Composite Quadratic Lyapunov Functions for Constrained Control Systems

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***Abstract*—A Lyapunov function based on a set of quadratic functions is introduced in this paper. We call this Lyapunov function a composite quadratic function. Some important properties of this Lyapunov function are revealed. We show that this function is continuously differentiable and its level set is the convex hull of a set of ellipsoids. These results are used to study the set invariance properties of continuous-time linear systems with input and state constraints. We show that, for a system under a given saturated linear feedback, the convex hull of a set of invariant ellipsoids is also invariant. If each ellipsoid in a set can be made invariant with a bounded control of the saturating actuators, then their convex hull can also be made invariant by the same actuators. For a set of ellipsoids, each invariant under a separate saturated linear feedback, we also present a method for constructing a nonlinear continuous feedback law which makes their convex hull invariant.**

***Index Terms*—Constrained control, invariant set, quadratic functions.**

I. Introduction

WE CONSIDER linear systems subject to input saturation and state constraint. Control problems for these systems have attracted tremendous attention in recent years because of their practical significance and the theoretical challenges (see, e.g., [1], [11], [20]–[22], and the references therein). For linear systems with input saturation, global and semiglobal stabilization results have been obtained for semistable systems[1] (see, e.g., [17], [18], and [24]–[27]) and systems with two anti-stable poles (see [11] and [14]). For more general systems with both input saturation and state constraint, there are numerous research reports on their stability analysis and design (see [4], [7], [8], [11], [20], [28], and the references therein). While analytical characterization of the domain of attraction and the maximal invariant set has been attempted and is believed to be extremely hard except for some special cases (see, e.g., [14]), most of the literature is dedicated to obtaining an estimate of the domain of attraction with reduced conservatism or to enlarging some invariant set inside the domain of attraction. Along this direction, the notion of set invariance has played a very important role (see, e.g., [2], [3], and [28]). The most commonly used invariant sets for continuous-time systems are invariant ellipsoids, resulting from the level sets of quadratic Lyapunov functions. The problem of estimating the domain of attraction by using invariant ellipsoids has been extensively studied, e.g., in [5]–[7], [9], [10], [19], and [28]. More recently, we developed a new sufficient condition for an ellipsoid to be invariant in [13] (see also [11]). It was shown that this condition is less conservative than the existing conditions resulting from the circle criterion or the vertex analysis. The most important feature of this new condition is that it can be expressed as linear matrix inequalities (LMIs) in terms of all the varying parameters and hence can be easily used for controller synthesis. A recent discovery makes this condition even more attractive. In [12], we showed that, for single input systems, this condition is also necessary. Thus, the largest invariant ellipsoid obtained with the LMI approach is actually the largest one.

In this paper, we will introduce a new type of Lyapunov function which is based on a set of quadratic functions. This is motivated by problems arising from estimating the domain of attraction and constructing controllers to enlarge the domain of attraction. Suppose that there are a set of invariant ellipsoids of the closed-loop system under a saturated feedback law. It is clear that the union of this set of ellipsoids is also an invariant set of the closed-loop system. The question whether the convex hull of this set of ellipsoids, a set potentially much larger than the union, is invariant remains unclear. Another problem is related to enlarging the domain of attraction by merging two or more feedback laws. Suppose that we have two ellipsoids, each of which is invariant under a separate feedback law. In [15], we showed that a switching feedback law can be constructed to make the union of the two ellipsoids invariant. We would further like to make the convex hull of these ellipsoids invariant, possibly with a continuous feedback law. Although the discontinuity of the switching feedback law in [15] does not cause chattering, a continuous feedback law would be more appealing.

Construction of Lyapunov functions is one of the most fundamental problems in system theory. One type of Lyapunov functions that are constructed from quadratic functions are piecewise quadratic functions [16], which may not be continuously differentiable and whose level sets may not be convex. For discrete-time systems, piecewise-linear and piecewise-affine Lyapunov functions are popular choices (see, e.g., [2] and [23]). In this paper, the Lyapunov function is defined in such a way that its level set is the convex hull of a set of ellipsoids. A nice feature of this function is that it is continuously differentiable. This makes it possible to construct continuous feedback laws based on the gradient of the function or on a given set of linear feedback laws.

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[1]A linear system is said to be semistable if all its poles are in the closed left-half plane.

The composite quadratic function is motivated from the study of control systems with saturating actuators and state constraints. It is a potential tool to handle more general nonlinearities.

This paper is organized as follows. In Section II, we introduce the composite quadratic Lyapunov function and show that this function is continuously differentiable and its level set is the convex hull of a set of ellipsoids. In Sections II–V, we use these properties of the Lyapunov function to study the set invariance of continuous-time linear systems with input and state constraints. In particular, we will show in Section III that under a given saturated linear feedback, the convex hull of a set of invariant ellipsoids is also invariant. In Section IV, we will study the controlled invariance of the convex hull. In Section V, we will present a method for constructing a nonlinear continuous controller which makes the convex hull invariant. Section VI draws the conclusions to this paper.

Notation: We use $\mathrm{sat}(\cdot)$ to denote the standard vector valued saturation function. For $u \in \mathbf{R}^m$, the ith component of $\mathrm{sat}(u)$ is $\{\mathrm{sat}(u)\}_i = \mathrm{sign}(u_i)\min\{1, |u_i|\}$. We use $|u|_\infty$ and $|u|_2$ to denote respectively the infinity norm and the 2-norm. For two integers $k_1, k_2, k_1 < k_2$, we denote $I[k_1, k_2] := \{k_1, k_1 + 1, \ldots, k_2\}$.

For a positive–definite (semidefinite) matrix P, we denote it as $P > 0$ $(P \geq 0)$. When we say positive–definite (semidefinite), it is implied that the matrix is symmetric. For a $P \in \mathbf{R}^{n\times n}$, $P > 0$, and a $\rho \in (0, \infty)$, denote

$$\mathcal{E}(P, \rho) := \{x \in \mathbf{R}^n : x^T P x \leq \rho\}.$$

For simplicity, we use $\mathcal{E}(P)$ to denote $\mathcal{E}(P, 1)$. For a matrix $F \in \mathbf{R}^{m\times n}$, denote the ith row of F as f_i and define

$$\mathcal{L}(F) := \{x \in \mathbf{R}^n : \ |f_i x| \leq 1, i \in I[1, m]\}.$$

If F is the feedback matrix, then $\mathcal{L}(F)$ is the region in the state space where the control $u = \mathrm{sat}(Fx)$ is linear in x. For an $x_0 \in \mathbf{R}^n$ and an $r \geq 0$, denote $\mathcal{B}(x_0, r) = \{x \in \mathbf{R}^n : \ |x - x_0|_2 \leq r\}$.

II. Composite Quadratic Lyapunov Function

A. *Definition and General Properties*

With a positive–definite matrix $P \in \mathbf{R}^{n\times n}$, a quadratic function can be defined as $V(x) = x^T P x$. For a positive number ρ, a level set of $V(\cdot)$, denoted $L_V(\rho)$, is

$$L_V(\rho) := \{x \in \mathbf{R}^n : V(x) \leq \rho\} = \mathcal{E}(P, \rho).$$

In this paper, we are interested in a function determined by a set of positive–definite matrices $P_1, P_2, \ldots, P_N \in \mathbf{R}^{n\times n}$. Let $Q_j = P_j^{-1}$, $j \in I[1, N]$. For a vector $\gamma \in \mathbf{R}^N$, define

$$Q(\gamma) := \sum_{j=1}^{N} \gamma_j Q_j \quad P(\gamma) := Q^{-1}(\gamma).$$

Let

$$\Gamma = \left\{\gamma \in \mathbf{R}^N : \sum_{j=1}^{N} \gamma_j = 1, \gamma_j \geq 0, j \in I[1, N]\right\}.$$

It is easy to see that $Q(\gamma), P(\gamma) > 0$ for all $\gamma \in \Gamma$ and these two matrix functions are analytic in $\gamma \in \Gamma$. The composite quadratic function is defined as

$$V_c(x) := \min_{\gamma\in\Gamma} x^T P(\gamma) x. \tag{1}$$

Clearly, $V_c(\cdot)$ is a positive–definite function. For $\rho > 0$, the level set of $V_c(\cdot)$ is

$$L_{V_c}(\rho) := \{x \in \mathbf{R}^n : V_c(x) \leq \rho\}.$$

A very useful property of this composite quadratic function is that its level set is the convex hull of the level sets of $x^T P_j x$, the ellipsoids $\mathcal{E}(P_j, \rho)$, $j \in I[1, N]$. Another nice property of $V_c(\cdot)$ is that it is continuously differentiable. In order to establish these results, we need some simple preliminaries which will be useful throughout this paper.

Fact 1 [11]: For a row vector $f_0 \in \mathbf{R}^{1\times n}$ and a matrix $P > 0$, $\mathcal{E}(P) \subset \mathcal{L}(f_0)$ if and only if

$$f_0 P^{-1} f_0^T \leq 1 \Longleftrightarrow \begin{bmatrix} 1 & f_0 P^{-1} \\ P^{-1} f_0^T & P^{-1} \end{bmatrix} \geq 0.$$

1) The equality $f_0 P^{-1} f_0^T = 1$ holds if and only if the ellipsoid $\mathcal{E}(P)$ touches the hyperplane $f_0 x = 1$ at $x_0 = P^{-1} f_0^T$ (the only intersection), i.e.,
$$1 = f_0 x_0 > f_0 x \quad \forall\, x \in \mathcal{E}(P) \setminus \{x_0\}.$$
2) If $f_0 P^{-1} f_0^T < 1$, then
$$\begin{bmatrix} 1 & f_0 P^{-1} \\ P^{-1} f_0^T & P^{-1} \end{bmatrix} > 0$$
and the ellipsoid $\mathcal{E}(P)$ lies strictly between the hyperplanes $f_0 x = 1$ and $f_0 x = -1$ without touching them.

