and the identifier output is $y_i(t) = 2.5\xi_4 + 10\xi_2 + (5 - \hat{\theta}_1)\xi_3 + (3 - \hat{\theta}_2)\xi_1$. The parameter update law is defined by the standard gradient algorithm in which $g_1 = g_2 = 7500$

$$\begin{aligned}\dot{\theta}_1 &= 7500(y_i(t) - y(t))\xi_3 \\ \dot{\theta}_2 &= 7500(y_i(t) - y(t))\xi_1.\end{aligned} \tag{20}$$

The estimations of ω_1 and ω_2 are then given by

$$\hat{\omega}_{1,2} = \sqrt{\frac{-\hat{\theta}_1 \pm \sqrt{\hat{\theta}_1^2 - 4\hat{\theta}_2}}{2}}.$$

The estimator consisting of (19) and (20) is a sixth-order one. In simulation, all initial conditions are set to be zero.

A simulation is also done where $y(t)$ is corrupted by a uniform random noise between -0.01 and 0.01.

Fig. 3 shows the convergence of the first estimated frequencies for both uncorrupted and corrupted version of $y(t)$. Fig. 4 shows the convergence of the second estimated frequencies for both uncorrupted and corrupted version of $y(t)$.

It can be observed that the estimations are accurate for both uncorrupted and corrupted signals. Simulation is also done for large corruptions, it is found that when corruptions are larger in magnitude, the steady state errors are bigger.

V. Conclusion

A design of adaptive identifiers to globally estimate the frequencies of a signal composed of n sinosuoidal components was shown. Convergence of the proposed estimator is proven. The new frequency estimator is of $3n$ order, comparing with the order $5n - 1$ of the estimator through Marino–Tomei observers. Results are demonstrated via simulation.

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On Semiglobal Stabilizability of Antistable Systems by Saturated Linear Feedback

Tingshu Hu and Zongli Lin

***Abstract*—It was recently established that a second-order antistable linear system can be semiglobally stabilized on its null controllable region by saturated linear feedback and a higher order linear system with two or more antistable poles can be semiglobally stabilized on its null controllable region by more general bounded feedback laws. We will show in this note that a system with three real-valued antistable poles cannot be semiglobally stabilized on its null controllable region by the simple saturated linear feedback.**

***Index Terms*—Actuator saturation, antistable systems, semiglobal stabilizability.**

I. Introduction

There has been a long history of exploring global or semiglobal stabilizability for linear systems with saturating actuators. In 1969, Fuller [1] studied global stabilizability of a chain of integrators of length greater than two by saturated linear feedback and obtained a negative result. This important problem also attracted the attention of Sussmann and Yang [9]. They obtained similar results independently in 1991. Because of the negative result on global stabilizability with saturated linear feedback, the only choice is to use general nonlinear feedback. In 1992, Teel [11] proposed a nested feedback design technique for designing nonlinear globally asymptotically stabilizing feedback laws for a chain of integrators. This technique was fully generalized by Sussman, Sontag and Yang [8] in 1994. Alternative solutions to global stabilization problem consisting of scheduling a parameter in an algebraic Riccati equation according to the size of the state vector were later proposed in [7], [10], and [12].

Another trend in the development, motivated by the objective of designing simple controllers, is semiglobal stabilizability with saturated linear feedback laws. The notion of semiglobal asymptotic stabilization for linear systems subject to actuator saturation was introduced in [5] and [6]. The semiglobal framework for stabilization requires feedback laws that yield a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes an *a priori* given (arbitrarily large) bounded subset of the state space. In [5] and [6], it was shown that, a linear system can be semiglobally stabilized by saturated

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linear feedback if it is stabilizable in the usual linear sense and has all its poles in the closed left-half plane.

It is notable that all the results mentioned above pertain to systems whose open-loop poles are all in the closed left-half plane. Such systems are said to be semistable. If a system has some open-loop poles in the open right-half plane, then it is exponentially unstable. A system with all its poles in the open right-half plane is said to be antistable. It is evident that the domain of attraction has to be a subset of the asymptotically null controllable region, the set of initial states that can be driven to the origin asymptotically with bounded controls delivered by the saturating actuators. Since the asymptotically null controllable region of a semistable system is the whole state space (if it is stabilizable in the linear sense), it is possible to stabilize it globally/semiglobally. However, the asymptotically null controllable region of an exponentially unstable system is not the whole state space, hence it cannot be globally/semiglobally stabilized with saturated feedback. For this reason, we generalized the notion of global/semiglobal stabilization, which was only suitable for semistable systems, by giving it a new meaning [2], [3]. A linear system subject to actuator saturation is globally stabilizable if there is a saturated feedback law such that the closed-loop system has a stability region which is equal to the asymptotically null controllable region; it is semiglobally stabilizable if, given an arbitrary compact subset of the asymptotically null controllable region, there is a saturated feedback law under which the closed-loop system has a stability region that includes this given compact set.

A first step toward global/semiglobal stabilization, which cannot be bypassed, is the characterization of the asymptotically null controllable region. We made this first step in [2], and then proceeded to construct stabilizing feedback laws for semiglobal stabilization. In [2] and [3], we developed simple feedback laws for systems with two antistable poles. For a second-order antistable system the controllers proposed are a family of saturated linear feedbacks of the form $u = \text{sat}(kF_0x)$ and for a high-order system with only two antistable poles, each controller in the family switches between two saturated linear feedbacks. In [2] and [4], we proposed a nonlinear switching feedback laws for more general systems. The controllers are more complicated than those of [3].

