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Stability Analysis of Linear Time-Delay Systems Subject to Input Saturation

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***Abstract*—This paper is devoted to stability analysis of linear systems with state delay and input saturation. The domain of attraction resulting from an *a priori* designed state feedback law is analyzed using Lyapunov–Razumikhin and Lyapunov–Krasovskii functional approach. Both delay-independent and delay-dependent estimation of the domain of attraction are presented using the linear matrix inequality technique. The problem of designing linear state feedback laws such that the domain of attraction is enlarged is formulated and solved as an optimization problem with LMI constraints. Numerical examples are used to demonstrate the effectiveness of the proposed design technique.**

***Index Terms*—Actuator saturation, domain of attraction, linear matrix inequality, time-delay.**

I. Introduction

Nonlinear systems with time-delay constitute basic mathematical models of real phenomena, for instance, in circuits theory, economics and mechanics. Not only dynamical systems with time-delay are common in chemical processes and long transmission lines in pneumatic, hydraulic, or rolling mill systems, but computer controlled systems requiring numerical computation have time-delays in control loops. The presence of time-delays in control loops usually degrades system performance and complicates the analysis and design of feedback controllers. Stability analysis and synthesis of retarded systems is an important issue addressed by many authors and for which surveys can be found in several monographs (see *e.g.*, [7], [9], [10], [13], [17], [20]).

Another common, but difficult, control problem is to deal with actuator saturation since all control devices are subject to saturation (limited in force, torque, current, flow rate, etc.). The analysis and synthesis of controllers for dynamic systems subject to actuator saturation have been attracting increasingly more attention (see, for example, [1], [11], [14], [15] and the references therein).

Actuator saturation and time-delays are often observed together in control systems. To deal with both problems effectively, appropriate design methods are required. Up to now, only a few methods were reported to deal with these problems simultaneously. Chen *et al.* [5] studied the stabilization problem of saturating time-delay system with state feedback and sampled-state feedback and they derived several sufficient conditions to ensure the system stability in terms of norm inequalities. Chou *et al.* [6] exploited a sufficient condition to stabilize a linear uncertain time-delay system containing input saturation. The problem of robust stabilization of uncertain time-delay systems containing a saturating actuator was addressed by Niculescu *et al.* [16] by a high gain approach. Oucheriah [18] considered a method to synthesize a globally stabilizing state feedback controller by means of an asymptotic observer for time-delay systems. In [19], a dynamic anti-windup method was presented for the systems with input delay and saturation. All of these works have mainly focused on the stabilizability of the systems.

In this paper, we will first analyze the stability and domain of attraction for linear systems with time-delay in state and actuator saturation. A less conservative estimate of the domain of attraction will be derived based on the Lyapunov–Razumikin and Lypunov–Krasovskii functional approaches. This estimate is then maximized over the choice of the feedback gains. It is known that the estimates of the domain of attraction made by small gain theorem, Popov criterion or circle criterion are sometimes very conservative. In [12], a less conservative analysis approach is proposed to analyze the stability and the domain of attraction for systems with actuator saturation. The idea is to formulate the analysis problem into a constrained optimization problem with constraints given by a set of linear matrix inequalities (LMI's). In this paper, we will further exploit the idea in [12] to arrive at an estimate of the domain of attraction for the linear systems subject to both delay in state and actuator saturation. An LMI optimization approach will be proposed to design the state feedback gain which maximizes this estimate of the domain of attraction.

The paper is organized as follows. Section II gives some preliminary results and states more precisely our problem formulation. Delay-dependent and delay-independent stability and domain of attraction of the closed-loop system with input saturation and state delay will be analyzed in Sections III and IV respectively. Numerical examples illustrating our design procedure and its effectiveness are given in Section V. The paper is concluded in Section VI.