A dual result, which will be useful, can be obtained by exchanging the roles of f_0 and x_0^T. Given x_0 and suppose that $x_0^T P x_0 = 1$, $f_0^T = P x_0$, then

$$1 = f_0 x_0 > f x_0 \quad \forall\, f^T \in \mathcal{E}(P^{-1}) \setminus \{f_0^T\}.$$

For an $F \in \mathbf{R}^{m\times n}$, $\mathcal{L}(F) = \cap_{i=1}^m \mathcal{L}(f_i)$. The relation $\mathcal{E}(P) \subset \mathcal{L}(F)$ holds if and only if $f_i P^{-1} f_i^T \leq 1$ for all $i \in I[1, m]$ [11]. Denote the convex hull of the ellipsoids $\mathcal{E}(P_j, \rho)$, $j \in I[1, N]$, as

$$\mathrm{co}\,\{\mathcal{E}(P_j, \rho), j \in I[1, N]\} := \left\{\sum_{j=1}^{N} \gamma_j x_j : \ x_j \in \mathcal{E}(P_j, \rho), \gamma \in \Gamma\right\}.$$

Then, we have the following.

Theorem 1:

a) $L_{V_c}(\rho) = \mathrm{co}\,\{\mathcal{E}(P_j, \rho), j \in I[1, N]\} = \bigcup_{\gamma\in\Gamma} \mathcal{E}(P(\gamma), \rho)$.
b) The function $V_c(\cdot)$ is continuously differentiable. Let $\gamma^*(x)$ be an optimal γ such that $x^T P(\gamma^*(x)) x = \min_{\gamma\in\Gamma} x^T P(\gamma) x$, then
$$\frac{\partial V_c}{\partial x} = 2P(\gamma^*(x))x.$$

Proof: See Appendix A.1. □

Remark 1: Let us justify our definition of the composite quadratic function $V_c(\cdot)$. With a set of matrices $P_1, P_2, \ldots, P_N > 0$, there are different ways to generate positive–definite functions. For example, we can define three other functions in a way similar to $V_c(\cdot)$ as follows:

$$V_{c1}(x) := \max_{\gamma\in\Gamma} x^T P(\gamma) x = \max_{\gamma\in\Gamma} x^T \left(\sum_{j=1}^{N} \gamma_j Q_j\right)^{-1} x$$

$$V_{c2}(x) := \min_{\gamma\in\Gamma} x^T \left(\sum_{j=1}^{N} \gamma_j P_j\right) x$$

$$V_{c3}(x) := \max_{\gamma\in\Gamma} x^T \left(\sum_{j=1}^{N} \gamma_j P_j\right) x.$$

It is easy to see that

$$V_{c2}(x) = \min\left\{x^T P_j x : j \in I[1,N]\right\}$$
$$V_{c3}(x) = \max\left\{x^T P_j x : j \in I[1,N]\right\}.$$

As to $V_{c1}(\cdot)$, we note that for a fixed x, $x^T P(\gamma)x$ is a convex function of γ (this can be verified by Schur complement). Hence, its maximum is attained at the vertices of Γ. It follows that $V_{c1}(\cdot) = V_{c3}(\cdot)$. The computation of these functions is easy and straightforward, but they are not well behaved as compared with $V_c(\cdot)$. It can be verified that the level set of $V_{c1}(\cdot)$ and $V_{c3}(\cdot)$ is the intersection of the ellipsoids $\mathcal{E}(P_j, \rho), j \in I[1,N]$, and the level set of $V_{c2}(\cdot)$ is the union of these ellipsoids. Both of these level sets have nonsmooth surfaces and the functions $V_{c1}(\cdot)$, $V_{c2}(\cdot)$ and $V_{c3}(\cdot)$ have nondifferentiable points. $\diamond$

B. Computational Issues

Next, we consider some computational issues with regard to the function $V_c(\cdot)$. From the definition of $V_c(\cdot)$, we have

$$V_c(x) = \min\{\alpha : \alpha \geq x^T P(\gamma)x \text{ for some } \gamma \in \Gamma\}.$$

By the Schur complement, we obtain

$$V_c(x) = \min_{\gamma} \alpha \quad (2)$$
$$\text{s.t.} \begin{bmatrix} \alpha & x^T \\ x & \sum_{j=1}^{N} \gamma_j Q_j \end{bmatrix} \geq 0$$
$$\sum_{j=1}^{N} \gamma_j = 1, \qquad \gamma_j \geq 0$$

which is an optimization problem with linear matrix inequality (LMI) constraints and can be easily solved with the techniques in [3].

We see that the optimal value of γ is $\gamma^*(x)$ such that $V_c(x) = x^T P(\gamma^*(x))x$. In some situations, the optimal value of γ is not unique. For example, this may happen if some Q_j can be expressed as the convex combination of other matrices in the set.

Fig. 1 illustrates a two-dimensional level set which is the convex hull of three ellipsoids. Fig. 2 plots the values of $\gamma^*(x) = (\gamma_1^*(x), \gamma_2^*(x), \gamma_3^*(x))$ as x varies along the boundary of $L_{V_c}(1)$ in the counterclockwise direction, where the abscissa is the angle of x (from 0 to π). From Fig. 1, we see that parts of $\partial L_{V_c}(1)$ overlap with segments of $\partial\mathcal{E}(P_i)$, $i = 1,2,3$. The overlapped segments correspond to the intervals in Fig. 2, where $\gamma_i^*(x) = 1$ for some i. Fig. 3 illustrates a three dimensional level set. It is also the convex hull of three ellipsoids.

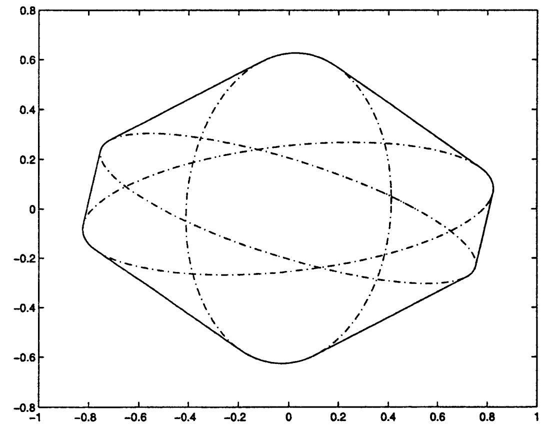

Fig. 1. Two-dimensional level set $L_{V_c}(1)$.

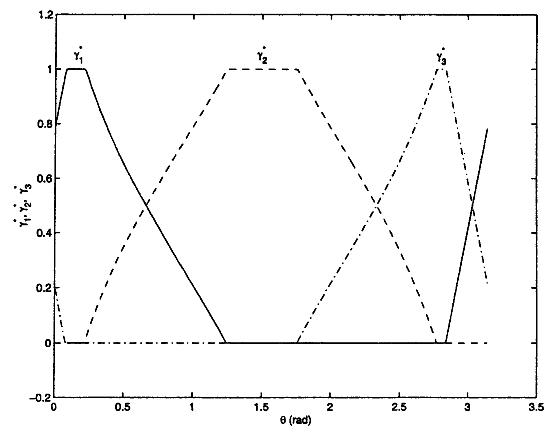

Fig. 2. γ^* along the boundary of $L_{V_c}(1)$.

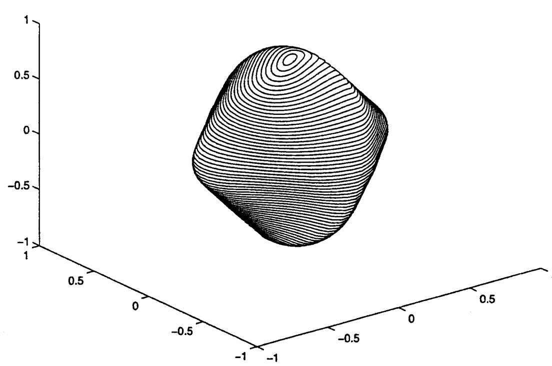

Fig. 3. Three-dimensional level set $L_{V_c}(1)$.

C. Special Case: Two Ellipsoids

If we only have two ellipsoids, there exists a more efficient way to obtain $V_c(x)$ through computing the generalized eigenvalues of certain matrices. In this case, we have

$$V_c(x) = \min_{\lambda\in[0,1]} x^T(\lambda Q_1 + (1-\lambda)Q_2)^{-1}x.$$

Denote $\alpha(\lambda, x) = x^T(\lambda Q_1 + (1-\lambda)Q_2)^{-1}x$.

Proposition 1: Assume that $Q_1 - Q_2$ is nonsingular. For every $x \in \mathbf{R}^n$, the function $\alpha(\cdot, x) : [0,1] \to \mathbf{R}$ is strictly convex and there exists a unique $\lambda^*(x) \in [0,1]$ such that $\alpha(\lambda^*(x), x) = \min_{\lambda \in [0,1]} \alpha(\lambda, x)$. Moreover, $\lambda^* : \mathbf{R}^n \to \mathbf{R}$ is a continuous function.