Given the results of [2]–[4], it is interesting to ask if a system with three or more antistable poles can be semiglobally stabilized with saturated linear feedback. In contrast to semistable systems, which can be semiglobally stabilized by saturated linear feedback, this note will show that a system with three or more antistable poles cannot be *semiglobally* stabilized by saturated linear feedback.

The remaining of this note is organized as follows. Section II reviews some results on the asymptotically null controllable region and develops some algebraic tools. Section III establishes the fact that a third-order antistable system cannot be semiglobally stabilized by saturated linear feedback. Some concluding remarks are given in Section IV.

II. Preliminary Results and Some Algebraic Tools

We recall from [2] a description of the asymptotically null controllable region. Consider a single-input linear system subject to actuator saturation

$$\dot{x} = Ax + bu, \quad x \in \mathbf{R}^n, u \in \mathbf{R}, |u| \le 1. \tag{1}$$

Assume that (A, b) is controllable in the usual linear sense. Then the asymptotically null controllable region is the same as the null controllable region, which is the set of initial states that can be driven to the origin in finite time. We use $\mathcal{C}$ to denote the null controllable region of system (1). It is known that if A is semistable, then $\mathcal{C} = \mathbf{R}^n$ and if A is antistable, then $\mathcal{C}$ is a bounded convex open set containing the origin in its interior. In this note, we will restrict our attention to third-order antistable systems with only real poles. For such a system, the boundary of the null controllable region is (see [2])

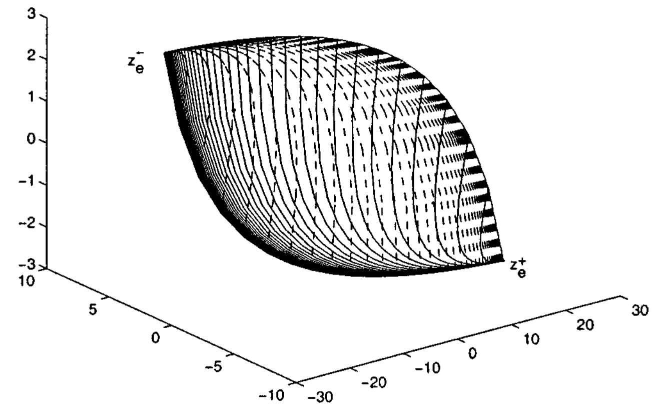

Fig. 1. $\partial\mathcal{C}$ of a third-order system.

$$\partial\mathcal{C} = \left\{ \pm \left(-2e^{-At_2} + 2e^{-At_1} - I \right) A^{-1}b : 0 \le t_1 \le t_2 \le \infty \right\}. \tag{2}$$

Fig. 1 illustrates a typical shape of $\mathcal{C}$ with a bunch of curves on $\partial\mathcal{C}$.

Denote

$$\partial\mathcal{C}^+ = \left\{ \left(-2e^{-At_2} + 2e^{-At_1} - I \right) A^{-1}b : 0 < t_1 < t_2 < \infty \right\}$$
$$\partial\mathcal{C}^- = -\,\partial\mathcal{C}^+$$
$$\partial\mathcal{C}^0 = \left\{ \left(\pm(-2e^{-At_2} + I \right) A^{-1}b : 0 \le t_2 \le \infty \right\}.$$

It can be verified that $\partial\mathcal{C} = \partial\mathcal{C}^+ \cup \partial\mathcal{C}^- \cup \partial\mathcal{C}^0$ and $\partial\mathcal{C}^+, \partial\mathcal{C}^-$ and $\partial\mathcal{C}^0$ are disjoint. In Fig. 1, the solid curves are on $\partial\mathcal{C}^+$ and the dashed ones are on $\partial\mathcal{C}^-$. The two smooth curves connecting z_e^+ and z_e^- (one on the highest boundary and the other on the lowest boundary) form $\partial\mathcal{C}^0$. It has been shown in [2] that all the curves are trajectories of the system (1) under controls that only take values of 1 and -1. They are called extremal trajectories. The solid curves on $\partial\mathcal{C}^+$ are trajectories of the system (1) under the control $u = 1$ and those on $\partial\mathcal{C}^-$ are trajectories under the control $u = -1$. The curve on the highest boundary is a trajectory going from z_e^- to z_e^+ under the control $u = -1$ and the one on the lowest boundary is a trajectory going from z_e^+ to z_e^- under the control $u = 1$.

From the definition, we see that $\partial\mathcal{C}^+$ and $\partial\mathcal{C}^-$ are smooth surfaces and $\partial\mathcal{C}^0$ is a closed curve connecting $\partial\mathcal{C}^+$ and $\partial\mathcal{C}^-$, i.e., $\partial\mathcal{C}^0$ is composed of all the common limit points of $\partial\mathcal{C}^+$ and $\partial\mathcal{C}^-$.

Since $\mathcal{C}$ is an open set, $\partial\mathcal{C} \cap \mathcal{C}$ is empty. Here, we summarize some facts from [2].

Fact 2.1: Under the constraint that $|u| \le 1$, the following hold true.

