In general of course, the objective is to have a short test interval. However, there is a tradeoff between the separability index and the length of the test period. As we have seen, γ^* is an increasing function of T.

VII. Conclusion

We have presented a methodology for error-free system identification in the situation where we have two candidate linear models subject to bounded energy noise, and where we have control over the input. The problem of selecting a best input signal over a test period (the minimum proper auxiliary signal design problem) has been solved and a solution given in terms of the solution to a boundary value system. The solution of this boundary value system also enables us to design a very efficient on-line identification scheme, the hyperplane test, that takes into account the fact that the input signal over the test period is known in advance.

A related problem which can be solved with the methodology presented here is the shortest test period problem where we fix the separability index and look for the shortest test period for perfect identification. The procedure is similar to the one presented in this paper, except the γ-iteration part should be replaced with T-iteration.

There are a number of possible extensions to the work presented here. One is to allow for additional inputs to the system, i.e., the models would have another input u, in addition to the auxiliary signal v, but this one measured online, just like the output y. This situation has been considered in [9].

Another possible extension is to allow for some nonlinearity. In particular, if the system is not linear in v, the Kalman filter implementation and the hyperplane test still apply. The problem is the optimization over v. An approach using optimization software is discussed in [3].

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Exact Characterization of Invariant Ellipsoids for Single Input Linear Systems Subject to Actuator Saturation

Tingshu Hu and Zongli Lin

***Abstract*—We present a necessary and sufficient condition for an ellipsoid to be an invariant set of a linear system under a saturated linear feedback. The condition is given in terms of linear matrix inequalities (LMIs) and can be easily used for optimization based analysis and design.**

***Index Terms*—Actuator saturation, contractive invariance, invariant ellipsoid.**

I. Introduction

In this paper, we will study the set invariance property for a linear system under saturated feedback

$$\dot{x} = Ax + B\mathrm{sat}(Fx). \tag{1}$$

We will be interested in invariant ellipsoids due to the popularity of the quadratic Lyapunov function and the simplicity of the results. In the literature, invariant ellipsoids have been used to estimate the domain of attraction for nonlinear systems (see, e.g., [1]–[4], [12]–[14], [16], and the references therein). The problem of estimating the domain of attraction for (1) has been a focus of study in recent years.

For a matrix $F \in \mathbf{R}^{m \times n}$, denote the ith row of F as f_i and define

$$\mathcal{L}(F) := \{x \in \mathbf{R}^n \colon |f_i x| \le 1,\ i = 1, 2, \ldots, m\}.$$

If F is a feedback gain matrix, then $\mathcal{L}(F)$ is the region where the feedback control $u = \mathrm{sat}(Fx)$ is linear in x. We call $\mathcal{L}(F)$ the linear region of the saturated feedback $\mathrm{sat}(Fx)$, or simply, the linear region of saturation.

Let $P \in \mathbf{R}^{n \times n}$ be a positive–definite matrix. For a positive number ρ, denote

$$\mathcal{E}(P, \rho) = \left\{x \in \mathbf{R}^n \colon x^{\mathrm{T}} P x \le \rho\right\}.$$

If

$$(A + BF)^{\mathrm{T}} P + P(A + BF) < 0$$

and $\mathcal{E}(P, \rho) \subset \mathcal{L}(F)$, then $\mathcal{E}(P, \rho)$ is an invariant ellipsoid inside the domain of attraction. The largest of these $\mathcal{E}(P, \rho)$s was used as an estimate of the domain of attraction in the earlier literature (see e.g., [4], [15]). This estimation method is simple, yet could be very conservative. Recent efforts have been made to extend the ellipsoid beyond the linear region $\mathcal{L}(F)$ (see, e.g., [7], [5], [6], [12], and [14]). In particular, simple and general methods have been derived by applying the absolute stability analysis tools, such as the circle and Popov criteria, where the saturation is treated as a locally sector bounded nonlinearity.

More recently, we developed a new sufficient condition for an ellipsoid to be invariant in [10] (see also [8]). It was shown that this condition is less conservative than the existing conditions resulting from the circle criterion or the vertex analysis. The most important feature of this new condition is that it can be expressed as LMI's in terms of

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all the varying parameters and hence can easily be used for controller synthesis. In this paper, we will further show that for the single input case, this condition is also necessary.

Notation: In this paper, we use $\mathrm{sat}\colon \mathbf{R} \to \mathbf{R}$ to denote the standard saturation function, i.e., $\mathrm{sat}(u) = \mathrm{sign}(u)\min\{1, |u|\}$.