Proof: See Appendix A.2. □

Remark 2: The assumption that $Q_1 - Q_2$ is nonsingular is without loss of generality. For the case where $n = 2$, $\det(Q_1 - Q_2) = 0$ implies that either $Q_1 \geq Q_2$ or $Q_1 \leq Q_2$. If $Q_1 \geq Q_2$, then $P_1 \leq P_2$ and $V_c(\cdot) = V_1(\cdot)$, which is trivial. ◇

By Proposition 1, $\gamma = (\lambda^*(x), 1 - \lambda^*(x))$ is the unique value such that $x^T P(\gamma) x = V_c(x)$ and hence $\gamma^*(x) = (\lambda^*(x), 1 - \lambda^*(x))$. Since $\lambda^*(\cdot)$ is continuous, $\gamma^*(\cdot)$ is also continuous. This property of $\gamma^*(\cdot)$ will be useful in Section V to our construction of continuous feedback laws. Here we provide a method for computing λ such that $\partial\alpha/\partial\lambda = 0$ for a given x. By Proposition 1, this will give us $\lambda^*(x)$ and $\gamma^*(x)$.

Proposition 2: Let $x \in \mathbf{R}^n$ and $Q_1, Q_2 > 0$ be given. Assume that $Q_1 - Q_2$ is nonsingular. Let $U \in \mathbf{R}^{n \times n}$ be such that $U^T U = U U^T = I$ and $U^T x x^T U = \operatorname{diag}\{x^T x, 0, \ldots, 0\}$. Let $\hat{Q}_1 = U^T Q_1 U$, $\hat{Q}_2 = U^T Q_2 U$ and partition $\hat{Q}_1$ and $\hat{Q}_2$ as

$$\hat{Q}_1 = [\hat{q}_1\ \hat{Q}_{12}] \quad \hat{Q}_2 = [\hat{q}_2\ \hat{Q}_{22}] \quad \hat{q}_1, \hat{q}_2 \in \mathbf{R}^{n \times 1}.$$

Then, $\partial\alpha/\partial\lambda = 0$ at $\lambda \in [0,1]$ if and only if

$$\det \begin{bmatrix} \lambda(\hat{Q}_{12} - \hat{Q}_{22}) + \hat{Q}_{22} & \hat{Q}_1 - \hat{Q}_2 \\ 0_{(n-1)\times(n-1)} & \lambda\left(\hat{Q}_{12} - \hat{Q}_{22}^T\right)^T + \hat{Q}_{22}^T \end{bmatrix} = 0. \tag{3}$$

Proof: See Appendix A.3. □

All the λ's satisfying (3) can be obtained by computing the generalized eigenvalues of the matrix pair (X, Y) where

$$X = \begin{bmatrix} \hat{Q}_{12} - \hat{Q}_{22} & 0_{n\times n} \\ 0_{(n-1)\times(n-1)} & \hat{Q}_{12}^T - \hat{Q}_{22}^T \end{bmatrix}$$

$$Y = \begin{bmatrix} \hat{Q}_{22} & \hat{Q}_1 - \hat{Q}_2 \\ 0_{(n-1)\times(n-1)} & \hat{Q}_{22}^T \end{bmatrix}.$$

By Propositions 1 and 2, (X, Y) has at most one generalized eigenvalue in $[0,1]$. If there is none in $[0,1]$, then $\lambda^* = 0$ or 1. Experience shows that computing the matrices X and Y and their generalized eigenvalues requires much less time than solving the LMI problem (2).

III. Invariant Sets Under a Given Saturated Linear Feedback

Consider the open-loop system

$$\dot{x} = Ax + Bu \tag{4}$$

where $x(t) \in \mathbf{R}^n$ is the state and $u(t) \in \mathbf{R}^m$ is the output of saturating actuators and is assumed to satisfy the bound $|u(t)|_\infty \leq 1$. The state constraint is represented by a convex set Ω_0, which contains the origin in its interior. It is required that the system operate in Ω_0 for all $t \geq 0$. Suppose that we have a stabilizing feedback law $u = \operatorname{sat}(Fx)$, under which the closed-loop system is

$$\dot{x} = Ax + B\operatorname{sat}(Fx). \tag{5}$$

Since Ω_0 is generally not an invariant set, we would like to determine a maximal subset of Ω_0 such that, for any initial state x_0 in this subset, the state trajectory of (5) will stay in it and converge to the origin. Because of the intrinsic difficulty involved in determining the maximal invariant set inside Ω_0, alternative problems have been formulated such as determining the invariant ellipsoids and searching for the largest invariant ellipsoid inside Ω_0.

In [13], we derived a sufficient condition for checking the invariance of a given ellipsoid. This condition turns out to be also necessary for single input systems [12]. We need some notation to state the set invariance condition of [13]. Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathcal{D}$. Suppose that each element of $\mathcal{D}$ is labeled as D_i, $i \in I[1, 2^m]$. Then, $\mathcal{D} = \{D_i : i \in I[1, 2^m]\}$. Denote $D_i^- = I - D_i$. Given two matrices $F, H \in \mathbf{R}^{m \times n}$

$$\{D_i F + D_i^- H : i \in I[1, 2^m]\}$$

is the set of matrices formed by choosing some rows from F and the rest from H.

Given a positive–definite matrix P, let $V(x) = x^T P x$. The ellipsoid $\mathcal{E}(P, \rho)$ is said to be contractively invariant if

$$\dot{V}(x) = 2x^T P(Ax + B\operatorname{sat}(Fx)) < 0 \tag{6}$$

for all $x \in \mathcal{E}(P, \rho) \setminus \{0\}$. The invariance of $\mathcal{E}(P, \rho)$ can be defined by replacing "$<$" in (6) with "$\leq$." Clearly, if $\mathcal{E}(P, \rho)$ is contractively invariant, then for every initial state $x_0 \in \mathcal{E}(P, \rho)$, the state trajectory will converge to the origin and $\mathcal{E}(P, \rho)$ is inside the domain of attraction.

Proposition 3 [11], [13]: Given an ellipsoid $\mathcal{E}(P, \rho)$, if there exists an $H \in \mathbf{R}^{m \times n}$ such that

$$(A + B(D_i F + D_i^- H))^T P + P(A + B(D_i F + D_i^- H)) \leq (<) 0 \quad \forall\, i \in I[1, 2^m] \tag{7}$$

and $\mathcal{E}(P, \rho) \subset \mathcal{L}(H)$, Then, $\mathcal{E}(P, \rho)$ is a (contractively) invariant set.

The condition in Proposition 3 is easy to check with the LMI method. To impose the state constraint, we only need to require that $\mathcal{E}(P, \rho) \subset \Omega_0$. In the case that Ω_0 is a symmetric polytope, there exists a matrix $G_0 \in \mathbf{R}^{\ell \times n}$ for some integer ℓ such that $\Omega_0 = \mathcal{L}(G_0)$. In light of Fact 1, the requirement that $\mathcal{E}(P, \rho) \subset \Omega_0$ can be easily transformed into LMIs. In [11]–[13], we also developed LMI methods for choosing the largest invariant ellipsoid with respect to some shape reference set, where the matrix P was taken as an optimizing parameter. The shape reference set could be a polygon or a fixed ellipsoid. It could also be a single point $x_0 \in \mathbf{R}^n$. In this case, the largest invariant ellipsoid inside Ω_0 is the one that includes αx_0 with the maximal $\alpha > 0$. By choosing different x_0, say, $x_{0,j}$, $j \in I[1, N]$, we can obtain N optimized invariant ellipsoids $\mathcal{E}(P_j, \rho_j) \subset \Omega_0$, $j \in I[1, N]$. It is easy to see that the union of these ellipsoids, $\bigcup_{j=1}^N \mathcal{E}(P_j, \rho_j)$, is also an invariant set inside Ω_0. But this union does not necessarily include the convex hull of $x_{0,j}$, $j \in I[1, N]$. What is desired here is that the convex hull of the ellipsoids, $\operatorname{co}\{\mathcal{E}(P_j, \rho_j), j \in I[1, N]\}$, is also an invariant set.

For simplicity and without loss of generality, we will consider a set of invariant ellipsoids $\mathcal{E}(P_j, \rho_j), j \in I[1, N]$, with $\rho_j = 1$. The following theorem says that if each $\mathcal{E}(P_j)$ satisfies the condition of Proposition 3, then their convex hull, $\text{co}\{\mathcal{E}(P_j), j \in I[1, N]\}$, is also invariant.

Theorem 2: Given a set of ellipsoids $\mathcal{E}(P_j)$, $j \in I[1, N]$. If there exist matrices H_j, $j \in I[1, N]$, such that

$$(A + B(D_iF + D_i^- H_j))^T P_j + P_j(A + B(D_iF + D_i^- H_j)) \leq 0 \quad \forall\, i \in I[1, 2^m], \; j \in I[1, N] \tag{8}$$

and $\mathcal{E}(P_j) \subset \mathcal{L}(H_j), j \in I[1, N]$, then $\text{co}\{\mathcal{E}(P_j), j \in I[1, N]\}$ is an invariant set. If "<" holds for each of the aforementioned inequalities, then for every initial state $x_0 \in \text{co}\{\mathcal{E}(P_j), j \in I[1, N]\}$, the state trajectory will converge to the origin.