Notations: The following notations will be used throughout the paper. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of nonnegative real numbers, $\mathbb{R}^n$ denotes the n dimensional Euclidean space and $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). $\mathcal{C}_{n,\tau} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,\ 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The following norms will be used: $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm; $\|\phi\|_c = \sup_{-\tau\leq t\leq 0}\|\phi(t)\|$ stands for the norm of a function $\phi \in \mathcal{C}_{n,\tau}$. Moreover, we denote by $\mathcal{C}^v_{n,\tau}$ the set $\mathcal{C}^v_{n,\tau} = \{\phi \in \mathcal{C}_{n,\tau}: \|\phi\|_c < v\}$, where v is a positive real number.

II. Problem Statement and Preliminaries

A. Problem Statement

Let us consider the linear system with time-delay in state and input saturation

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + B\sigma(u(t)) \tag{1}$$

$$x(t) = \psi(t), \quad t \in [-\tau, 0] \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, τ a constant and A, A_d and B are known matrices. Assume that the initial condition ψ is a continuous vector-valued function, *i.e.*, $\psi \in \mathcal{C}_{n,\tau}$. The function $\sigma: \mathbb{R}^m \to \mathbb{R}^m$ is the standard saturation function defined as follows:

$$\sigma(u) = [\sigma(u_1) \quad \sigma(u_2)\cdots\sigma(u_m)]^T$$

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where $\sigma(u_i) = \text{sign}(u_i)\min\{1, |u_i|\}$. Here we have slightly abused the notation by using σ to denote both the scalar valued and the vector valued saturation functions. Also, note that it is without loss of generality to assume unity saturation level. We use $x_t \in \mathcal{C}_{n,\tau}$ to denote the restriction of $x(t)$ to the interval $[t-\tau, t]$ translated to $[-\tau, 0]$, that is, $x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0]$.

In this paper, we consider the control of the system (1) using a linear state feedback $u = Fx$. The closed-loop system under this feedback is given by

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + B\sigma(Fx(t)), \quad x_0 = \psi \in \mathcal{C}_{n,\tau}. \tag{3}$$

We will be interested in the stability analysis and design for (3). For an initial condition $x_0 \in \mathcal{C}_{n,\tau}$, denote the state trajectory of the system (3) as $x(t, x_0)$. Suppose that the solution $x(t) \equiv 0$ is asymptotically stable, then the domain of attraction of the origin is

$$\mathcal{S} := \left\{ x_0 \in \mathcal{C}_{n,\tau}: \quad \lim_{t\to\infty} x(t, x_0) = 0 \right\}.$$

A set $\mathcal{X} \subset \mathcal{C}_{n,\tau}$ is said to be invariant if

$$x_0 \in \mathcal{X} \Longrightarrow x_t \in \mathcal{X} \qquad \forall t \geq 0.$$

In general, given a stabilizing state feedback $u = Fx$, it is impossible to determine exactly the domain of attraction of the origin. The objective of this paper is to obtain an estimate of the domain of attraction for (3). The problems to be studied in this paper are the following.

Problem 1: Given a state feedback matrix F and a set of initial conditions $\mathcal{D}$, determine if $\mathcal{D} \subset \mathcal{S}$.

Problem 2: Design an F such that an estimate of the domain of attraction is maximized.

B. Razumikhin Theorem and Krasovskii Theorem

For stability analysis of systems with time-delay, the Razumikhin Theorem and Krasovskii Theorem are used extensively. In what follows, we give a brief summary of the two theorems simplified to autonomous systems.

Consider the retarded functional differential equation

$$\dot{x}(t) = f(x_t), \quad t \geq 0 \tag{4}$$
$$x(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{5}$$

Assume that $\psi \in \mathcal{C}_{n,\tau}$ and the map $f(\psi): \mathcal{C}_{n,\tau} \longmapsto \mathbb{R}^n$ is continuous and Lipschitzian in ψ and $f(0) = 0$. Also denote the solution of the functional differential (4) with the initial condition $x_0 \in \mathcal{C}_{n,\tau}$ as $x(t, x_0)$.