1) All the states in $\mathcal{C}$ can be driven to the origin.
2) All the states outside of $\mathcal{C} \cup \partial\mathcal{C}$ will grow unbounded no matter what control is applied.
3) All the states on $\partial\mathcal{C}$ cannot be driven to the interior of $\mathcal{C}$. The only way to keep them bounded is to make them stay on $\partial\mathcal{C}$ with a control $u = 1$ or $u = -1$. For $x(0) \in \partial\mathcal{C}^+$, the only control to keep $x(t), t \in [0, \varepsilon)$ for some $\varepsilon > 0$ on $\partial\mathcal{C}$ is $u = 1$ and for $x(0) \in \partial\mathcal{C}^-$, the only control is $u = -1$.

From Fact 2.1, we know that $\mathcal{C} \cup \partial\mathcal{C}$ is the largest bounded set that can be rendered invariant by means of admissible controls.

The basic fact we will use to prove our main result is that any segment on $\partial\mathcal{C}^0$ is three dimensional, i.e., it cannot be fit into any plane. Before proving this fact, we need an algebraic result which will be used several times in this note.

Lemma 2.1: Suppose that (A, b) is controllable, $A \in \mathbf{R}^{3\times 3}$ is antistable and A has no complex eigenvalues. Let t_1, t_2, t_3 be distinct real numbers. Then, for all $(k_1, k_2, k_3) \neq (0, 0, 0)$

$$\left(k_1 e^{At_1} + k_2 e^{At_2} + k_3 e^{At_3}\right) b \neq 0. \tag{3}$$

Proof: See Appendix A.

We note that, if A has complex eigenvalues $\alpha \pm j\beta$ and $(t_1, t_2, t_3) = (\pi N_1/\beta, \pi N_2/\beta, \pi N_3/\beta)$, where N_1, N_2 and N_3 are integers, there may exist $(k_1, k_2, k_3) \neq (0, 0, 0)$ that satisfy (3). For instance, suppose that A has eigenvalues $1, 1 \pm j1$. Let $t_1 = 0, t_2 = \pi, t_3 = 2\pi$, $k_1 = 1, k_2 = 0, k_3 = -e^{-2\pi}$, then $k_1 e^{At_1} + k_2 e^{At_2} + k_3 e^{At_3} = 0$.

Proposition 2.1: Let

$$x(t) = \left(-2e^{-At} + I\right) A^{-1} b.$$

If t_1, t_2, t_3 and t_4 are distinct numbers, then $x(t_i), i = 1, 2, 3, 4$, are not in the same plane.

Proof: For simplicity, assume that $t_1 < t_2 < t_3 < t_4$. We first show that $x(t_i), i = 1, 2, 3$, are not on the same straight line. Suppose, on the contrary, that they are, then

$$x(t_3) - x(t_2) = c(x(t_2) - x(t_1))$$

for some c, i.e.,

$$\left(e^{-At_3} - e^{-At_2}\right) A^{-1} b - c\left(e^{-At_2} - e^{-At_1}\right) A^{-1} b = 0.$$

This can be written as

$$A^{-1} e^{-At_3}\left(I - (1+c)e^{-A(t_3-t_2)} + ce^{A(t_3-t_1)}\right) b = 0$$

which contradicts Lemma 2.1. Here, we note that A and e^{At} commute.

Now that $x(t_i), i = 1, 2, 3$, are not on the same straight line, they uniquely determine a plane. Let this plane be $fx = 1$. Suppose, on the contrary, that $x(t_4)$ is also in this plane, then

$$fx(t_1) = fx(t_2) = fx(t_3) = fx(t_4) = 1.$$

By mean value theorem, there exist $t_1' \in (t_1, t_2), t_2' \in (t_2, t_3)$ and $t_3' \in (t_3, t_4)$ such that

$$f\dot{x}(t_1') = f\dot{x}(t_2') = f\dot{x}(t_3') = 0$$

which is equivalent to

$$fe^{-At_1'} b = fe^{-At_2'} b = fe^{-At_3'} b = 0.$$

This equality can be written as

$$f\left[e^{-At_1'} b \quad e^{-At_2'} b \quad e^{-At_3'} b\right] = 0$$

which implies that the 3×3 matrix

$$M = \left[e^{-At_1'} b \quad e^{-At_2'} b \quad e^{-At_3'} b\right]$$

is singular. This again contradicts Lemma 2.1. Hence we conclude that $x(t_i), i = 1, 2, 3, 4$, are not in the same plane. □

Recalling that

$$\partial\mathcal{C}^0 = \left\{\pm\left(-2e^{-At} + I\right) A^{-1} b : 0 \leq t \leq \infty\right\}$$

Proposition 2.1 implies that any segment of $\partial\mathcal{C}^0$ is three dimensional and cannot be placed in one plane.

III. Main Results

For an $x_0 \in \mathbf{R}^3$ and a positive number r, denote

$$\mathcal{B}(x_0, r) = \left\{x \in \mathbf{R}^3 : \|x - x_0\| \leq r\right\}$$

where $\|\cdot\|$ is the Euclidean norm. We use sat$(\cdot)$ to denote the standard saturation function, i.e., $\text{sat}(u) = \text{sign}(u)\min\{1, |u|\}$. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two subsets of $\mathbf{R}^n$. Then their distance is defined as

$$d(\mathcal{X}_1, \mathcal{X}_2) := \inf_{x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2} \|x_1 - x_2\|.$$

In terms of $\partial\mathcal{C}^+, \partial\mathcal{C}^-$ and $\partial\mathcal{C}^0$, the space $\mathbf{R}^3$ can be divided into three subsets

$$\mathbf{E}^+ = \left\{\gamma x : \gamma \in (0, \infty), x \in \partial\mathcal{C}^+\right\}$$
$$\mathbf{E}^- = -\mathbf{E}^+$$
$$\mathbf{E}^0 = \left\{\gamma x : \gamma \in [0, \infty), x \in \partial\mathcal{C}^0\right\}.$$

We see that $\mathbf{E}^+$ and $\mathbf{E}^-$ are simply connected open sets and $\mathbf{E}^0$ is a surface consisting of the common limit points of $\mathbf{E}^+$ and $\mathbf{E}^-$. In other words, $\mathbf{E}^+$ and $\mathbf{E}^-$ are connected by $\mathbf{E}^0$.