Proof: Let $Q_j = P_j^{-1}$ and $Z_j = H_j Q_j$. The inequalities in (8) are equivalent to

$$\begin{aligned} &Q_j(A + BD_iF)^T + (A + BD_iF)Q_j \\ &\quad + Z_j^T D_i^- B^T + BD_i^- Z_j \\ &\leq 0 \quad \forall\, i \in I[1, 2^m], \; j \in I[1, N]. \end{aligned} \tag{9}$$

The condition $\mathcal{E}(P_j) \subset \mathcal{L}(H_j)$, $j \in I[1, N]$, can be written as

$$\begin{bmatrix} 1 & z_{jk} \\ z_{jk}^T & Q_j \end{bmatrix} \geq 0, \qquad j \in I[1, N], \;\; k \in I[1, m] \tag{10}$$

where z_{jk} is the kth row of the matrix Z_j. Consider $x_0 \in \text{co}\{\mathcal{E}(P_j), j \in I[1, N]\}$. There exists $x_j \in \mathcal{E}(P_j)$ and $\gamma_j \geq 0, j \in I[1, N]$, such that $\gamma_1 + \gamma_2 + \cdots + \gamma_N = 1$ and $x_0 = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_N x_N$. Let $Q = \gamma_1 Q_1 + \gamma_2 Q_2 + \cdots + \gamma_N Q_N$ and $P = Q^{-1}$. Then, by Theorem 1, $\mathcal{E}(P) \subset \text{co}\{\mathcal{E}(P_j), j \in I[1, N]\}$. From $x_j \in \mathcal{E}(P_j)$, we have $x_j^T P_j x_j \leq 1$, which is equivalent to

$$\begin{bmatrix} 1 & x_j^T \\ x_j & Q_j \end{bmatrix} \geq 0, \qquad j \in I[1, N].$$

By the convexity, we have

$$\begin{bmatrix} 1 & x_0^T \\ x_0 & Q \end{bmatrix} \geq 0$$

which implies that $x_0^T P x_0 \leq 1$ and $x_0 \in \mathcal{E}(P)$.

Let $Z = \gamma_1 Z_1 + \gamma_2 Z_2 + \cdots + \gamma_N Z_N$, and z_k be the kth row of Z, then by (9), (10), and the convexity, we have

$$Q(A + BD_iF)^T + (A + BD_iF)Q + Z^T D_i^- B^T + BD_i^- Z \leq 0 \quad \forall\, i \in I[1, 2^m] \tag{11}$$

and

$$\begin{bmatrix} 1 & z_k \\ z_k^T & Q \end{bmatrix} \geq 0, k \in I[1, m]. \tag{12}$$

Let $H = ZQ^{-1} = ZP$. The inequalities in (11) and (12) can be rewritten as

$$(A + B(D_iF + D_i^- H))^T P + P(A + B(D_iF + D_i^- H)) \leq 0, \quad i \in I[1, 2^m] \tag{13}$$

and

$$\begin{bmatrix} 1 & h_k P^{-1} \\ P^{-1} h_k^T & P^{-1} \end{bmatrix} \geq 0 k \in I[1, m] \Longleftrightarrow \mathcal{E}(P) \subset \mathcal{L}(H). \tag{14}$$

The inequalities in (13) and the condition (14) jointly show that $\mathcal{E}(P)$ is an invariant set by Proposition 3. Hence, a trajectory starting from x_0 will stay inside of $\mathcal{E}(P)$, which is a subset of $\text{co}\,\{\mathcal{E}(P_j) : j \in I[1, N]\}$. Since x_0 is an arbitrary point inside $\text{co}\,\{\mathcal{E}(P_j) : j \in I[1, N]\}$, it follows that this convex hull is an invariant set. If "<" holds for all the inequalities in (8), then we also have "<" in (13), which guarantees that the trajectory starting form x_0 will converge to the origin. □

For single input systems, it was shown in [12] that the set invariance condition in Proposition 3 is also necessary. Hence, if each ellipsoid $\mathcal{E}(P_j)$ is contractively invariant, then $\text{co}\{\mathcal{E}(P_j), j \in I[1, N]\}$ is an invariant set inside the domain of attraction.

IV. Controlled Invariant Sets

In this section, we investigate the possibility that a level set can be made invariant with controls delivered by the saturating actuators. Given a positive–definite function $V(x)$, suppose that the level set $L_V(1)$ is bounded and $V(kx) = k^2 V(x)$. A level set $L_V(\rho)$ is said to be controlled contractively invariant if for every $x \in L_V(\rho) \setminus \{0\}$, there exists a $u \in \mathbf{R}^m$, $|u|_\infty \leq 1$, such that

$$\dot{V}(x, u) = \left(\frac{\partial V}{\partial x}\right)^T (Ax + Bu) < 0.$$

The controlled invariance can be defined by replacing "<" with "≤." Since $V(kx) = k^2 V(x)$, we have $(\partial V/\partial x)|_{x = kx_0} = k(\partial V/\partial x)|_{x = x_0}$. Hence, if $L_V(\rho)$ is controlled (contractively) invariant, then $L_V(\rho_1)$ is for all $\rho_1 \leq \rho$. Therefore, to determine the controlled (contractive) invariance of $L_V(\rho)$, it suffices to check all the points in $\partial L_V(\rho)$. For the composite quadratic Lyapunov function $V_c(x)$ defined in (1), we have

Theorem 3: Suppose that each of the ellipsoids $\mathcal{E}(P_j)$, $j \in I[1, N]$, is controlled (contractively) invariant, then $L_{V_c}(1)$ is controlled (contractively) invariant.

Proof: We only prove controlled invariance. The controlled contractive invariance can be shown similarly.

Denote $V_j(x) = x^T P_j x$. The condition implies that for all $x \in \partial\mathcal{E}(P_j)$, there exists a $u \in \mathbf{R}^m$, $|u|_\infty \leq 1$, such that

$$\dot{V}_j(x, u) = 2x^T P_j (Ax + Bu) \leq 0. \tag{15}$$

Now, we consider an arbitrary $x_0 \in \partial L_{V_c}(1)$. If $x_0 \in \partial\mathcal{E}(P_j)$ for some $j \in I[1, N]$, then $(\partial V_c/\partial x)|_{x = x_0} = 2P_j x_0$ and $\dot{V}_c(x, u) = \dot{V}_j(x, u) \leq 0$ follows from (15). Hence we assume that $x_0 \notin \partial\mathcal{E}(P_j)$ for any j. Then, there exist an integer $N_0 \leq N$, some numbers $\alpha_j \in (0, 1)$ and vectors $x_j \in \mathcal{E}(P_j), j \in I[1, N_0]$, such that

$$\sum_{j=1}^{N_0} \alpha_j = 1 \quad x_0 = \sum_{j=1}^{N_0} \alpha_j x_j.$$

(Here, we have assumed for simplicity that x_0 is only related to the first N_0 ellipsoids. Otherwise, the ellipsoids can be reordered to meet this assumption). Let $h_0 = (1/2)\left((\partial V_c/\partial x)|_{x = x_0}\right)^T$, then by Theorem 1, $h_0 x_0 = x_0^T P(\gamma^*(x_0))x_0 = 1$. It follows that the hyperplane $h_0 x = 1$ is tangential to the convex set $L_{V_c}(1)$ at $x = x_0$. Hence $L_{V_c}(1)$

lies between $h_0x = 1$ and $h_0x = -1$, i.e., $L_{V_c}(1) \subset \mathcal{L}(h_0)$. Therefore

$$\mathcal{E}(P_j) \subset \mathcal{L}(h_0) \quad \forall\, j \in I[1, N_0] \tag{16}$$

and

$$1 = h_0x_0 \geq h_0x_j \quad \forall\, j \in I[1, N_0].$$

We claim that $h_0x_j = 1$ for all $j \in I[1, N_0]$. Suppose on the contrary that $h_0x_j < 1$ for some j, say, $h_0x_1 < 1$, then

$$\begin{aligned} 1 =& h_0x_0 = \alpha_1 h_0x_1 + \sum_{j=2}^{N_0} \alpha_j h_0x_j \\ \leq& \alpha_1 h_0x_1 + \sum_{j=2}^{N_0} \alpha_j < \sum_{j=1}^{N_0} \alpha_j = 1 \end{aligned}$$

which is a contradiction. Because of (16) and $x_j \in \mathcal{E}(P_j)$, the equality $h_0x_j = 1$ implies that $\mathcal{E}(P_j)$ touches the hyperplane $h_0x = 1$ at $x = x_j$. Hence, the hyperplane $h_0x = 1$ is tangential to $\mathcal{E}(P_j)$ at x_j for every $j \in I[1, N_0]$. It follows from Fact 1 that

$$h_0^T = P_jx_j \quad \forall\, j \in I[1, N_0].$$

By assumption, there exists a $u_j \in \mathbf{R}^m$, $|u_j|_\infty \leq 1$, such that

$$\dot{V}_j(x_j, u_j) = 2x_j^T P_j(Ax_j + Bu_j) \leq 0$$

i.e., $2h_0(Ax_j + Bu_j) \leq 0$, for all $j \in I[1, N_0]$. Let $u_0 = \sum_{j=1}^{N_0} \alpha_j u_j$. Then, $|u_0|_\infty \leq 1$ and by the convexity, we have

$$\dot{V}_c(x_0, u_0) = 2h_0(Ax_0 + Bu_0) \leq 0.$$

Since x_0 is an arbitrary point in $\partial L_{V_c}(1)$, this implies that the level set $L_{V_c}(1)$ is controlled invariant. □

If a level set $L_{V_c}(1)$ is controlled contractively invariant, a simple feedback law to make it contractively invariant is

$$u_i = -\mathrm{sign}\left(b_i^T \frac{\partial V_c}{\partial x}\right), \qquad i \in I[1, m] \tag{17}$$

where b_i is the ith column of B. However, due to the discontinuity of the sign function, the closed-loop system under this control may be not well behaved. For instance, the closed-loop differential equation may have no solution. It can be shown with methods in [11, Ch. 11] that there exists a positive number k such that $L_{V_c}(1)$ is contractively invariant under the saturated linear feedback law

$$u = -\mathrm{sat}\left(kB^T \frac{\partial V_c}{\partial x}\right).$$

This control is continuous in x since both $\mathrm{sat}(\cdot)$ and $(\partial V_c/\partial x)$ are continuous. The value of k may be however difficult to determine. In the next section, we will provide a method for constructing a controller from a set of saturated linear feedback laws.