Definition 1: The trivial solution $x(t) \equiv 0$ of (4) and (5) is said to be asymptotically stable if

1) for every $\delta > 0$ there exists an $\epsilon = \epsilon(\delta)$ such that for any $\psi \in \mathcal{B}(0, \epsilon)$ the solution $x(t, \psi)$ of (4) and (5) satisfies $x_t \in \mathcal{B}(0, \delta)$ for all $t \geq 0$.
2) for every $\eta > 0$ there exist a $T(\eta)$ and a $v_0 > 0$ independent of η such that $\psi \in \mathcal{B}(0, v_0)$ implies that $\|x_t\|_c < \eta, \forall t \geq T(\eta)$.

The Krasovskii Theorem and the Razumikhin Theorem give conditions for $x(t) \equiv 0$ to be asymptotically stable. Actually, more information about invariant set and regional stability is contained in the proofs for these theorems in [9]. The additional information is incorporated in the following statement of these theorems.

Theorem 1 (Krasovskii Stability Theorem): Suppose that the function $f: \mathcal{C}_{n,\tau} \longmapsto \mathbb{R}^n$ takes bounded sets of $\mathcal{C}_{n,\tau}$ in bounded sets of $\mathbb{R}^n$ and suppose that $u(s)$, $v(s)$ and $w(s)$ are scalar, continuous, positive and nondecreasing functions. If there is a continuous function $V: \mathcal{C}_{n,\tau} \longmapsto \mathbb{R}^+$ and a positive number ρ such that for all $x_t \in L_V(\rho) := \{\psi \in \mathcal{C}_{n,\tau} : V(\psi) \leq \rho\}$, the following conditions hold.

1) $u(\|x_t(0)\|) \leq V(x_t) \leq v(\|x_t\|_c)$.
2) $\dot{V}(x_t) \leq -w(\|x_t(0)\|)$.

Then, the solution $x(t) \equiv 0$ of the (4) and (5) is asymptotically stable. Moreover, the set $L_V(\rho)$ is an invariant set inside the domain of attraction.

Theorem 2 (Razumikhin Stability Theorem): Suppose that $u(s)$, $v(s)$, $w(s)$ and $p(s) \in \mathbb{R}^+ \longmapsto \mathbb{R}^+$ are scalar, continuous and nondecreasing functions, $u(s)$, $v(s)$, $w(s)$ positive for $s > 0$, $u(0) = v(0) = 0$ and $p(s) > s$ for $s > 0$. If there is a continuous function $V: \mathbb{R}^n \longmapsto \mathbb{R}$ and a positive number ρ, such that for all $x_t \in M_V(\rho) := \{\psi \in \mathcal{C}_{n,\tau}: V(\psi(\theta)) \leq \rho, \ \forall\theta \in [-\tau, 0]\}$, the following conditions hold.

1) $u(\|x\|) \leq V(x) \leq v(\|x\|)$.
2) $\dot{V}(x(t)) \leq -w(\|x(t)\|)$, *if* $V(x(t+\theta)) < p(V(x(t))), \forall\theta \in [-\tau, 0]$.

Then, the solution $x(t) \equiv 0$ of the (4) and (5) is asymptotically stable. Moreover, the set $M_V(\rho)$ is an invariant set inside the domain of attraction.

C. Some Mathematical Tools

Let f_i be the i-th row of the matrix F. We define the symmetric polyhedron

$$\mathcal{L}(F) = \{x \in \mathbb{R}^n: |f_i x| \leq 1, \quad i = 1, \ldots, m\}.$$

If the control u does not saturate for all $i = 1, \ldots, m$, that is $x \in \mathcal{L}(F)$, then the nonlinear system (3) admits the following linear representation:

$$\dot{x}(t) = (A + BF)x(t) + A_d x(t-\tau). \tag{6}$$

Let $P \in \mathbb{R}^{n\times n}$ be a positive-definite matrix. For a number $\rho > 0$, the ellipsoid $\Omega(P, \rho)$ is defined by

$$\Omega(P, \rho) := \left\{ x \in \mathbb{R}^n: x^T P x \leq \rho \right\}.$$

Let $\mathcal{V}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. Then there are 2^m elements in $\mathcal{V}$. Suppose that each element of $\mathcal{V}$ is labeled as D_i, $i = 1, 2, \ldots, 2^m$ and denote $D_i^- = I - D_i$. Clearly, D_i^- is also an element of $\mathcal{V}$ if $D_i \in \mathcal{V}$.