II. Review of the Existing Results

Consider the linear system with a single saturating input

$$\dot{x} = Ax + Bu, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}, \quad |u|_\infty \le 1. \tag{2}$$

Under a saturated linear feedback $u = \mathrm{sat}(Fx)$, the closed-loop system is

$$\dot{x} = Ax + B\mathrm{sat}(Fx). \tag{3}$$

Given a positive definite matrix P, let $V(x) = x^{\mathrm{T}}Px$. The ellipsoid $\mathcal{E}(P, \rho)$ is said to be (contractively) invariant if

$$\dot{V}(x) = 2x^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) \le (<) 0$$

for all $x \in \mathcal{E}(P, \rho) \setminus \{0\}$. Clearly, if $\mathcal{E}(P, \rho)$ is contractively invariant, then it is inside the domain of attraction.

A. Conditions for Set Invariance

We developed a sufficient condition for set invariance in [10] (see also [8]) for multi-input systems. For single input systems, the condition can be stated as follows.

Theorem 1: Consider system (3). Given an ellipsoid $\mathcal{E}(P, \rho)$, suppose that

$$(A + BF)^{\mathrm{T}}P + P(A + BF) < 0. \tag{4}$$

If there exists an $H \in \mathbf{R}^{1\times n}$ such that

$$(A + BH)^{\mathrm{T}}P + P(A + BH) < 0 \tag{5}$$

and $\mathcal{E}(P, \rho) \subset \mathcal{L}(H)$, i.e., $|Hx| \le 1, \forall\, x \in \mathcal{E}(P, \rho)$, then $\mathcal{E}(P, \rho)$ is a contractively invariant set and hence inside the domain of attraction.

For ρ sufficiently small, we always have $\mathcal{E}(P, \rho) \subset \mathcal{L}(F)$. Hence, (4) is a necessary condition. Before determining the contractive invariance of an ellipsoid $\mathcal{E}(P, \rho)$, it is necessary to make sure that (4) is satisfied. Since (5) is very simple and easily tractable, it is desirable that (5) is also a necessary condition. Although we will finally show that this is indeed the case, the proof of the necessity is much more involved than the sufficiency. We have to approach it through some more existing results.

In [8], we obtained a necessary and sufficient condition for set invariance.

Theorem 2: Given an ellipsoid $\mathcal{E}(P, \rho)$ and an $F \in \mathbf{R}^{1\times n}$, suppose that

$$(A + BF)^{\mathrm{T}}P + P(A + BF) < 0. \tag{6}$$

Then $\mathcal{E}(P, \rho)$ is contractively invariant under $u = \mathrm{sat}(Fx)$ if and only if there exists a function $h(x)\colon \mathbf{R}^n \to \mathbf{R}$, $|h(x)| \le 1$ for all $x \in \mathcal{E}(P, \rho)$, such that $\mathcal{E}(P, \rho)$ is contractively invariant under the control $u = h(x)$, i.e.,

$$x^{\mathrm{T}}P(Ax + Bh(x)) < 0, \qquad \forall\, x \in \mathcal{E}(P, \rho) \setminus \{0\}. \tag{7}$$

B. Existence of Control Law for Set Invariance

The condition of Theorem 2, the existence of $h(x)$ satisfying (7), was studied in [8], [11]. The results are summarized as follows.

Fact 1: The following two statements are equivalent:

a) the ellipsoid $\mathcal{E}(P, \rho)$ can be made contractively invariant for
$$\dot{x} = Ax + Bu \tag{8}$$
with a bounded control $u = h(x)$, $|h(x)| \le 1$;

b) the ellipsoid $\mathcal{E}(P, \rho)$ is contractively invariant for (8) under the control $u = -\mathrm{sign}(B^{\mathrm{T}}Px)$, i.e., the following condition is satisfied,
$$\begin{aligned}\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - 2x^{\mathrm{T}}PB\,\mathrm{sign}(B^{\mathrm{T}}Px) < 0,&\\ \forall\, x \in \mathcal{E}(P, \rho) \setminus \{0\}.&\end{aligned} \tag{9}$$

For an arbitrary $P > 0$, there may exist no $\rho > 0$ such that $\mathcal{E}(P, \rho)$ can be made contractively invariant with a bounded control $u = h(x)$, $|h(x)| \le 1$. In the sequel, we simply say that $\mathcal{E}(P, \rho)$ can be made contractively invariant. Let P be given, the following proposition gives a condition for the existence of $\rho > 0$ such that $\mathcal{E}(P, \rho)$ can be made contractively invariant.