Theorem 3.1: A third-order antistable system with real poles cannot be semiglobally stabilized by saturated linear feedback on its null controllable region $\mathcal{C}$. Specifically, for any saturated linear feedback law $u = \text{sat}(Fx)$, there exists a ball $\mathcal{B}(x_*, r_*) \subset \mathcal{C}$, where $r_* > 0$ is independent of F, such that all the trajectories starting from $\mathcal{B}(x_*, r_*)$ will grow unbounded.

To prove Theorem 3.1, we first examine the difference between the control under a saturated linear feedback and the control that is required to keep the state bounded. Unlike the case for a second order system, here for a third order system, there is always a ball of fixed size where the difference between the two controls is greater than a fixed positive number. The result is stated in the following lemma.

Lemma 3.1: There exist positive numbers r_0 and d_0 such that for any $F \in \mathbf{R}^{1\times 3}$, there is a ball $\mathcal{B}(x_0, r_0)$, where $x_0 \in \partial\mathcal{C}^+$ such that $d(\mathcal{B}(x_0, r_0), \mathbf{E}^0) \geq d_0$ and $\text{sat}(Fx) \leq 1/2$ for all $x \in \mathcal{B}(x_0, r_0)$.

Proof: For an $F \in \mathbf{R}^{1\times 3}$, denote the distance between the two planes $Fx = 1$ and $Fx = -1$ by $g(F)$. By Proposition 2.1, $\partial\mathcal{C}^0$ is a three dimensional closed curve. So there exists a minimal distance g_0 between $Fx = 1$ and $Fx = -1$ such that $\partial\mathcal{C}^0$ lies completely between these two planes. In other words, if $g(F) = g_1 < g_0$, then the total length of the segments of $\partial\mathcal{C}^0$ which are in the half space $Fx \leq -1$ must be greater than a positive number depending only on g_1. Also by Proposition 2.1, there are no more than two segments of $\{x(t) = (-2e^{-At} + I)A^{-1}b : 0 \leq t \leq \infty\}$ which are in the half space, otherwise, there would be four points $x(t_i), i = 1, 2, 3, 4$, in the plane $Fx = -1$. Hence, there is a segment of $\partial\mathcal{C}^0$, with length greater than a fixed positive number, that is completely in the half space $Fx \leq -1$.

Let us first consider an F such that $g(F) \leq (1/2)g_0$. By the foregoing arguments, there is a segment of $\partial\mathcal{C}^0$, with length greater than $l_0 > 0$, that is completely in the half space $Fx \leq -1$. Since $\partial\mathcal{C}^0$ is a compact set and any segment on it is three dimensional, the largest distance from a point of the segment to the plane $Fx = -1$ is greater than a fixed positive number. Recalling that $\partial\mathcal{C}^0$ is the closed curve that connects $\partial\mathcal{C}^+$ and $\partial\mathcal{C}^-$, we know that there is a simply connected region on the surface of $\partial\mathcal{C}^+$ that is in the half space $Fx \leq -1$ and the surface area of this region is greater than a fixed-positive number. It follows that there exists a ball $\mathcal{B}(x_1, r_1)$ with $x_1 \in \partial\mathcal{C}^+$ and r_1 a fixed-positive number such that $\mathcal{B}(x_1, r_1)$ is in the half space $Fx \leq -1$ and $d(\mathcal{B}(x_1, r_1), \mathbf{E}^0) \geq d_1$, where d_1 is also a fixed positive number. Here, for $x \in \mathcal{B}(x_1, r_1)$, $\text{sat}(Fx) = -1$.

Next, we consider an F such that $g(F) > (1/2)g_0$. Then there exists a segment of $\partial\mathcal{C}^0$, of length greater than a fixed-positive number, which is between the two planes $Fx = -1/2$ and $Fx = 1/2$. Following similar arguments as in the previous paragraph, there is a ball $\mathcal{B}(x_2, r_2)$ with $x_2 \in \partial\mathcal{C}^+$ and r_2 a fixed-positive number such that $\mathcal{B}(x_2, r_2)$ is between the two planes $Fx = -1/2$ and $Fx = 1/2$ and $d(\mathcal{B}(x_2, r_2), \mathbf{E}^0) \geq d_2$, where d_2 is also a fixed positive number. Here, for $x \in \mathcal{B}(x_2, r_2)$, $|\text{sat}(Fx)| \leq 1/2$.