Lemma 1 [11]: Let $F, H \in \mathbb{R}^{m\times n}$ be given. For $x \in \mathbb{R}^n$, if $\|Hx\|_\infty \leq 1$, then

$$\sigma(Fx) \in \text{co}\left\{ D_i F x + D_i^- H x: i \in [1, 2^m] \right\}$$

where co$\{\cdot\}$ *denotes the convex hull of a set.*

Lemma 2 [3]: For any $x, y \in \mathbb{R}^n$ and a matrix $M > 0$ with compatible dimensions, the following inequality holds

$$2x^T y \leq x^T M x + y^T M^{-1} y.$$

III. Delay-Independent Analysis

In this section, we will give methods for estimating the domain of attraction for the system (3) with invariant sets. We will first give conditions for a set to be an invariant set inside the domain of attraction and then use optimization approach to enlarge the invariant set by choosing the feedback gain matrix F and the Lyapunov function.

A. Razumikhin Functional Approach

Theorem 3: Let $F \in \mathbb{R}^{m\times n}$ be given. For a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a number $\rho > 0$, consider the set

$$M_V(\rho) = \{\psi \in \mathcal{C}_{n,\tau}: \psi(\theta) \in \Omega(P, \rho) \qquad \forall\theta \in [-\tau, 0]\}.$$

If there exist two matrices $H \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{n\times n}$, $W > 0$ such that

$$\begin{aligned} &\left(A + B\left(D_i F + D_i^- H\right)\right)^T P \\ &\quad + P\left(A + B\left(D_i F + D_i^- H\right)\right) \\ &\quad + P A_d W A_d^T P + P \\ &< 0, \ i \in [1, 2^m], \end{aligned} \tag{7}$$

$$P \geq W^{-1} \tag{8}$$

and $\Omega(P,\rho) \subset \mathcal{L}(H)$, i.e., $|h_i x| \le 1$ for all $x \in \Omega(P,\rho)$, $i = 1,2,\ldots,m$, then the solution $x(t) \equiv 0$ is asymptotically stable for the system (3) and the set $M_V(\rho)$ is an invariant set inside the domain of attraction.

Proof: Given $P > 0$, consider a quadratic Lyapunov function candidate $V(x) = x^T P x$. First, we have $\varepsilon_1 \|x\|^2 \le V(x) \le \varepsilon_2 \|x\|^2$, where $\varepsilon_1 = \lambda_{\min}(P)$, $\varepsilon_2 = \lambda_{\max}(P)$. The derivative of V along the solutions of (3) is

$$\dot V(x(t)) = 2x(t)^T PAx(t) + 2x^T(t)PA_d x(t-\tau) + 2x^T(t)PB\sigma(Fx(t)).$$

In what follows, we will be interested in $x_t \in M_V(\rho)$. In this case, $x(t) \in \Omega(P,\rho)$. Since $|h_i x| \le 1$ for all $x \in \Omega(P,\rho)$, $i = 1,2,\ldots,m$, by Lemma 1, for every $x(t) \in \Omega(P,\rho)$

$$\sigma(Fx(t)) \in \text{co}\left\{\left(D_i F + D_i^- H\right)x(t)\text{: } i = 1,\ldots,2^m\right\}.$$

It follows that for every $x(t) \in \Omega(P,\rho)$, we have

$$\dot V(x(t)) \le \max_{i\in[1,2^m]} 2x^T(t)P\left(A + B\left(D_i F + D_i^- H\right)\right)x(t) + 2x^T(t)PA_d x(t-\tau).$$