V. Construction of Continuous Feedback Laws

Suppose that we have a set of ellipsoids $\mathcal{E}(P_j)$, $j \in I[1, N]$, each of which is (contractively) invariant under a corresponding saturated linear feedback $u = \mathrm{sat}(F_jx)$. It was shown in [11] and [15] that a switching feedback law can be constructed such that the union $\cup_{j=1}^N \mathcal{E}(P_j)$ is invariant. In this section, we would like to construct a continuous feedback law from these F_j's such that the convex hull of the ellipsoids, $\mathrm{co}\{\mathcal{E}(P_j) : \ j \in I[1, N]\} = L_{V_c}(1)$, is invariant.

Theorem 4: Given ellipsoids $\mathcal{E}(P_j)$ and feedback matrices $F_j \in \mathbf{R}^{m \times n}$, $j \in I[1, N]$. Suppose that there exist $H_j \in \mathbf{R}^{m \times n}$ such that $\mathcal{E}(P_j) \subset \mathcal{L}(H_j)$ and

$$\begin{aligned} (A + B(D_iF_j + D_i^- H_j))^T P_j + P_j(A + B(D_iF_j + D_i^- H_j)) \\ \leq 0 (< 0) \end{aligned} \tag{18}$$

for all $i \in I[1, 2^m]$ and $j \in I[1, N]$. Let $Q_j = P_j^{-1}$, $Y_j = F_jQ_j$. Let $\gamma^*(x)$ be such that $x^T P(\gamma^*(x))x = V_c(x)$. Define $F(\gamma) := Y(\gamma)Q^{-1}(\gamma)$, where

$$Y(\gamma) = \sum_{j=1}^N \gamma_j Y_j \quad Q(\gamma) = \sum_{j=1}^N \gamma_j Q_j.$$

Then, $L_{V_c}(1)$ is (contractively) invariant under the feedback $u = \mathrm{sat}(F(\gamma^*(x))x)$. Moreover, if the vector function $\gamma^*(\cdot)$ is continuous, then $u = \mathrm{sat}(F(\gamma^*(x))x)$ is a continuous feedback law.

Proof: In the following, we only prove the invariance of $L_{V_c}(1)$. The contractive invariance follows from similar arguments. Let $Z_j = H_jQ_j$. Denote $Z(\gamma) = \sum_{j=1}^N \gamma_j Z_j$ and $H(\gamma) = Z(\gamma)Q^{-1}(\gamma)$. We see that (18) can be rewritten as

$$\begin{aligned} Q_jA^T + AQ_j + (D_iY_j + D_i^- Z_j))^T B^T \\ + B(D_iY_j + D_i^- Z_j) \leq 0 \end{aligned}$$

for all $i \in I[1, 2^m]$ and $j \in I[1, N]$. It follows from the convexity that $\mathcal{E}(P(\gamma)) \subset \mathcal{L}(H(\gamma))$ (see the proof of Theorem 2, (14), where the dependence on γ is suppressed) and

$$\begin{aligned} Q(\gamma)A^T + AQ(\gamma) + (D_iY(\gamma) + D_i^- Z(\gamma)))^T B^T \\ + B(D_iY(\gamma) + D_i^- Z(\gamma))) \leq 0 \end{aligned}$$

for all $i \in I[1, 2^m]$ and $\gamma \in \Gamma$. The previous inequality is equivalent to

$$\begin{aligned} &(A + B(D_iF(\gamma) + D_i^- H(\gamma)))^T P(\gamma) \\ &\quad + P(\gamma)(A + B(D_iF(\gamma) + D_i^- H(\gamma))) \leq 0 \\ &\forall\, i \in I[1, 2^m]. \end{aligned}$$

By Proposition 3, this inequality along with $\mathcal{E}(P(\gamma)) \in \mathcal{L}(H(\gamma))$ ensures that $\mathcal{E}(P(\gamma))$ is invariant under the control of $u = \mathrm{sat}(F(\gamma)x)$, i.e.,

$$2x^T P(\gamma)(Ax + B\mathrm{sat}(F(\gamma)x)) \leq 0 \quad \forall\, x \in \mathcal{E}(P(\gamma)). \tag{19}$$

For an arbitrary $x_0 \in \partial L_{V_c}(1)$, let $\gamma_0 = \gamma^*(x_0)$. Then, $x_0^T P(\gamma_0)x_0 = 1$, i.e., $x_0 \in \partial\mathcal{E}(P(\gamma_0))$, and from Theorem 1

$$\left.\frac{\partial V_c}{\partial x}\right|_{x=x_0} = 2P(\gamma_0)x_0.$$

It follows from (19) that

$$\dot{V}_c(x_0) = 2x_0^T P(\gamma_0)(Ax_0 + B\mathrm{sat}(F(\gamma_0)x_0))) \leq 0.$$

This shows that $L_{V_c}(1)$ is invariant under the control of $u = \mathrm{sat}(F(\gamma^*(x))x)$.

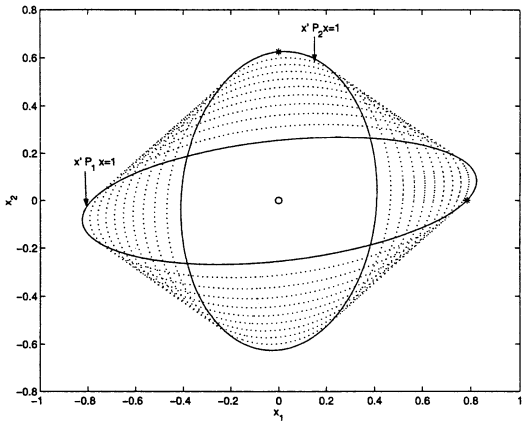

Fig. 4. Convex hull of two ellipsoids.

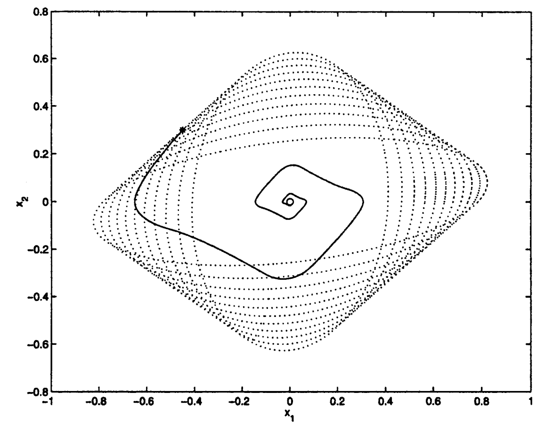

Fig. 5. Trajectory and the invariant set $L_{V_c}(1)$.

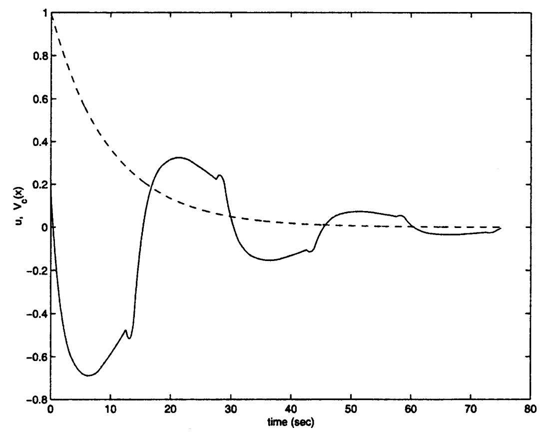

Fig. 6. Control signal $u(t)$ and the Lyapunov function $V_c(x(t))$.