Proposition 1: For a given positive definite matrix P, $\mathcal{E}(P, \rho)$ can be made contractively invariant for some $\rho > 0$ if and only if there exists a $k > 0$ such that

$$A^{\mathrm{T}}P + PA - kPBB^{\mathrm{T}}P < 0. \tag{10}$$

Note that if (10) is satisfied for $k = k_0$, then it is satisfied for all $k > k_0$.

Theorem 3: Given $P > 0$, assume that $\mathcal{E}(P, \rho)$ can be made contractively invariant for some $\rho > 0$. Now we consider a given ρ. Let $\lambda_1, \lambda_2, \ldots, \lambda_{\mathcal{J}} > 0$ be real numbers such that

$$\det\begin{bmatrix}\lambda_j P - A^{\mathrm{T}}P - PA & P\\ \rho^{-1}PBB^{\mathrm{T}}P & \lambda_j P - A^{\mathrm{T}}P - PA\end{bmatrix} = 0 \tag{11}$$

and

$$B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda_j P)^{-1}PB > 0. \tag{12}$$

Then, $\mathcal{E}(P, \rho)$ is contractively invariant under the control $u = -\mathrm{sign}(B^{\mathrm{T}}Px)$ if and only if

$$\begin{aligned}\lambda_j\rho - B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda_j P)^{-1}PB < 0,&\\ \forall\, j = 1, 2, \ldots, \mathcal{J}.&\end{aligned} \tag{13}$$

If there exists no $\lambda_j > 0$ satisfying (11) and (12), then $\mathcal{E}(P, \rho)$ is contractively invariant.

Here we note that all the λ_js satisfying (11) are the eigenvalues of the matrix shown at the bottom of the page. Hence the condition of Theorem 3 can be easily checked. It is shown in [8] and [11] that there is a $\rho^* > 0$ such that $\mathcal{E}(P, \rho)$ is contractively invariant if and only if $\rho < \rho^*$. Therefore, the maximum value ρ^* can be obtained by checking the condition of Theorem 3 using a bisection method.

For the single input case, if P is fixed, then we can combine Theorems 2 and 3 to find the largest ρ such that $\mathcal{E}(P, \rho)$ is invariant under a given saturated feedback $u = \mathrm{sat}(Fx)$. However, if P is an unknown parameter for optimization (for example, in the design problem of enlarging the domain of attraction), then the condition of Theorem 3 would not be easy to deal with. For this reason, we would like to use Theorem 1 since its condition leads to LMI constraints (see, e.g., [2]). The only concern is that we might not find the optimal invariant ellipsoid since the condition of Theorem 1 was only shown to be sufficient. This paper is intended to

$$\begin{bmatrix}P^{-(1/2)}A^{\mathrm{T}}P^{1/2} + P^{1/2}AP^{-(1/2)} & -I\\ -\rho^{-1}P^{1/2}BB^{\mathrm{T}}P^{1/2} & P^{-(1/2)}A^{\mathrm{T}}P^{1/2} + P^{1/2}AP^{-(1/2)}\end{bmatrix}.$$

close this gap. We will show in the next section that for the single input case, the condition in Theorem 1 is also necessary. If P is fixed, then the largest ρ satisfying the condition of Theorem 1 is the same as the largest ρ satisfying the condition of Theorem 3.

III. Main Results

Theorem 4: Given an ellipsoid $\mathcal{E}(P, \rho)$ and an $F \in \mathbf{R}^{1\times n}$, assume that

$$(A+BF)^{\mathrm{T}}P + P(A+BF) < 0. \tag{14}$$

Then $\mathcal{E}(P, \rho)$ is contractively invariant under the feedback $u = \mathrm{sat}(Fx)$ if and only if there exists an $H \in \mathbf{R}^{1\times n}$ such that

$$(A+BH)^{\mathrm{T}}P + P(A+BH) < 0 \tag{15}$$

and $\mathcal{E}(P, \rho) \subset \mathcal{L}(H)$.

Recall from [8], [9], $\mathcal{E}(P, \rho) \subset \mathcal{L}(H)$ is equivalent to

$$\rho H P^{-1} H^{\mathrm{T}} \le 1.$$

Lemma 1: Suppose that $\mathcal{E}(P, \rho)$ can be made contractively invariant for some $\rho > 0$. Let ρ^* be the supremum of all the ρs such that $\mathcal{E}(P, \rho)$ is contractively invariant under $u = -\mathrm{sign}(B^{\mathrm{T}}Px)$ (i.e., satisfying the conditions in Theorem 3). Then there exists a $\lambda > 0$ such that

$$\det \begin{bmatrix} \lambda P - A^{\mathrm{T}}P - PA & P \\ \dfrac{1}{\rho^*} PBB^{\mathrm{T}}P & \lambda P - A^{\mathrm{T}}P - PA \end{bmatrix} = 0 \tag{16}$$

and

$$\lambda \rho^* = B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB. \tag{17}$$