If we let $d_0 = \min\{d_1, d_2\}$ and $r_0 = \min\{r_1, r_2\}$, then the result of Lemma 3.1 readily follows. □

To prove Theorem 3.1, we need to show that, under the control of any feedback $u = \text{sat}(Fx)$, there exists a ball in $\mathcal{C}$ of radius greater than a fixed positive number, such that all the trajectories starting from the ball will go out of $\mathcal{C}$ and diverge. We will use Lyapunov function analysis to show this result. The Lyapunov function is defined in terms of $\partial\mathcal{C}$ as follows:

$$V(x) := \gamma \geq 0, \text{ such that } \frac{x}{\gamma} \in \partial\mathcal{C} \quad (\text{or } x \in \gamma\partial\mathcal{C}). \tag{4}$$

Since $\mathcal{C}$ is a bounded convex open set, any ray starting from the origin has a unique intersection with $\partial\mathcal{C}$ and hence any vector in the state space can be uniquely scaled to be exactly on $\partial\mathcal{C}$. Therefore, $V(x)$ is a well-defined positive–definite function.

Clearly, $V(x) = 1$ for all $x \in \partial\mathcal{C}$ and $V(x) < 1$ for all $x \in \mathcal{C}$. We also see that $V(\alpha x) = \alpha V(x)$ for any $\alpha > 0$. Moreover, if $\partial V/\partial x$ exists at some x_0, then

$$\left.\frac{\partial V}{\partial x}\right|_{x=\alpha x_0} = \left.\frac{\partial V}{\partial x}\right|_{x=x_0}. \tag{5}$$

To see this, note that

$$\frac{V\left(\alpha x_0 + \begin{bmatrix}\Delta\\0\\0\end{bmatrix}\right) - V(\alpha x_0)}{\Delta} = \frac{V\left(x_0 + \begin{bmatrix}\frac{1}{\alpha}\Delta\\0\\0\end{bmatrix}\right) - V(x_0)}{\frac{1}{\alpha}\Delta}$$

and the aforementioned equality is also true if we replace $\begin{bmatrix}\Delta\\0\\0\end{bmatrix}$ with $\begin{bmatrix}0\\\Delta\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\\Delta\end{bmatrix}$.

Lemma 3.2: The Lyapunov function $V(x)$ is continuously differentiable in x for $x \in \mathbf{E}^+$. For all $x \in \mathbf{E}^+$, $b^T(\partial V/\partial x) \neq 0$.

Proof: Every $x \in \mathbf{E}^+$ can be expressed as

$$x = \gamma\left(-2e^{-At_2} + 2e^{-At_1} - I\right)A^{-1}b$$

for some $\gamma \in (0,\infty)$, $0 < t_1 < t_2 < \infty$. It follows from the definition that $V(x) = \gamma$. We see that x is analytic in γ, t_1, and t_2. Moreover

$$dx = T\begin{bmatrix}d\gamma\\dt_1\\dt_2\end{bmatrix}$$

where

$$\begin{aligned}T &= \left[\frac{\partial x}{\partial\gamma}\,\frac{\partial x}{\partial t_1}\,\frac{\partial x}{\partial t_2}\right]\\ &= \left[\left(-2e^{-At_2} + 2e^{-At_1} - I\right)A^{-1}b \,\vdots - 2\gamma e^{-At_1}b \vdots 2\gamma e^{-At_2}b\right].\end{aligned}$$

We claim that T is nonsingular. This can be seen as follows. For simplicity, consider an $x_0 \in \partial\mathcal{C}^+$, then $\gamma = 1$. Applying Lemma 2.1, we see that $\partial x/\partial t_1 = -2e^{-At_1}b$ and $\partial x/\partial t_2 = 2e^{-At_2}b$ are independent and they determine a plane $fx = 1$ that contains x_0. Since $\partial\mathcal{C}^+$ is smooth, this plane is tangential to $\partial\mathcal{C}^+$ at x_0. Since $\mathcal{C}$ is a bounded convex set containing the origin in its interior, this plane $fx = 1$ does not contain the origin. Hence, the vector from x_0 to the origin, $-x_0 = -(-2e^{-At_2} + 2e^{-At_1} - I)A^{-1}b$, must be independent of the two vectors that determine the plane. That is, the three column vectors in T are independent.

Now that T is nonsingular, we have

$$\begin{bmatrix}d\gamma\\dt_1\\dt_2\end{bmatrix} = T^{-1}dx.$$

It is also clear that T and T^{-1} are continuous. Hence, γ, t_1 and t_2 are continuously differentiable in x for $x \in \mathbf{E}^+$. Therefore, $V(x) = \gamma$ is continuously differentiable in x for $x \in \mathbf{E}^+$.

Noting that $V(x) = \gamma$, we have

$$\frac{\partial V}{\partial x} = \left([1 \quad 0 \quad 0]\,T^{-1}\right)^T$$

and

$$\left(\frac{\partial V}{\partial x}\right)^T b = [1 \quad 0 \quad 0]\,T^{-1}b.$$

Suppose on the contrary that $(\partial V/\partial x)^T b = 0$, then

$$T^{-1}b = \begin{bmatrix}0\\k_1\\k_2\end{bmatrix}$$

for some $(k_1, k_2) \neq (0, 0)$. It follows that:

$$b = T\begin{bmatrix}0\\k_1\\k_2\end{bmatrix} = -2k_1\gamma e^{-At_1}b + 2k_2\gamma e^{-At_2}b$$

which contradicts Lemma 2.1. Therefore, we must have $(\partial V/\partial x)^T b \neq 0$ for all $x \in \mathbf{E}^+$. □

Since $(\partial V/\partial x)^T b$ is continuous in $\mathbf{E}^+$, from Lemma 3.2, we conclude that $(\partial V/\partial x)^T b$ either "> 0" or "< 0" in $\mathbf{E}^+$ and for any compact subset of $\mathbf{E}^+$, $|(\partial V/\partial x)^T b|$ is greater than a positive number. Now we are ready to prove the main result of this note.