From Lemma 2 and (8), we further have

$$\begin{aligned}\dot V(x(t)) \le \max_{i\in[1,2^m]} x^T(t)\Bigg(&\left(A + B\left(D_i F + D_i^- H\right)\right)^T P \\ &+ P\left(A + B\left(D_i F + D_i^- H\right)\right) \\ &+ PA_d W A_d^T P\Bigg)x(t) + V(x(t-\tau)).\end{aligned} \tag{9}$$

By Razumikhin Theorem, to prove that $M_V(\rho)$ is an invariant set inside the domain of attraction, it suffices to show that there exist an $\varepsilon > 1$ and a $\delta > 0$ such that

$$\begin{aligned}&\dot V(x(t)) \le -\delta V(x(t)),\\ &\text{if}\quad V(x(t+\theta)) < \varepsilon V(x(t)) \qquad \forall\theta \in [-\tau, 0].\end{aligned} \tag{10}$$

In the remainder of the proof, we will construct such ε and δ and show that they satisfy (10).

From (7), we see that there exists a $\delta > 0$ such that

$$\begin{aligned}&\left(A + B\left(D_i F + D_i^- H\right)\right)^T P\\ &\quad + P\left(A + B\left(D_i F + D_i^- H\right)\right)\\ &\quad + PA_d W A_d^T P + (1+2\delta)P\\ &<0, \quad i \in [1, 2^m].\end{aligned}$$

Let $\varepsilon = 1 + \delta$. Now suppose that $V(x(t+\theta)) < \varepsilon V(x(t))$, $\forall\theta \in [-\tau, 0]$. Then from (9), we have

$$\begin{aligned}\dot V(x) \le& \max_{i\in[1,2^m]} x^T\Bigg(\left(A + B\left(D_i F + D_i^- H\right)\right)^T P\\ &+ P\left(A + B\left(D_i F + D_i^- H\right)\right) + PA_d W A_d^T P + \varepsilon P\Bigg)x\\ <& -\delta V(x(t)).\end{aligned}$$

This completes the proof. ∎

Note that the condition of Theorem 3 does not include any information of time-delay, *i.e.*, the theorem provides a delay-independent condition for regional stability of linear time-delay systems with input saturation in terms of the feasibility of several linear matrix inequalities. This result can also be easily extended to systems with multiple time-varying time-delays in state [2].

Remark 1: In practice, we may be interested in the stability region in which the asymptotic stability of closed-loop system (3) is guaranteed under saturation and the linear closed-loop system (6) (i.e., unsaturated closed-loop system) is β-stable. As shown in [17], β-stability is equivalent to

$$\dot y(t) = (A + BF + \beta I)y(t) + e^{\beta\tau} A_d y(t-\tau)$$

which is stable. This can be guaranteed by the following matrix inequality:

$$(A+BF)^T P + P(A+BF) + e^{2\beta\tau} PA_d W A_d^T P + 2\beta P < 0. \tag{11}$$

Remark 2: If the matrix A_d is rank deficiency, i.e. there exists a decomposition $A_d = D_d E_d$, where $D_d \in \mathbb{R}^{n\times p}$, $E_d \in \mathbb{R}^{p\times n}$, $p < n$, then we can prove with similar arguments that $M_V(\rho)$ is an invariant set inside the domain of attraction if there exist two matrices $H \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{p\times p} > 0$ satisfying the matrix inequalities

$$\begin{aligned}&\left(A + B\left(D_i F + D_i^- H\right)\right)^T P\\ &\quad + P\left(A + B\left(D_i F + D_i^- H\right)\right)\\ &\quad + PD_d W D_d^T P + P\\ &<0, \quad i \in [1, 2^m], \qquad E_d^T W^{-1} E_d \le P.\end{aligned}$$