Let us now address the continuity of the feedback law. As we have noted earlier, the matrix $Q(\gamma)$ is nonsingular for all $\gamma \in \Gamma$. Hence, $Q^{-1}(\gamma)$ is continuous in $\gamma \in \Gamma$. Since the saturation function $\mathrm{sat}(\cdot)$ is continuous and by assumption $\gamma^*(\cdot)$ is continuous, it follows that the feedback law $u = \mathrm{sat}(F(\gamma^*(x))x)$ is continuous. □

Remark 3: In Theorem 4, we assumed that the function $\gamma^*(\cdot)$ is continuous. For the case where $N = 2$ and $Q_1 - Q_2$ is nonsingular, $\gamma^*(\cdot)$ is continuous by Proposition 1 and can be computed with Proposition 2. If $N > 2$, as we have noted earlier, $\gamma^*(x)$ could be nonunique in some special cases where one of the Q_j is the convex combination of other matrices in the set. Such special cases may be considered as degenerated. We may exclude such special cases by assuming that none of the $\mathcal{E}(P_j)$'s is in the convex hull of other ellipsoids. Whether this assumption will lead to the uniqueness and continuity of $\gamma^*(\cdot)$ is an interesting problem and needs further study. ◇

Example: Consider system (4) with

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

We have designed two feedback matrices

$$F_1 = [\,0.9471 \quad 1.6000\,] \quad F_2 = [\,-0.1600 \quad 1.6000\,]$$

along with two ellipsoids $\mathcal{E}(P_1)$ and $\mathcal{E}(P_2)$, where

$$P_1 = \begin{bmatrix} 1.6245 & -1.5364 \\ -1.5364 & 15.3639 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 5.9393 & -0.2561 \\ -0.2561 & 2.5601 \end{bmatrix}.$$

The matrices P_1 and F_1 are designed such that the value α_1 is maximized, where $\alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathcal{E}(P_1)$ and $V_1(x) = x^T P_1 x$ has a guaranteed convergence rate inside $\mathcal{E}(P_1)$ under $u = \mathrm{sat}(F_1 x)$. The matrices P_2 and F_2 are designed such that the value α_2 is maximized, where $\alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{E}(P_2)$ and $V_2(x) = x^T P_2 x$ has a guaranteed convergence rate inside $\mathcal{E}(P_2)$ under $u = \mathrm{sat}(F_2 x)$ (see [11] for the detailed design method). In Fig. 4, the boundaries of the two ellipsoids are plotted in solid curves. The dotted curves are the boundaries of $\mathcal{E}(P(\gamma))$ as γ varies in the set Γ. The shape of $L_{V_c}(1) = \bigcup_{\gamma \in \Gamma} \mathcal{E}(P(\gamma))$ can be seen from these dotted curves. It can be verified that $Q_1 - Q_2 = P_1^{-1} - P_2^{-1}$ is nonsingular. So we can use the method in Proposition 2 to compute $\gamma^*(x)$, which is guaranteed to be continuous by Proposition 1. Simulation is carried out under the feedback law $u = \mathrm{sat}(F(\gamma^*(x))x)$, where

$$\begin{aligned} F(\gamma) &= (\gamma_1 Y_1 + \gamma_2 Y_2)(\gamma_1 Q_1 + \gamma_2 Q_2)^{-1} \\ Y_1 &= [0.75270.1794] \quad Y_2 = [0.00000.6250] \end{aligned}$$

and

$$Q_1 = \begin{bmatrix} 0.6799 & 0.0680 \\ 0.0680 & 0.0719 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 0.1691 & 0.0169 \\ 0.0169 & 0.3923 \end{bmatrix}.$$

In Fig. 5, a trajectory starting from $\partial L_{V_c}(1)$ is plotted. Fig. 6 plots the control signal $u(t)$ (in solid curve) and the composite quadratic function $V_c(x(t))$ (in dashed curve).

VI. Conclusion

This paper introduced the composite quadratic Lyapunov function and showed that it is a powerful tool to handle saturation nonlinearity. The composite quadratic Lyapunov function is continuously differentiable and its level set is the convex hull of a set of ellipsoids. Using these results, we studied some set invariance properties of continuous-time linear systems with input saturation. In particular, we showed that if every ellipsoid in a set is invariant under a saturated feedback, then their convex hull is also invariant. Similar results on controlled invariance have also been established. We also proposed a method to construct a continuous feedback law based on a set of saturated linear feedback laws to make the convex hull of a set of ellipsoids invariant.

There still exist some interesting problems about the composite quadratic Lyapunov function. For example, the computation of this function is carried out by solving an LMI problem. The simplification for the case $N=2$ motivates us to find a more efficient way to compute this function for $N>2$. Also, we have only given condition for the continuity of $\gamma^*(\cdot)$ for $N=2$ and the condition for more general cases remains unknown. Nevertheless, the composite quadratic function is relatively easier to handle than a general nonlinear Lyapunov function and we expect to use it to study more general nonlinear systems.

APPENDIX
PROOFS OF THEOREM 1 AND PROPOSITIONS 1 AND 2

A. Proof of Theorem 1

a) It is obvious that $V_c(kx) = k^2 V_c(x)$, so $L_{V_c}(\rho) = \sqrt{\rho} L_{V_c}(1)$. Since $\mathcal{E}(P,\rho) = \sqrt{\rho}\mathcal{E}(P)$, it suffices to show that

$$L_{V_c}(1) = \mathrm{co}\,\{\mathcal{E}(P_j), j \in I[1,N]\} = \bigcup_{\gamma\in\Gamma} \mathcal{E}(P(\gamma)).$$

We first show that $\mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\} \subset L_{V_c}(1)$. Suppose $x \in \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}$, then there exists a $\gamma\in\Gamma$ and $x_j \in \mathcal{E}(P_j), j\in I[1,N]$, such that

$$x = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_N x_N.$$

Since $x_j \in \mathcal{E}(P_j)$, we have $x_j^T P_j x_j \le 1$, which is equivalent (by the Schur complement) to

$$\begin{bmatrix} 1 & x_j^T \\ x_j & Q_j \end{bmatrix} \ge 0, \qquad j \in I[1,N].$$

Recalling that $Q(\gamma) = \gamma_1 Q_1 + \gamma_2 Q_2 + \cdots + \gamma_N Q_N$, we have, by the convexity

$$\begin{bmatrix} 1 & x^T \\ x & Q(\gamma) \end{bmatrix} \ge 0$$

which implies that $x^T P(\gamma) x \le 1$. It follows that $V_c(x)\le 1$ and $x \in L_{V_c}(1)$.

We next prove that $\bigcup_{\gamma\in\Gamma}\mathcal{E}(P(\gamma)) \subset \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}$. It suffices to show that $\mathcal{E}(P(\gamma)) \subset \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}$ for every $\gamma\in\Gamma$. Let γ be any vector in the set Γ. One way for the proof is to show that for any $x\in\mathcal{E}(P(\gamma))$, there exist a $\hat{\gamma}\in\Gamma$ and a set of $x_j \in \mathcal{E}(P_j), j\in I[1,N]$, such that $x = \hat{\gamma}_1 x_1 + \cdots + \hat{\gamma}_N x_N$. However, this approach seems to be not easy. Instead, we will prove $\mathcal{E}(P(\gamma)) \subset \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}$ by using Fact 1. Suppose on the contrary that there exists an $x_0 \in \mathcal{E}(P(\gamma))$ and $x_0 \notin \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}$. Then, there exists a vector $h\in\mathbf{R}^{1\times n}$ such that

$$hx_0 > hx \quad \forall\, x\in \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}.$$

Let $x_* \in \mathcal{E}(P(\gamma))$ be the point such that $hx_* = \max\{hx : x\in\mathcal{E}(P(\gamma))\}$ and let $h_* = h/(hx_*)$, then $\max\{h_* x : x\in \mathcal{E}(P(\gamma))\} = 1$ and the hyperplane $h_* x = 1$ touches the ellipsoid $\mathcal{E}(P(\gamma))$ at only one point x_*. It is obvious that

$$h_* x < h_* x_0 \le h_* x_* = 1 \quad \forall\, x\in \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}.$$

This implies that each ellipsoid $\mathcal{E}(P_j)$ is between the hyperplanes $h_* x = 1$ and $h_* x = -1$ without touching them. By Fact 1, we have

$$\begin{bmatrix} 1 & h_* Q_j \\ Q_j h_*^T & Q_j \end{bmatrix} > 0, \qquad j\in I[1,N]. \tag{20}$$

By the convexity, we should have

$$\begin{bmatrix} 1 & h_* Q(\gamma) \\ Q(\gamma) h_*^T & Q(\gamma) \end{bmatrix} > 0 \Longleftrightarrow h_* Q(\gamma) h_*^T < 1. \tag{21}$$

However, since the hyperplane $h_* x = 1$ touches the ellipsoid $\mathcal{E}(P(\gamma))$, we also have

$$h_* Q(\gamma) h_*^T = 1$$

which contradicts (21). Therefore, we conclude that there exists no $x_0 \in \mathcal{E}(P(\gamma))$ such that $x_0 \notin \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}$. This proves that $\mathcal{E}(P(\gamma)) \subset \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}$.

Finally, we show that $L_{V_c}(1) \subset \bigcup_{\gamma\in\Gamma}\mathcal{E}(P(\gamma))$. Suppose that $x\in L_{V_c}(1)$. Then there exists a $\gamma\in\Gamma$ such that $x^T P(\gamma) x \le 1$. It follows that $x\in\mathcal{E}(P(\gamma)) \subset \bigcup_{\gamma\in\Gamma}\mathcal{E}(P(\gamma))$.