Proof of theorem 3.1: Let $F \in \mathbf{R}^{1\times 3}$ be an arbitrary feedback gain matrix. From Lemma 3.1, we know that there always exists a ball $\mathcal{B}(x_0, r_0)$, $x_0 \in \partial\mathcal{C}^+$, such that $\text{sat}(Fx) \leq 1/2$ for all $x \in \mathcal{B}(x_0, r_0)$ and $d(\mathcal{B}(x_0, r_0), \mathbf{E}^0) \geq d_0$. Here, d_0 and r_0 are independent of F.

Since $\mathcal{B}(x_0, r_0)$ contains one point x_0 in $\mathbf{E}^+$ and has a distance greater than d_0 from $\mathbf{E}^0$, we have $\mathcal{B}(x_0, r_0) \subset \mathbf{E}^+$. Without loss of generality, assume that $\mathcal{B}(x_0, r_0) \subset \mathbf{E}^+ \cap (5/4)\mathcal{C} \setminus (3/4)\mathcal{C}$. Otherwise, we can choose a smaller r_0. Let $\mathcal{M}$ be the maximal compact set in $\mathbf{E}^+ \cap (5/4)\mathcal{C} \setminus (3/4)\mathcal{C}$ such that $d(\mathcal{M}, \mathbf{E}^0) = d_0$. By Lemma 3.2, there is a positive number η such that

$$\min\left\{\left|\left(\frac{\partial V}{\partial x}\right)^T b\right| : x \in \mathcal{M}\right\} \geq \eta.$$

Since $d(\mathcal{B}(x_0, r_0), \mathbf{E}^0) \geq d_0$, we must have $\mathcal{B}(x_0, r_0) \subset \mathcal{M}$. Therefore

$$\left|\left(\frac{\partial V}{\partial x}\right)^T b\right| \geq \eta, \ \forall x \in \mathcal{B}(x_0, r_0). \tag{6}$$

Consider the derivative of the Lyapunov function $V(x)$ along the trajectory of the system

$$\dot{x} = Ax + b\,u.$$

We have

$$\dot{V}(x, u) = \left(\frac{\partial V}{\partial x}\right)^T Ax + \left(\frac{\partial V}{\partial x}\right)^T bu.$$

From Fact 2.1, we know that if a control $u = 1$ is applied at $x \in \partial\mathcal{C}^+$, the trajectory will stay on $\partial\mathcal{C}^+$ and $V(x)$ will remain to be 1. Hence, for $x \in \partial\mathcal{C}^+$

$$\dot{V}(x, 1) = \left(\frac{\partial V}{\partial x}\right)^T Ax + \left(\frac{\partial V}{\partial x}\right)^T b = 0. \tag{7}$$

Since $(\partial V/\partial x)^T b \neq 0$ and any control $u < 1$ is unable to bring a state on $\partial\mathcal{C}^+$ into a smaller level set $\gamma_1\mathcal{C}$, $\gamma_1 < 1$, we must have

$$\dot{V}(x, u) = \left(\frac{\partial V}{\partial x}\right)^T Ax + \left(\frac{\partial V}{\partial x}\right)^T bu > 0$$

for all $u < 1$. This implies that $(\partial V/\partial x)^T b < 0$ for $x \in \partial\mathcal{C}^+$ and also for $x \in \mathbf{E}^+$. It follows from (6) that:

$$\left(\frac{\partial V}{\partial x}\right)^T b < -\eta, \quad \forall\, x \in \mathcal{B}(x_0, r_0). \tag{8}$$

By (5) and (7), if a control $u = \gamma = V(x)$ is applied at $x \in \gamma\partial\mathcal{C}^+$, we will have

$$\left(\frac{\partial V}{\partial x}\right)^T Ax + \left(\frac{\partial V}{\partial x}\right)^T bV(x) = 0. \tag{9}$$

Now consider the system under the saturated linear feedback $u = \text{sat}(Fx)$. Recalling from Lemma 3.1 that $\text{sat}(Fx) \le 1/2$ and $V(x) \ge 3/4$ for all $x \in \mathcal{B}(x_0, r_0)$, we have

$$\text{sat}(Fx) - V(x) \le \frac{1}{2} - \frac{3}{4} = -\frac{1}{4}. \tag{10}$$

It follows from (9), (8), and (10) that:

$$\begin{aligned}\dot V(x, \text{sat}(Fx)) &= \left(\frac{\partial V}{\partial x}\right)^T Ax + \left(\frac{\partial V}{\partial x}\right)^T b\,\text{sat}(Fx)\\ &= \left(\frac{\partial V}{\partial x}\right)^T Ax + \left(\frac{\partial V}{\partial x}\right)^T bV(x)\\ &\quad + \left(\frac{\partial V}{\partial x}\right)^T b(\text{sat}(Fx) - V(x))\\ &= \left(\frac{\partial V}{\partial x}\right)^T b(\text{sat}(Fx) - V(x))\\ &> \frac{\eta}{4}\end{aligned}$$

for all $x \in \mathcal{B}(x_0, r_0)$. We see that there exists a positive number N such that $\|\dot x\| \le N$ for all $x \in (5/4)\mathcal{C}$ under any saturated feedback control. Hence, there exists a $t_0 \in (0, 4/\eta)$ and an $r_1 \in (0, r_0)$ independent of x_0, such that all the trajectories starting from $\mathcal{B}(x_0, r_1)$ will stay inside $\mathcal{B}(x_0, r_0)$ for $t \in [0, t_0]$. Therefore, $V(x(t_0)) - V(x(0)) \ge \eta t_0/4$. Also, there exists a $r_2 \in (0, r_1)$ such that

$$\mathcal{B}(x_0, r_2) \subset \mathbf{E}^+ \setminus \left(1 - \frac{\eta t_0}{4}\right)\mathcal{C}.$$