Proof: Let $V(x) = x^{\mathrm{T}}Px$. Under the control $u = -\mathrm{sign}(B^{\mathrm{T}}Px)$, the derivative of $V(x)$ along the trajectory of the closed-loop system is

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - 2x^{\mathrm{T}}PB\,\mathrm{sign}(B^{\mathrm{T}}Px).$$

Let

$$g(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - 2x^{\mathrm{T}}PB.$$

Define

$$\gamma(\rho) = \max\left\{g(x) \colon x^{\mathrm{T}}Px = \rho,\ B^{\mathrm{T}}Px \ge 0\right\}.$$

It is shown in [8] and [11] that if $\gamma(\rho) < 0$, then $\gamma(\rho_1) < 0$ for all $\rho_1 \in (0, \rho)$. Hence, $\mathcal{E}(P, \rho)$ is contractively invariant under $u = -\mathrm{sign}(B^{\mathrm{T}}Px)$ if and only if $\gamma(\rho) < 0$. Since the function $g(x)$ is uniformly continuous on any compact set, $\gamma(\rho)$ is continuous in ρ. By the definition of ρ^*, we have

$$\gamma(\rho) < 0, \qquad \forall \rho \in (0, \rho^*)$$

and $\gamma(\rho^*) = 0$. It is clear that $\mathcal{E}(P, \rho^*)$ is invariant but not contractively invariant. By Theorem 3, there exists a $\lambda > 0$ satisfying (16) and

$$B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB > 0 \tag{18}$$

and

$$\lambda\rho^* - B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB \ge 0. \tag{19}$$

It can be shown with algebraic manipulation (see the Appendix) that (16) is equivalent to

$$B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB^{\mathrm{T}} = \rho^*. \tag{20}$$

We claim that only "=" is possible in (19). Otherwise, if we let $x = (A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB$, then from (20), we have $x^{\mathrm{T}}Px = \rho^*$, and from (18), we have $B^{\mathrm{T}}Px > 0$. Thus

$$\begin{aligned} g(x) &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - 2x^{\mathrm{T}}PB \\ &= \lambda x^{\mathrm{T}}Px - x^{\mathrm{T}}PB \\ &= \lambda\rho^* - B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB > 0. \end{aligned}$$

This means that $\gamma(\rho^*) > 0$, which is a contradiction. Therefore, we must have

$$\lambda\rho^* - B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB = 0.$$

□

Lemma 2: Suppose that $\mathcal{E}(P, \rho)$ can be made contractively invariant for some $\rho > 0$. Let ρ^* be defined in Lemma 1 with λ satisfying

$$\det \begin{bmatrix} \lambda P - A^{\mathrm{T}}P - PA & P \\ \dfrac{1}{\rho^*} PBB^{\mathrm{T}}P & \lambda P - A^{\mathrm{T}}P - PA \end{bmatrix} = 0 \tag{21}$$

and

$$\lambda \rho^* = B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB. \tag{22}$$

Let

$$H_0 = -\frac{1}{\rho^*} B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}P$$

then $\rho^* H_0 P^{-1} H_0^{\mathrm{T}} = 1$, i.e., $\mathcal{E}(P, \rho^*) \subset \mathcal{L}(H_0)$ and

$$(A + BH_0)^{\mathrm{T}}P + P(A + BH_0) \le 0. \tag{23}$$

Proof: Since $\mathcal{E}(P, \rho)$ can be made contractively invariant for some $\rho > 0$, by Proposition 1, there exists a $k_0 > 0$ such that

$$A^{\mathrm{T}}P + PA - kPBB^{\mathrm{T}}P < 0 \tag{24}$$

for all $k > k_0$.

From Fact 2 in the Appendix, (21) is equivalent to

$$B^{\mathrm{T}}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}P(A^{\mathrm{T}}P + PA - \lambda P)^{-1}PB = \rho^* \tag{25}$$

it follows that $\rho^* H_0 P^{-1} H_0^{\mathrm{T}} = 1$, which implies that $\mathcal{E}(P, \rho^*) \subset \mathcal{L}(H_0)$.