Combining the previous set inclusion results, we obtain

$$L_{V_c}(1) \subset \bigcup_{\gamma\in\Gamma}\mathcal{E}(P(\gamma)) \subset \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\} \subset L_{V_c}(1).$$

Therefore

$$L_{V_c}(1) = \bigcup_{\gamma\in\Gamma}\mathcal{E}(P(\gamma)) = \mathrm{co}\,\{\mathcal{E}(P_j), j\in I[1,N]\}.$$

b) Let us first establish a property about the differentiability which will simplify the proof. Suppose that $V_c(x)$ is differentiable at x_0 with partial derivative $(\partial V_c/\partial x)|_{x=x_0}$ and let k be given. Since $V_c(kx) = k^2 V_c(x)$ for all $x\in\mathbf{R}^n$, we have

$$\begin{aligned} V_c(kx_0+\Delta x) - V_c(kx_0) =& k^2\left(V_c\left(x_0 + \frac{\Delta x}{k}\right) - V_c(x_0)\right) \\ =& k^2\left(\left(\left.\frac{\partial V_c}{\partial x}\right|_{x=x_0}\right)^T \frac{\Delta x}{k} \right. \\ & \left. + o\left(\left|\frac{\Delta x}{k}\right|\right)\right) \\ =& k\left(\left.\frac{\partial V_c}{\partial x}\right|_{x=x_0}\right)^T \Delta x \\ & + o(|\Delta x|), \end{aligned}$$

where $|\cdot|$ can be any norm. It follows that

$$\left.\frac{\partial V_c}{\partial x}\right|_{x=kx_0} = k\left.\frac{\partial V_c}{\partial x}\right|_{x=x_0}.$$

In view of this equality, we only need to consider those x on the boundary of $L_{V_c}(1)$, $\partial L_{V_c}(1)$. Here, we use "∂" to denote both the boundary of a set and the partial derivative.

Since the set $L_{V_c}(1)$ is convex, for each $x_0\in\partial L_{V_c}(1)$, there exists a vector $h_0\in\mathbf{R}^{1\times n}$ such that

$$1 = h_0 x_0 \ge h_0 x \quad \forall\, x\in L_{V_c}(1) \tag{22}$$

which implies that $L_{V_c}(1) \subset \mathcal{L}(h_0)$. Let γ^* be an optimal γ such that

$$x_0^T P(\gamma^*) x_0 = \min_{\gamma \in \Gamma} x_0^T P(\gamma) x_0 = 1.$$

Since $\mathcal{E}(P(\gamma^*)) \subset L_{V_c}(1)$, it follows that $\mathcal{E}(P(\gamma^*)) \subset \mathcal{L}(h_0)$ and

$$h_0 x_0 \geq h_0 x \quad \forall\, x \in \mathcal{E}(P(\gamma^*)).$$

By Fact 1, the hyperplane $h_0 x = 1$ is tangential to the ellipsoid $\mathcal{E}(P(\gamma^*))$ at x_0. Therefore, such a vector h_0 is uniquely determined by x_0. To simplify the notation, denote $P_0 = P(\gamma^*)$. Combining the aforementioned results, we have

$$\mathcal{E}(P_0) \subset L_{V_c}(1) \subset \mathcal{L}(h_0). \tag{23}$$

By Fact 1, we also have

$$x_0 = P_0^{-1} h_0^T \quad h_0 P_0^{-1} h_0^T = 1 \tag{24}$$

and

$$h_0 x_0 > h x_0, \qquad \forall\, h^T \in \mathcal{E}(P_0^{-1}) \setminus \{h_0^T\}. \tag{25}$$

Now, we show that

$$\left.\frac{\partial V_c}{\partial x}\right|_{x=x_0} = 2h_0^T = 2P_0 x_0.$$

From (23), it follows that for all $x \in \partial L_{V_c}(1)$, $V_c(x) = 1$, $h_0 x \leq 1$ and $x^T P_0 x \geq 1$, i.e.,

$$\begin{aligned} h_0 x \leq 1 =& V_c^{1/2}(x) = 1 \leq (x^T P_0 x)^{1/2} \\ &\forall\, x \in \partial L_{V_c}(1). \end{aligned}$$

Since $V_c^{1/2}(kx) = kV_c^{1/2}(x), h_0(kx) = kh_0 x$ and $((kx)^T P_0 (kx))^{1/2} = k(x^T P_0 x)^{1/2}$, we have

$$\begin{aligned} h_0(kx) \leq& V_c^{1/2}(kx) \leq ((kx)^T P_0 (kx))^{1/2} \\ &\forall\, k \geq 0, x \in \partial L_{V_c}(1). \end{aligned}$$

Since every point in $\mathbf{R}^n$ can be written as kx for some $k \geq 0$ and $x \in \partial L_{V_c}(1)$, we have

$$h_0 x \leq V_c^{1/2}(x) \leq (x^T P_0 x)^{1/2} \quad \forall\, x \in \mathbf{R}^n. \tag{26}$$

Recalling that $(x_0^T P_0 x_0)^{1/2} = 1$ and from (24), we have

$$\left.\frac{\partial (x^T P_0 x)^{1/2}}{\partial x}\right|_{x=x_0} = P_0 x_0 = h_0^T.$$

Hence

$$\left((x_0 + \Delta x)^T P_0 (x_0 + \Delta x)\right)^{1/2} = 1 + h_0 \Delta x + o(|\Delta x|). \tag{27}$$

Recalling from (22) that $h_0 x_0 = 1$, we have

$$h_0(x_0 + \Delta x) = 1 + h_0 \Delta x. \tag{28}$$

Combining (26)–(28) and that $V_c^{1/2}(x_0) = 1$, we obtain

$$\begin{aligned} V_c^{1/2}(x_0 + \Delta x) =& 1 + h_0 \Delta x + o(|\Delta x|) \\ =& V_c^{1/2}(x_0) + h_0 \Delta x + o(|\Delta x|) \end{aligned}$$

which implies that $V_c^{1/2}(x)$ is differentiable at $x = x_0$ and the partial derivative is given by h_0^T. It follows that $V_c(x)$ is differentiable at $x = x_0$ with the partial derivative given by

$$\left.\frac{\partial V_c}{\partial x}\right|_{x=x_0} = 2V_c^{1/2}(x_0) h_0^T = 2h_0^T = 2P_0 x_0 = 2P(\gamma^*) x_0.$$

In the rest of the proof, we show that $\partial V_c / \partial x$ is continuous in x. Since

$$\left.\frac{\partial V_c}{\partial x}\right|_{x=kx_0} = k \left.\frac{\partial V_c}{\partial x}\right|_{x=x_0}$$

it suffices to prove the continuity on the surface $\partial L_{V_c}(1)$. Let $x_0 \in \partial L_{V_c}(1)$ with h_0 and P_0 defined as before. Then, we have $h_0^T = P_0 x_0$ and $h_0 P_0^{-1} h_0^T = h_0 x_0 = 1$. Consider $v \in \partial L_{V_c}(1)$. Let

$$h(v) = \frac{1}{2}\left(\left.\frac{\partial V_c}{\partial x}\right|_{x=v}\right)^T.$$

Then, $h(x_0) = h_0$ and by the foregoing proof, we have $h(v)v = 1$, $L_{V_c}(1) \subset \mathcal{L}(h(v))$ and, hence, $\mathcal{E}(P_0) \subset \mathcal{L}(h(v))$. By Fact 1

$$\begin{aligned} h(v) P_0^{-1} h^T(v) \leq& 1 \Longleftrightarrow h^T(v) \in \mathcal{E}(P_0^{-1}) \\ &\forall\, v \in \partial L_{V_c}(1). \end{aligned} \tag{29}$$

It follows that there exists a positive number d_0 such that $|h(v)|_2 \leq d_0$ for all $v \in \partial L_{V_c}(1)$. Now, suppose on the contrary that $h(v)$ is not continuous at $v = x_0$ on the surface $\partial L_{V_c}(1)$. Then, there exists a positive number ε such that for any arbitrarily small number $\delta > 0$, there exists a $v \in \mathcal{B}(x_0, \delta) \cap \partial L_{V_c}(1)$ satisfying $h^T(v) \notin \mathcal{B}(h_0, \varepsilon)$. Note that ε is fixed and δ can be arbitrarily small. What we will show next is that the assumption of the existence of such ε will cause contradiction.

From above, we have

$$\begin{aligned} h_0 P_0^{-1} h_0^T =& 1 \quad h(v) P_0^{-1} h(v)^T \leq 1 \\ h_0 x_0 =& 1, \qquad h(v)v = 1. \end{aligned}$$

By Fact 1, we know that

$$h_0 x_0 > h x_0 \quad \forall\, h^T \in \mathcal{E}(P_0^{-1}) \setminus h_0^T.$$

Then, it is clear that

$$\sup_{h^T \in \mathcal{E}(P_0^{-1}) \setminus \mathcal{B}(h_0, \varepsilon)} h x_0 =: k^* < 1. \tag{30}$$

Hence, for all $h^T \in \mathcal{E}(P_0^{-1}) \setminus \mathcal{B}(h_0, \varepsilon)$, $h x_0 \leq k^* < 1$. On the other hand, for all $v \in \mathcal{B}(x_0, \delta) \cap \partial L_{V_c}(1)$, we have $h(v)v = 1$ and

$$|h(v)x_0 - h(v)v| \leq |h(v)|_2 \delta \leq d_0 \delta.$$

Hence

$$h(v)x_0 \geq h(v)v - d_0\delta = 1 - d_0\delta.$$

Let δ be chosen such that $d_0\delta < 1 - k^*$. Then

$$h(v)x_0 \geq 1 - d_0\delta > k^* \quad \forall\, v \in \mathcal{B}(x_0, \delta) \cap \partial L_{V_c}(1). \tag{31}$$

By assumption, there exists a $v \in \mathcal{B}(x_0,\delta)\cap\partial L_{V_c}(1)$ such that $h^T(v)\in\mathcal{E}(P_0^{-1})\backslash\mathcal{B}(h_0,\varepsilon)$ (note that $h^T(v)\in\mathcal{E}(P_0^{-1})$ is from (29)). It follows from (30) that $h(v)x_0\le k^*$, which contradicts (31). Therefore, $h(v)$ must be continuous at x_0.