Clearly, for all $x(0) \in \mathcal{B}(x_0, r_2)$, $V(x(0)) > 1 - \eta t_0/4$. Hence, for any trajectory starting from $\mathcal{B}(x_0, r_2)$, we will have $V(x(t_0)) > V(x(0)) + \eta t_0/4 > 1$, which means that the trajectory has gone out of $\partial\mathcal{C}$ at t_0 and will diverge by Fact 2.1.

It is easy to see that there exists a ball $\mathcal{B}(x_*, r_*) \subset \mathcal{B}(x_0, r_2) \cap \mathcal{C}$, with r_* greater than a fixed positive number. In summary, no matter what F is, there always exists a ball $\mathcal{B}(x_*, r_*) \subset \mathcal{C}$ from which the trajectories will diverge under the saturated linear feedback $u = \text{sat}(Fx)$. This completes the proof. □

IV. Conclusion

We have shown in this note that a third-order antistable system with real eigenvalues cannot be semiglobally stabilized with saturated linear feedback. The study is based on examining a Lyapunov function defined in terms of the null controllable region. The level sets of the Lyapunov function are the null controllable region scaled by positive numbers. The main idea is to show the existence of a ball inside the null controllable region, with radius greater than a fixed-positive number, from which the Lyapunov function will grow unbounded. The increasing of the Lyapunov function is caused by the difference between the control $u = \text{sat}(Fx)$ and the one that is required to keep the state within a level set. The difference between the two controls cannot be reduced to an arbitrarily small level because the two surfaces $\partial\mathcal{C}^+$ and $\partial\mathcal{C}^-$ cannot be separated with a plane, or, the closed curve that connects these two surfaces is three dimensional. For systems with complex eigenvalues, if we define $\partial\mathcal{C}^+$ to be the set of states on $\partial\mathcal{C}$ which can only be kept on $\partial\mathcal{C}$ by $u = 1$ and $\partial\mathcal{C}^-$ to be the set of states on $\partial\mathcal{C}$ which can only be kept on $\partial\mathcal{C}$ by $u = -1$, then intuitively, these two surfaces are not separable with a plane. With a similar procedure, the negative result in this note can be extended to systems with complex eigenvalues, although it is somewhat harder to characterize the curve that separates $\partial\mathcal{C}^+$ and $\partial\mathcal{C}^-$.

Appendix
Proof of Lemma 2.1

For simplicity, we assume that the smallest t_i is t_3 and $t_3 = 0$. Otherwise, we can multiply (3) from left with e^{-At_3}. We also assume that $t_2 > t_1 > 0$.

We will first show that

$$k_1 e^{At_1} + k_2 e^{At_2} + k_3 I \ne 0,\ \forall\,(k_1, k_2, k_3) \ne (0, 0, 0). \tag{11}$$

We assume that A has three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$, with $0 < \lambda_1 < \lambda_2 < \lambda_3$. For the case where A has two or three identical eigenvalues, we can prove the result in a simpler way using similar ideas. We further assume that $A = \text{diag}[\lambda_1, \lambda_2, \lambda_3]$. Then (11) can be reorganized as

$$\begin{bmatrix} e^{\lambda_1 t_1} & e^{\lambda_1 t_2} & 1\\ e^{\lambda_2 t_1} & e^{\lambda_2 t_2} & 1\\ e^{\lambda_3 t_1} & e^{\lambda_3 t_2} & 1\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} \ne 0,\ \forall(k_1, k_2, k_3) \ne (0, 0, 0). \tag{12}$$

This is equivalent to

$$\det\begin{bmatrix} e^{\lambda_1 t_1} & e^{\lambda_1 t_2} & 1\\ e^{\lambda_2 t_1} & e^{\lambda_2 t_2} & 1\\ e^{\lambda_3 t_1} & e^{\lambda_3 t_2} & 1\end{bmatrix} \ne 0. \tag{13}$$

Direct computation shows that

$$\det\begin{bmatrix} e^{\lambda_1 t_1} & e^{\lambda_1 t_2} & 1\\ e^{\lambda_2 t_1} & e^{\lambda_2 t_2} & 1\\ e^{\lambda_3 t_1} & e^{\lambda_3 t_2} & 1\end{bmatrix} = \left(e^{\lambda_2 t_1} - e^{\lambda_1 t_1}\right)\left(e^{\lambda_3 t_2} - e^{\lambda_1 t_2}\right) - \left(e^{\lambda_2 t_2} - e^{\lambda_1 t_2}\right)\left(e^{\lambda_3 t_1} - e^{\lambda_1 t_1}\right).$$

We claim that

$$\frac{e^{\lambda_3 t_2} - e^{\lambda_1 t_2}}{e^{\lambda_3 t_1} - e^{\lambda_1 t_1}} > \frac{e^{\lambda_2 t_2} - e^{\lambda_1 t_2}}{e^{\lambda_2 t_1} - e^{\lambda_1 t_1}} \tag{14}$$

from which (13) and (11) will follow.