We next proceed to prove (23). Consider a state transformation of the form $z = Tx$, where

$$T = U\left(\frac{P}{\rho^*}\right)^{1/2}$$

for some unitary matrix $U \in \mathbf{R}^{n\times n}$, $UU^{\mathrm{T}} = I$. U is to be specified later. Let

$$\overline{B} = TB, \quad \overline{A} = TAT^{-1}, \quad \overline{H}_0 = H_0T^{-1}$$

and

$$Q = \overline{A}^{\mathrm{T}} + \overline{A}.$$

Then

$$\begin{aligned}
&\overline{B}^{\mathrm{T}}(Q-\lambda I)^{-1}(Q-\lambda I)^{-1}\overline{B}\\
&= B^{\mathrm{T}}\left(\frac{P}{\rho^*}\right)^{1/2}U^{\mathrm{T}}\\
&\quad\cdot\left(U\left(\frac{P}{\rho^*}\right)^{-(1/2)}A^{\mathrm{T}}\left(\frac{P}{\rho^*}\right)^{1/2}U^{\mathrm{T}}\right.\\
&\qquad\left.+\,U\left(\frac{P}{\rho^*}\right)^{1/2}A\left(\frac{P}{\rho^*}\right)^{-(1/2)}U^{\mathrm{T}}-\lambda I\right)^{-1}UU^{\mathrm{T}}\\
&\quad\cdot\left(U\left(\frac{P}{\rho^*}\right)^{-(1/2)}A^{\mathrm{T}}\left(\frac{P}{\rho^*}\right)^{1/2}U^{\mathrm{T}}+U\left(\frac{P}{\rho^*}\right)^{1/2}\right.\\
&\qquad\left.\cdot\,A\left(\frac{P}{\rho^*}\right)^{-(1/2)}U^{\mathrm{T}}-\lambda I\right)^{-1}U\left(\frac{P}{\rho^*}\right)^{1/2}B\\
&= B^{\mathrm{T}}\left(\frac{P}{\rho^*}\right)^{1/2}\left(P^{-(1/2)}A^{\mathrm{T}}P^{1/2}\right.\\
&\qquad\left.+\,P^{1/2}AP^{-(1/2)}-\lambda I\right)^{-1}\\
&\quad\cdot\left(P^{-(1/2)}A^{\mathrm{T}}P^{1/2}\right.\\
&\qquad\left.+P^{1/2}AP^{-(1/2)}-\lambda I\right)^{-1}\left(\frac{P}{\rho^*}\right)^{1/2}B\\
&= \frac{1}{\rho^*}B^{\mathrm{T}}P(A^{\mathrm{T}}P+PA-\lambda P)^{-1}\\
&\quad\cdot P(A^{\mathrm{T}}P+PA-\lambda P)^{-1}PB=1,
\end{aligned}$$

where the last "=" follows from (25). For easy reference, we rewrite the above equation as

$$\overline{B}^{\mathrm{T}}(Q-\lambda I)^{-1}(Q-\lambda I)^{-1}\overline{B}=1. \tag{26}$$

Similarly, from (22), we obtain

$$\overline{B}^{\mathrm{T}}(Q-\lambda I)^{-1}\overline{B}=\lambda. \tag{27}$$

Also, we have

$$\overline{H}_0=-\overline{B}^{\mathrm{T}}(Q-\lambda I)^{-1}. \tag{28}$$

From (26) and (28), we have $\overline{H}_0\overline{H}_0^{\mathrm{T}}=1$. Recall that

$$\overline{H}_0=H_0T^{-1}=H_0\left(\frac{P}{\rho^*}\right)^{-(1/2)}U^{\mathrm{T}}.$$

By choosing U, we can make

$$-\overline{H}_0=\overline{B}^{\mathrm{T}}(Q-\lambda I)^{-1}=\begin{bmatrix}1 & 0_{1\times(n-1)}\end{bmatrix}. \tag{29}$$

Partition $\overline{B}$ and Q as follows

$$\overline{B}=\begin{bmatrix}b_1\\ b_2\end{bmatrix},\quad Q=\begin{bmatrix}q_1 & q_{21}^{\mathrm{T}}\\ q_{21} & Q_2\end{bmatrix},\qquad b_1,\,q_1\in\mathbf{R}.$$

From (29), we have

$$\overline{B}^{\mathrm{T}}=\begin{bmatrix}1 & 0_{1\times(n-1)}\end{bmatrix}(Q-\lambda I).$$

It follows that:

$$q_1-\lambda=b_1,\qquad q_{21}=b_2.$$

From (27) and (29), we have $b_1=\lambda$. In summary

$$\overline{B}=\begin{bmatrix}\lambda\\ b_2\end{bmatrix},\qquad Q=\begin{bmatrix}2\lambda & b_2^{\mathrm{T}}\\ b_2 & Q_2\end{bmatrix}.$$