With all the $M_V(\rho)$ satisfying the set invariance condition, we would like to choose the "largest" one to obtain the least conservative estimate of the domain of attraction by the method introduced in [12]. We see that the "size" of the set $M_V(\rho)$ is proportional to the size of $\Omega(P,\rho)$. Here we would like to take the shape of $\Omega(P,\rho)$ into consideration. For this purpose, we introduced the notion of shape reference set as in [12]. Let $\mathcal{X}_\mathrm{R} \subset \mathbb{R}^n$ be a prescribed bounded convex set containing origin. For a set $S \subset \mathbb{R}^n$ containing origin, define the size of S with respect to $\mathcal{X}_\mathrm{R}$ as

$$\alpha_\mathrm{R}(S) := \sup\{\alpha > 0\text{: } \alpha\mathcal{X}_\mathrm{R} \subset S\}.$$

Obviously, if $\alpha_\mathrm{R}(S) \ge 1$, then $\mathcal{X}_\mathrm{R} \subset S$. Two typical types of $\mathcal{X}_\mathrm{R}$ are the ellipsoid

$$\mathcal{X}_\mathrm{R} = \{x \in \mathbb{R}^n\text{: } x^T R x \le 1\} \quad R > 0$$

and the polyhedron

$$\mathcal{X}_\mathrm{R} = \text{co}\{x_1, x_2, \ldots, x_l\}$$

where $x_1, x_2, \ldots, x_l$ are some given points in $\mathbb{R}^n$.

Theorem 3 gives a condition for a set $M_V(\rho)$ to be inside the domain of attraction for the closed-loop time-delay system subject to input saturation (3). With a given shape reference set, we can choose from all the $\Omega(P,\rho)$'s that satisfy the condition such that the quantity $\alpha_\mathrm{R}(\Omega(P,\rho))$ is maximized. This problem can be formulated as

$$\begin{aligned}&\sup_{P\ge W^{-1}>0,\rho,H} \alpha, \quad \text{s.t.}\\ &a)\ \alpha\mathcal{X}_\mathrm{R} \subset \Omega(P,\rho),\\ &b)\ \left(A + B\left(D_i F + D_i^- H\right)\right)^T P + P\left(A + B\left(D_i F + D_i^- H\right)\right)\\ &\qquad + PA_d W A_d^T P + P < 0, \quad i \in [1, 2^m],\\ &c)\ |h_i x| \le 1,\ \forall x \in \Omega(P,\rho), \quad i \in [1, m].\end{aligned} \tag{12}$$

Let $Q = (\rho^{-1}P)^{-1}$, $\gamma = 1/\alpha^2$ and $G = HQ$. With similar procedure as in [12], we can transform the above optimization problem to an LMI problem. That is, if we substitute ρW with W, then for the case that $\mathcal{X}_\mathrm{R}$ is a polyhedron, the optimization problem (12) can be rewritten as follows:

$$\begin{aligned}&\inf_{W\ge Q>0,G} \gamma, \quad \text{s.t.}\\ &a)\ \begin{bmatrix}\gamma & x_i^T\\ x_i & Q\end{bmatrix} \ge 0,\ i \in [1, l].\\ &b)\ QA^T + AQ + B\left(D_i FQ + D_i^- G\right)\\ &\quad + \left(D_i FQ + D_i^- G\right)^T B^T + A_d W A_d^T\\ &\quad + Q < 0, i \in [1, 2^m].\\ &c)\ \begin{bmatrix}1 & g_i\\ g_i^T & Q\end{bmatrix} \ge 0, \quad i \in [1, m].\end{aligned} \tag{13}$$

If $\mathcal{X}_\mathrm{R}$ is an ellipsoid, then we need to replace $a)$ in (13) with

$$\alpha^{-2} R \ge \rho^{-1} P \Longleftrightarrow R^{-1} \le \gamma Q.$$

As is proven in [12], for systems without delay, *i.e.*, $A_d = 0$, solving the above LMI optimization problem will give a less conservative estimate of the domain of attraction than other methods resulting from, for example, the circle criterion.