Finally, we note that the partial derivative is continuous at $x=0$ with $\partial V_c/\partial x|_{x=0}=0$.

B. Proof of Proposition 1

The first and the second partial derivatives of $\alpha(\lambda,x)$ with respect to λ are

$$\frac{\partial\alpha}{\partial\lambda}=x^T(\lambda Q_1+(1-\lambda)Q_2)^{-1}(Q_2-Q_1)(\lambda Q_1+(1-\lambda)Q_2)^{-1}x$$

and

$$\begin{aligned}\frac{\partial^2\alpha}{\partial\lambda^2}=&\,2x^T(\lambda Q_1+(1-\lambda)Q_2)^{-1}(Q_2-Q_1)(\lambda Q_1\\&+(1-\lambda)Q_2)^{-1}(Q_2-Q_1)(\lambda Q_1+(1-\lambda)Q_2)^{-1}x.\end{aligned}$$

Since $(\lambda Q_1+(1-\lambda)Q_2)^{-1}>0$ for all $\lambda\in[0,1]$ and Q_2-Q_1 is nonsingular, we have $\partial^2\alpha/\partial\lambda^2>0$ for all $\lambda\in[0,1]$ and $x\in\mathbf{R}^n$. This shows that $\alpha(\cdot,x):[0,1]\to\mathbf{R}$, is strictly convex and establishes the uniqueness of $\lambda^*(x)\in[0,1]$ such that $\alpha(\lambda^*(x),x)=\min_{\lambda\in[0,1]}\alpha(\lambda,x)$. Consider $x_0\in\mathbf{R}^n$. If $\lambda^*(x_0)\in(0,1)$, then for x in a neighborhood of x_0, $\alpha(\lambda,x)$ has a unique minimum at $\lambda^*(x)\in(0,1)$ satisfying

$$\begin{aligned}\left.\frac{\partial\alpha}{\partial\lambda}\right|_{\lambda=\lambda^*(x)}=&\,x^T(\lambda^*(x)Q_1+(1-\lambda^*(x))Q_2)^{-1}\\&\times(Q_2-Q_1)(\lambda^*(x)Q_1+(1-\lambda^*(x))Q_2)^{-1}x=0.\end{aligned}$$

Since $(\partial^2\alpha/\partial\lambda^2)\neq0$, by implicit function theorem, $\lambda^*(x)$ is continuously differentiable at x_0. For those x_0 such that $\lambda^*(x_0)=0$ (or 1), we have two possibilities.

1) $\partial\alpha/\partial\lambda|_{\lambda=0,x=x_0}=0$. Then as x varies in a neighborhood of x_0, $\partial\alpha/\partial\lambda=0$ occurs in a neighborhood of $\lambda=0$. In this neighborhood of x_0, if $\partial\alpha/\partial\lambda=0$ for some $\lambda<0$, then we must have $\lambda^*(x)=0$ and, if $\partial\alpha/\partial\lambda=0$ for some $\lambda>0$, then $\lambda^*(x)>0$ and must be in a neighborhood of $\lambda=0$. These show the continuity of $\lambda^*(x)$ for this case.
2) $\partial\alpha/\partial\lambda|_{\lambda=0,x=x_0}>0$. Then we have $\partial\alpha/\partial\lambda|_{\lambda=0}>0$ for all x in a neighborhood of x_0. By the convexity, we have $\lambda^*(x)=0$ for all x in this neighborhood of x_0, which also confirms the continuity of $\lambda^*(x)$.

C. Proof of Proposition 2

In the proof, we will use the following algebraic fact. Suppose that X_1,X_4 and $\begin{bmatrix}X_1&X_2\\X_3&X_4\end{bmatrix}$ are square matrices. If X_1 is nonsingular, then

$$\det\begin{bmatrix}X_1&X_2\\X_3&X_4\end{bmatrix}=\det(X_1)\det(X_4-X_3X_1^{-1}X_2)\tag{32}$$

and if X_4 is nonsingular, then

$$\det\begin{bmatrix}X_1&X_2\\X_3&X_4\end{bmatrix}=\det(X_4)\det(X_1-X_2X_1^{-1}X_3).\tag{33}$$

Let $Q(\lambda)\in\mathbf{R}^{n\times n}$ be a matrix function. Suppose that $Q(\lambda)$ is nonsingular, then

$$\frac{dQ^{-1}(\lambda)}{d\lambda}=-Q^{-1}(\lambda)\frac{dQ(\lambda)}{d\lambda}Q^{-1}(\lambda).\tag{34}$$

In what follows, we use I to denote the $n\times n$ unit matrix and 0 to denote a zero matrix of appropriate dimension. For simplicity, we use $Q(\lambda)$ to denote $\lambda Q_1+(1-\lambda)Q_2$.

If $\alpha(\lambda^*(x),x)=V_c(x)$ and $\lambda^*(x)\in(0,1)$, then we must have $(\partial\alpha/\partial\lambda)|_{\lambda=\lambda^*(x)}=0$. From (34), we have

$$x^TQ^{-1}(\lambda^*(x))(Q_1-Q_2)Q^{-1}(\lambda^*(x))x=0$$

which can be written as

$$\det\left(1-x^TQ^{-1}(\lambda^*(x))(Q_1-Q_2)Q^{-1}(\lambda^*(x))x\right)=1.$$

By applying (32) and (33), we obtain a sequence of equivalent relations

$$\begin{aligned}&\det\begin{bmatrix}1&x^TQ^{-1}(\lambda^*(x))\\Q^{-1}(\lambda^*(x))x&(Q_1-Q_2)^{-1}\end{bmatrix}=\det(Q_1-Q_2)^{-1}\\
&\Updownarrow\\
&\det\left(\begin{bmatrix}1&0\\0&(Q_1-Q_2)^{-1}\end{bmatrix}-\begin{bmatrix}x^T&0\\0&I\end{bmatrix}\right.\\
&\qquad\left.\times\begin{bmatrix}Q^{-1}(\lambda^*(x))&0\\0&Q^{-1}(\lambda^*(x))\end{bmatrix}\begin{bmatrix}0&I\\x&0\end{bmatrix}\right)\\
&=\det(Q_1-Q_2)^{-1}\\
&\Updownarrow\\
&\det\left(\begin{bmatrix}Q(\lambda^*(x))&0\\0&Q(\lambda^*(x))\end{bmatrix}-\begin{bmatrix}0&I\\x&0\end{bmatrix}\right.\\
&\qquad\left.\times\begin{bmatrix}1&0\\0&(Q_1-Q_2)\end{bmatrix}\begin{bmatrix}x^T&0\\0&I\end{bmatrix}\right)\\
&=\det\begin{bmatrix}Q(\lambda^*(x))&0\\0&Q(\lambda^*(x))\end{bmatrix}\\
&\Updownarrow\\
&\det\begin{bmatrix}Q(\lambda^*(x))&Q_1-Q_2\\xx^T&Q(\lambda^*(x))\end{bmatrix}\\
&=\det\begin{bmatrix}Q(\lambda^*(x))&0\\0&Q(\lambda^*(x))\end{bmatrix}.\end{aligned}$$

Multiplying the matrices on both sides from left with $\begin{bmatrix}U^T&0\\0&U^T\end{bmatrix}$ and from right with $\begin{bmatrix}U&0\\0&U\end{bmatrix}$, we obtain

$$\begin{aligned}&\det\begin{bmatrix}\hat Q_2+\lambda^*(x)(\hat Q_1-\hat Q_2)&\hat Q_1-\hat Q_2\\\begin{bmatrix}x^Tx&0\\0&0_{n-1}\end{bmatrix}&\hat Q_2+\lambda^*(x)(\hat Q_1-\hat Q_2)\end{bmatrix}\\
&=\det\begin{bmatrix}\hat Q_2+\lambda^*(x)(\hat Q_1-\hat Q_2)&\hat Q_1-\hat Q_2\\0&\hat Q_2+\lambda^*(x)(\hat Q_1-\hat Q_2)\end{bmatrix}.\end{aligned}\tag{35}$$

Here, we notice that $\hat Q_1-\hat Q_2$ has been added to the upper-right block of the right-hand side matrix. This does not change the value of the determinant.

The two determinants in (35) are different only at the first column. By using the property of determinant on column addition and the partition of $\hat Q_1$ and $\hat Q_2$, we have the equation shown at the top of the next page. The last equality is (3).

$$\det\begin{bmatrix} 0 & \lambda^*(x)(\hat{Q}_{12}-\hat{Q}_{22})+\hat{Q}_{22} & \hat{Q}_1-\hat{Q}_2 \\ x^Tx & 0 & \lambda^*(x)(\hat{q}_1-\hat{q}_2)^T+\hat{q}_2^T \\ 0 & 0 & \lambda^*(x)(\hat{Q}_{12}-\hat{Q}_{22})^T+\hat{Q}_{22}^T \end{bmatrix}=0$$

$$\Updownarrow$$

$$\det\begin{bmatrix} \lambda^*(x)(\hat{Q}_{12}-\hat{Q}_{22})+\hat{Q}_{22} & \hat{Q}_1-\hat{Q}_2 \\ 0 & \lambda^*(x)(\hat{Q}_{12}-\hat{Q}_{22})^T+\hat{Q}_{22}^T \end{bmatrix}=0.$$

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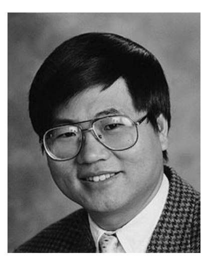

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