We now proceed to prove (14). Define

$$f(\lambda) := \frac{e^{\lambda t_2} - e^{\lambda_1 t_2}}{e^{\lambda t_1} - e^{\lambda_1 t_1}} \times \frac{e^{\lambda_1 t_1}}{e^{\lambda_1 t_2}} = \frac{e^{(\lambda-\lambda_1)t_2} - 1}{e^{(\lambda-\lambda_1)t_1} - 1}.$$

It suffices to show that $f(\lambda)$ is an increasing function of λ for $\lambda > \lambda_1$, or, equivalently, that

$$f_1(\lambda) = \frac{e^{\lambda t_2} - 1}{e^{\lambda t_1} - 1}$$

is an increasing function of λ for $\lambda > 0$. Let $g(\lambda) = (df_1/d\lambda)\left(e^{\lambda t_1} - 1\right)^2$, then

$$\begin{aligned} g(\lambda) &= (t_2 - t_1)e^{\lambda(t_2+t_1)} - t_2 e^{\lambda t_2} + t_1 e^{\lambda t_1}\\ &= e^{\lambda t_1}\left((t_2 - t_1)e^{\lambda t_2} - t_2 e^{\lambda(t_2 - t_1)} + t_1\right).\end{aligned}$$

Let

$$g_1(\lambda) = g(\lambda)e^{-\lambda t_1} = (t_2 - t_1)e^{\lambda t_2} - t_2 e^{\lambda(t_2 - t_1)} + t_1.$$

Then

$$\frac{dg_1}{d\lambda} = (t_2 - t_1)t_2\left(e^{\lambda t_2} - e^{\lambda(t_2 - t_1)}\right) > 0,\ \forall\,\lambda > 0.$$

Since $g_1(0) = 0$, it follows that $g_1(\lambda) > 0$ and hence $g(\lambda) > 0$ for all $\lambda > 0$. Therefore, $df_1/d\lambda > 0$ for all $\lambda > 0$ and hence $f_1(\lambda)$ is an increasing function of λ. It follows that (14) and (11) are true.

We next show (3). Suppose, on the contrary, that there exist $(k_1, k_2, k_3) \neq (0, 0, 0)$ such that

$$\left(k_1 e^{At_1} + k_2 e^{At_2} + k_3 I\right) b = 0.$$

Noting that

$$k_1 e^{At_1} + k_2 e^{At_2} + k_3 I = \gamma_1 A^2 + \gamma_2 A + \gamma_3 I$$

for some $(\gamma_1, \gamma_2, \gamma_3) \neq (0, 0, 0)$, we would have

$$[A^2 b \quad Ab \quad b] \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = 0$$

which contradicts the assumption that (A, b) is controllable. □

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Adaptive Control of Robots With an Improved Transient Performance

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***Abstract*—By using a robust control technique, this note proposes an adaptive control for rigid robots with the following important features: under a parameter-dependent persistent excitation (PE) condition, it gives a guaranteed transient performance of tracking a smooth desired trajectory while assuring the parameter estimation error to go to a residual set of the origin arbitrarily fast. Simulations are included to support the theoretical results.**

***Index Terms*—Adaptive control, robot manipulators, transient performance.**

I. Introduction

The dynamics of rigid robots can be described by a set of nonlinear differential equations. In order to be able to carry out accurate tracking control, the knowledge of the robot model parameters is necessary. However, it is a rather difficult task to calculate the parameter vector accurately. Fortunately, the nonlinear model of rigid robots is linear in its parameters [2], [3]. Thus, adaptive control of robots has received considerable attention during the last two decades (see [4]–[9]). Since the main goal of robot control is to achieve accurate tracking of desired trajectories, many globally stable algorithms have been developed that result in zero tracking error in the steady state (see [10] and [11]). Nevertheless, parameter convergence does not necessarily take place. In fact, even if there is persistent excitation (PE) it may take long time before the estimated parameters tend to the real ones, what decreases the transient performance of the tracking error. Also, without the PE condition being satisfied, in the presence of external perturbations and/or unmodeled dynamics most of the existing adaptation algorithms may present parameter drifting phenomena similar to those observed in adaptive controllers studied in the 1980s. A solution to this consists in modifying the adaptation algorithms.

Since robot manipulators constitute a class of passive systems, many authors have exploited this property in order to design and prove the stability of their adaptive control approaches [12]–[14]. In fact, the passivity property can be exploited in a very general framework to design adaptive algorithms, i.e., it can be shown that any adaptive algorithm which is passive can be used to have zero tracking error in the steady state [15]. This fact was used in [16] to slightly relax the general PE condition required for most algorithms in order to guarantee parameter convergence by taking advantage of the transient response of the system. Since a well-known problem in most adaptive controllers is the poor transient response observed when the adaptation is initiated, [17]–[19] present adaptive schemes which give a guaranteed transient performance. In the absence of disturbances, these algorithms are able to guarantee that the tracking errors will tend to zero as well. The transient performance is improved arbitrarily by a proper choice of some control gains. However, tuning these control gains too large to improve the transient performance usually implies that the output torques/forces

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