If the unsaturated system is required to have some stability margin, *i.e.*, it is required to be β-stable, based on Remark 1, the additional LMI constraint (11) needs to be added to optimization problem (13), leading to the following LMI optimization problem:

$$\begin{aligned}&\inf_{W \geq Q > 0, G} \gamma, \quad \text{s.t.}\\ &a)\ b) \text{ and } c) \text{ in } (13).\\ &d)\ QA^T + AQ + BFQ + (BFQ)^T\\ &\qquad + e^{2\beta\tau} A_d W A_d^T + 2\beta Q < 0. \end{aligned} \tag{14}$$

The problem of designing a feedback matrix F such that the estimate of the domain of attraction is enlarged can be formulated by simply taking the parameter F in (13) as a variable for optimization. To do so, we just need to replace $Y = FQ$ in (13b) with a new varaiable Y.

B. Krasovskii Functional Approach

In this subsection, we will consider the following Lyapunov-Krasovskii functional:

$$V(x_t) := x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Wx(s)ds \tag{15}$$

where $P > 0$ and $W > 0$. This type of functional has been widely used for stability analysis of time-delay systems (see, *e.g.*, [9]).

Theorem 4: Let the feedback gain $F \in \mathbb{R}^{m\times n}$ be given. For given $P, W > 0$ and $\rho > 0$, consider the set

$$L_V(\rho) = \left\{ \psi \in \mathcal{C}_{n,\tau}\colon \psi^T(0)P\psi(0) + \int_{-\tau}^{0} \psi^T(s)W\psi(s)ds \leq \rho \right\}. \tag{16}$$

If there exists a matrix $H \in \mathbb{R}^{m\times n}$ such that we get (17) shown at the bottom of the page and $\Omega(P,\rho) \subset \mathcal{L}(H)$, then the solution $x(t) \equiv 0$ of the system (3) is asymptotically stable. Moreover, the set $L_V(\rho)$ is an invariant set inside the domain of attraction.

Proof: Consider the Lyapunov functional given by (15). First, we have

$$\varepsilon_1 \|x_t(0)\|^2 \leq V(x_t) \leq \varepsilon_2 \|x_t\|_c^2$$

where $\varepsilon_1 = \lambda_{\min}(P)$, $\varepsilon_2 = \lambda_{\max}(P) + \tau\lambda_{\max}(W)$. Then

$$\begin{aligned}\dot{V}(x_t) = x^T(t)(A^TP + PA + W)x(t) + 2x^T(t)PA_d x(t-\tau)\\ +2x^T(t)PB\sigma(Fx(t)) - x^T(t-\tau)Wx(t-\tau).\end{aligned}$$

We will be interested in $x_t \in L_V(\rho)$. In this case, $x(t) \in \Omega(P,\rho) \subset \mathcal{L}(H)$ and we have

$$\sigma(Fx(t)) \in \text{co}\left\{ \left(D_iF + D_i^- H\right)x(t)\colon i \in [1, 2^m] \right\}.$$

With similar arguments as in the proof of Theorem 3, we get the second equation shown at the bottom of the page where $\xi^T(t) = \left[x^T(t)\ x^T(t-\tau)\right]$. Under the condition (17), there exists a $\delta > 0$ such that we get the third equation shown at the bottom of the page. It follows that

$$\dot{V}(x_t) < -\delta x(t)^T Px(t) \leq -\delta\varepsilon_1 \|x(t)\|_2^2.$$

By Krasovskii Stability Theorem, $L_V(\rho)$ is an invariant set inside the domain of attraction. ■

As an estimate of the domain of attraction, the invariant set $L_V(\rho)$ in Theorem 4 depends not only on the P matrix, but also on an integration over $[-\tau, 0]$. This makes the structure of the set $L_V(\rho)$ much more complicated than the invariant set $M_V(\rho)$ in Theorem 3 based on Lyapunov-Razumikhin functional approach. Hence, it is not easy to measure the size of the set $L_V(\rho)$. Because of this, we would like to determine a subset of $L_V(\rho)$ which is of a more regular shape, say, like $M_V(\rho)$ in Theorem 3.

