{T}}\xi) : i = 1,2\} \leq -0.25V_c(\xi)$ is satisfied for all ξ. In Fig. 6, the boundary of a level set and a trajectory starting from a point marked with "$*$" are plotted. This trajectory also enters a sliding mode after some time.

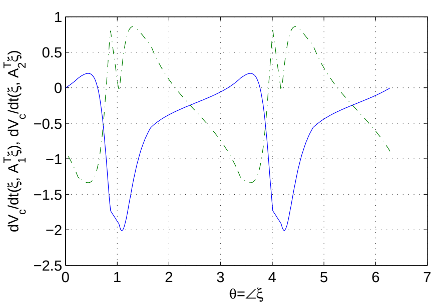

Fig. 5. Dual system: Directional derivatives along the boundary of L_{V_c}

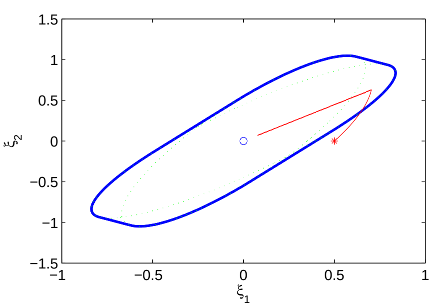

Fig. 6. Dual system: Boundary of L_{V_c} and a converging trajectory

V. Conclusions

The three types of Lyapunov functions studied in this paper are directly constructed from a family of quadratic functions (without additional parameters) and are natural extensions of quadratic functions. They can be used to approximate a wide variety of convex/non-convex functions which are homogeneous of degree 2. They lead to matrix inequalities as conditions for stability/stabilization. Since switched systems are intrinsically nonlinear and discontinuous, a nondifferentiable and non-convex Lyapunov function may work better than convex and/or differentiable Lyapunov functions. Furthermore, non-convex optimization problems and tools are expected to incorporate much greater design freedoms than an overly simplified convex optimization problem. As numerical examples demonstrate in this paper, composite quadratic functions can effectively reduce the conservatism in stability conditions via solving BMI problems.

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