Multiplying (24) from left with $(T^{-1})^{\mathrm{T}}$ and from right with T^{-1}, we obtain

$$Q-k\rho^*\overline{B}\overline{B}^{\mathrm{T}}<0$$

for all $k>k_0$. That is

$$\begin{bmatrix}k\rho^*\lambda^2-2\lambda & (k\rho^*\lambda-1)b_2^{\mathrm{T}}\\ (k\rho^*\lambda-1)b_2 & k\rho^*b_2b_2^{\mathrm{T}}-Q_2\end{bmatrix}>0,\qquad\forall k>k_0.$$

By Schur complements, this implies

$$k\rho^*b_2b_2^{\mathrm{T}}-Q_2>0,$$
$$k\rho^*b_2b_2^{\mathrm{T}}-Q_2-\frac{(k\rho^*\lambda-1)^2}{k\rho^*\lambda^2-2\lambda}b_2b_2^{\mathrm{T}}>0,\qquad\forall k>k_0.$$

The second inequality can be rewritten as

$$-\frac{1}{k\rho^*\lambda^2-2\lambda}b_2b_2^{\mathrm{T}}-Q_2>0.$$

Since this inequality is true for all $k>k_0$, we must have $Q_2\le 0$.

Let's finally examine the term $(A+BH_0)^{\mathrm{T}}P+P(A+BH_0)$. Recall that in the new coordinate

$$\overline{H}_0=-\begin{bmatrix}1 & 0_{1\times(n-1)}\end{bmatrix}.$$

Hence

$$(\overline{A}+\overline{B}\,\overline{H}_0)^{\mathrm{T}}+\overline{A}+\overline{B}\,\overline{H}_0=Q-\begin{bmatrix}2\lambda & b_2^{\mathrm{T}}\\ b_2 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & Q_2\end{bmatrix}\le 0$$

which is equivalent to

$$(T^{-1})^{\mathrm{T}}(A+BH_0)T^{\mathrm{T}}+T(A+BH_0)T^{-1}\le 0.$$

Multiplying both sides from left with T^{T} and from right with T, we have

$$(A+BH_0)T^{\mathrm{T}}T+T^{\mathrm{T}}T(A+BH_0)\le 0.$$

Noting that $T^{\mathrm{T}}T=P/\rho^*$, we finally obtain (23). □

Proof of Theorem 4: The sufficiency follows from Theorem 1. Now we prove the necessity. That is, suppose that $\mathcal{E}(P,\rho)$ is contractively invariant, then there exists an $H\in\mathbf{R}^{1\times n}$ such that

$$(A+BH)^{\mathrm{T}}P+P(A+BH)<0$$

and $\mathcal{E}(P,\rho)\subset\mathcal{L}(H)$.

Here, we have

$$(A+BF)^{\mathrm{T}}P+P(A+BF)<0.$$

Define $H(\alpha)=(1-\alpha)H_0+\alpha F$, where H_0 is defined in Lemma 2. It follows from (23) in Lemma 2 that:

$$(A+BH(\alpha))^{\mathrm{T}}P+P(A+BH(\alpha))<0 \tag{30}$$

for all $\alpha\in(0,1]$. Also from Lemma 2

$$\rho^*H_0P^{-1}H_0^{\mathrm{T}}=1.$$

Since $\mathcal{E}(P,\rho)$ is contractively invariant, we must have $\rho<\rho^*$. It follows that:

$$\rho H_0P^{-1}H_0^{\mathrm{T}}<1.$$

Since $H_0=H(0)$, by the continuity of $H(\alpha)$, there exists a sufficiently small $\alpha>0$ such that

$$\rho H(\alpha)P^{-1}H(\alpha)^{\mathrm{T}}<1 \tag{31}$$

i.e., $\mathcal{E}(P,\rho)\subset\mathcal{L}(H(\alpha))$. This completes the proof by observing (30) and letting $H=H(\alpha)$. □

The results in Theorems 2 and 4 and Fact 1 can be summarized in the following proposition.

Proposition 2: Given a $P>0$, assume that $\mathcal{E}(P,\rho)$ can be made contractively invariant for some $\rho>0$. Let $\rho>0$ be given. The following statements are equivalent:

$$\det(\rho - B^{\mathrm{T}} P \Phi^{-1} P \Phi^{-1} PB) = 0$$
$$\Updownarrow$$
$$\det \begin{bmatrix} \rho & -B^{\mathrm{T}} P \Phi^{-1} \\ -\Phi^{-1} PB & P^{-1} \end{bmatrix} = 0$$
$$\Updownarrow$$
$$\det \left(\begin{bmatrix} \rho & 0 \\ 0 & P^{-1} \end{bmatrix} - \begin{bmatrix} B^{\mathrm{T}} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & \Phi^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ PB & 0 \end{bmatrix} \right) = 0$$
$$\Updownarrow$$
$$\det \left(\begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} - \begin{bmatrix} 0 & I \\ PB & 0 \end{bmatrix} \begin{bmatrix} \rho^{-1} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} B^{\mathrm{T}} P & 0 \\ 0 & I \end{bmatrix} \right) = 0$$
$$\Updownarrow$$
$$\det \begin{bmatrix} \lambda P - A^{\mathrm{T}} P - PA & P \\ \rho^{-1} PBB^{\mathrm{T}} P & \lambda P - A^{\mathrm{T}} P - PA \end{bmatrix} = 0.$$

a) $\mathcal{E}(P, \rho)$ can be made contractively invariant with some $u = h(x)$, $|h(x)| \le 1$;
b) $\mathcal{E}(P, \rho)$ is contractively invariant under the control $u = -\mathrm{sign}(B^{\mathrm{T}} Px)$;
c) $\mathcal{E}(P, \rho)$ is contractively invariant under $u = \mathrm{sat}(Fx)$, where F satisfies
$$(A + BF)^{\mathrm{T}} P + P(A + BF) < 0; \tag{32}$$
d) there exists $H \in \mathbf{R}^{1\times n}$ satisfying
$$(A + BH)^{\mathrm{T}} P + P(A + BH) < 0$$
and $\mathcal{E}(P, \rho) \subset \mathcal{L}(H)$.

The equivalence results in Proposition 2 are somewhat counter intuitive. Condition a) seems to be the weakest and condition d) seems to be the strongest. Yet they are all equal. The equivalence of a) and d) implies that if an ellipsoid can be made contractively invariant with a control $u = h(x)$, $|h(x)| \le 1$, then there exists a control law linear in x in the ellipsoid to make it contractively invariant.

Of the four statements in Proposition 2, b) can be checked with Theorem 3 but there is no direct method to check c). The last condition d) is the most easily tractable and can be flexibly incorporated into analysis and design problems such as estimation of the domain of attraction and enlarging the domain of attraction (see [8]–[10]). It can also be extended for the purpose of disturbance rejection (see [8], [10]).

Let us illustrate the application of Proposition 2 with the following example. Suppose that we are given a shape reference set X_{R}. We want to design a controller $u = \mathrm{sat}(f(x))$ such that the scaled reference set αX_{R} is inside some invariant ellipsoid $\mathcal{E}(P, \rho)$ of the closed-loop system
$$\dot{x} = Ax + B\mathrm{sat}(f(x)).$$
The objective is to maximize the quantity α with design parameters P, ρ and the control law $u = \mathrm{sat}(f(x))$. This problem can be referred to as one of enlarging the domain of attraction as in [9]. In [9], we restricted the control law to be linear in the ellipsoid $\mathcal{E}(P, \rho)$. That is, $u = \mathrm{sat}(Fx)$ and $|Fx| \le 1$ for all $x \in \mathcal{E}(P, \rho)$, which is equivalent to $\mathcal{E}(P, \rho) \subset \mathcal{L}(F)$. In view of Proposition 2, this restriction to linear control law will not affect the resulting maximal value of α. The great advantage of the restriction is that the optimization problem can be easily solved with LMI technique (see [8] and [9]).

IV. Conclusion

This paper presents a complete characterization of invariant ellipsoids for a single input linear system subject to actuator saturation. In particular, we obtained several equivalent conditions for an ellipsoid to be invariant under a certain saturated linear feedback. One of the condition is stated in terms of linear matrix inequality, which can be easily used for stability analysis and controller design.

Appendix
An Algebraic Fact

Fact 2: Assume that P and $(\lambda P - A^{\mathrm{T}} P - PA)$ are nonsingular, then
$$\det \begin{bmatrix} \lambda P - A^{\mathrm{T}} P - PA & P \\ \rho^{-1} PBB^{\mathrm{T}} P & \lambda P - A^{\mathrm{T}} P - PA \end{bmatrix} = 0 \tag{33}$$
is equivalent to
$$B^{\mathrm{T}} P(A^{\mathrm{T}} P + PA - \lambda P)^{-1} P(A^{\mathrm{T}} P + PA - \lambda P)^{-1} PB = \rho. \tag{34}$$

Proof: Denote
$$\Phi = \lambda P - A^{\mathrm{T}} P - PA,$$
then the equation (34) can be written as
$$B^{\mathrm{T}} P \Phi^{-1} P \Phi^{-1} PB = \rho. \tag{35}$$
By invoking the equalities
$$\det \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \det(X_1)\det(X_4 - X_3 X_1^{-1} X_2) \tag{36}$$
and
$$\det \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \det(X_4)\det(X_1 - X_2 X_1^{-1} X_3) \tag{37}$$
we see that (35) is equivalent to the equations shown at the top of the page. The last equation is (33).

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Solving a Nonlinear Output Regulation Problem: Zero Miss Distance of Pure PNG

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***Abstract*—This note presents a solution to the output regulation problem of a nonlinear system with time-varying disturbance: the system represents the well-known missile-target pursuit situation where the missile is guided by the pure proportional navigation guidance (PPNG) law while the target maneuvers with time-varying normal acceleration, and the problem is to prove the zero miss distance property of PPNG, which has been studied for decades without satisfactory success. To solve this problem, we construct a function by which a time sequence of the missile-to-target range is upper-bounded, and prove that the function is strictly decreasing, which is also proven to guarantee that there is always a subsequence that asymptotically converges to zero. The solution is given in the form of a necessary and sufficient condition guaranteeing zero miss distance of PPNG.**

***Index Terms*—Nonlinear output regulation, pure proportional navigation guidance (PPNG), time-varying disturbance, zero miss distance.**

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NOMENCLATURE

$v_m (v_t)$	Missile (target) speed.
$a_m (a_t)$	Missile (target) acceleration.
θ_L	Euler angle from reference coordinate system to LOS coordinate system.
θ_m	Euler angle from LOS coordinate system to missile body coordinate system.
θ_t	Euler angle from LOS coordinate system to target body coordinate system.
N	Navigation constant.
r	Missile-to-target range.
ρ	v_t/v_m.
$s\theta_i,\ c\theta_i$	$\sin\theta_i,\ \cos\theta_i$.
$a.e.$	*almost everywhere*.

I. INTRODUCTION

The pure proportional navigation guidance (PPNG) law has been widely adopted in many tactical missile systems because of its simplicity and high capturability, which has been proven in practice. As a consequence, a lot of research has been carried out to confirm mathematically the high capturability of the PPNG [1]–[6].

Specifically, it has been shown in [2], and [4]–[6] that a missile guided by PPNG can always intercept a target, which is maneuvering with time-varying normal acceleration, under some conditions on the navigation constant, the initial heading error, the initial missile-to-target range, the magnitude of target acceleration, and the ratio of missile speed to target speed. These conditions are equivalent to the conditions for the missile-to-target range to be *strictly decreasing* after a finite time. In other words, the prior research excludes the case when the missile-to-target range has a *fluctuating* time-profile caused by the target maneuver. This exclusion results in significant discrepancies between the mathematical analysis results and the actual capturability of the PPNG.

In fact, it is quite challenging to mathematically analyze the capturability of PPNG for the case when the missile-to-target range is doomed to fluctuate. Interestingly, the problem that we are dealing with can be viewed as an output regulation problem of a nonlinear system with time-varying perturbations: the nonlinear system is the missile-target pursuit dynamics, the feedback controller is the PPNG law, the output is the missile-to-target range, and the perturbation is the time-varying normal acceleration of the target. However, since our output zeroing manifold does not contain any equilibrium points, it is impossible to directly apply the results of earlier research [7]–[11] to this problem.

This note describes the construction of an asymptotic time-function by which the missile-to-target range is always upper-bounded. This approach can provide a necessary and sufficient condition under which a missile, which is launched toward the target with $\rho < 1$ and guided by the PPNG law, can always capture a target maneuvering arbitrarily with time-varying normal acceleration. Specifically, it is shown that a navigation constant larger than 1 is the only condition required to achieve zero miss distance. In our analysis, the nonlinear dynamics of pursuit situations are taken into full account.

II. PRELIMINARIES

The two-dimensional (2-D) pursuit situation can be represented as follows [2], [4]:

$$\dot{r} = (\rho c\theta_t - c\theta_m)v_m \tag{1}$$

$$\dot{\theta}_m = \frac{a_m}{v_m} - \dot{\theta}_L \tag{2}$$

$$\dot{\theta}_t = \frac{a_t}{v_t} - \dot{\theta}_L \tag{3}$$

where a_m, missile acceleration generated by the PPNG, and $\dot{\theta}_L$, LOS rate, are given by

$$\dot{\theta}_L = \frac{(\rho s\theta_t - s\theta_m)v_m}{r} \tag{4}$$

$$a_m = N v_m \dot{\theta}_L. \tag{5}$$