Let $z(t) = P^{1/2}x(t)$. Then

$$\begin{aligned}\|z_t\|_c &= \sup_{-\tau\leq\theta\leq 0} \|z(t)\|\\ &= \sup_{-\tau\leq\theta\leq 0} \left(x^T(t)Px(t)\right)^{1/2}\end{aligned}$$

$$\text{and} \quad V(x_t) \leq \left(1 + \tau\lambda_{\max}\left(P^{-1/2}WP^{-1/2}\right)\right)\|z_t\|_c^2.$$

Let

$$\rho_1 = \frac{\rho}{1 + \tau\lambda_{\max}\left(P^{-1/2}WP^{-1/2}\right)}.$$

Then, we have

$$M(\rho_1) = \left\{ \psi \in \mathcal{C}_{n,\tau}\colon \psi(\theta)^T P\psi(\theta) \leq \rho_1, \quad \forall\theta \in [-\tau, 0] \right\} \subset L_V(\rho).$$

On the other hand, let

$$\delta = \frac{\rho}{\lambda_{\max}(P) + \tau\lambda_{\max}(W)}$$

then the ball $\mathcal{B}(\delta) = \{\psi \in \mathcal{C}_{n,\tau}\colon \|\psi\|_c^2 < \delta\}$ is inside the domain of attraction. We see that the size of $M(\rho_1)$ is proportional to the size of $\Omega(P,\rho_1)$. With a given $\mathcal{X}_{\rm R}$, we can choose from all the $\Omega(P,\rho_1)$'s

$$\begin{bmatrix} \left(A + B\left(D_iF + D_i^- H\right)\right)^T P + P\left(A + B\left(D_iF + D_i^- H\right)\right) + W & PA_d \\ A_d^T P & -W \end{bmatrix} < 0,\ i \in [1, 2^m] \tag{17}$$

$$\dot{V}(x_t) \leq \max_{i\in[1,2^m]} \xi(t)^T \begin{bmatrix} \left(A + B\left(D_iF + D_i^- H\right)\right)^T P + P\left(A + B\left(D_iF + D_i^- H\right)\right) + W & PA_d \\ A_d^T P & -W \end{bmatrix} \xi(t)$$

$$\begin{bmatrix} \left(A + B\left(D_iF + D_i^- H\right)\right)^T P + P\left(A + B\left(D_iF + D_i^- H\right)\right) + W & PA_d \\ A_d^T P & -W \end{bmatrix} < -\begin{bmatrix} \delta P & 0 \\ 0 & 0 \end{bmatrix}, \quad i \in [1, 2^m].$$

such that the quantity $\alpha_{\mathrm{R}}(\Omega(P,\rho_1))$ is maximized. This problem can be formulated as shown in (18) at the bottom of the page.

As in Section III-A, we can cast the problem into the LMI framework. Let $Q=(\rho^{-1}P)^{-1}$, $\gamma=1/\alpha^2$, $G=HQ$ and substitute ρW^{-1} with X, we can reduce the optimization problem (18) to the one with LMI constraints as shown in (19) at the bottom of the page. If $\mathcal{X}_{\mathrm{R}}$ is an ellipsoid, then $a)$ should be replaced with

$$\begin{aligned}\alpha^{-2}R-\left(1+\tau\lambda_{\max}\left(P^{-1/2}WP^{-1/2}\right)\right)\frac{P}{\rho} &>0\\ \Leftrightarrow (1+\tau\varepsilon)R^{-1} &<\gamma Q.\end{aligned}$$

Also, as in Section III-A, a controller design problem can be readily formulated by taking F in (19) as an optimizing parameter.

IV. Delay-Dependent Analysis

To reduce conservativeness in the analysis when the information on the delay is available, in this section, we will establish a delay-dependent stability result for the time-delay system (3) with input saturation. For simplicity, denote $\hat{A}_i := A+A_d+B(D_iF+D_i^-H)$.

Theorem 5: Let the state feedback gain F be given. Consider the ellipsoid $\Omega(P,\rho)$. If there exist matrices $H\in\mathbb{R}^{m\times n}$, $P_1,P_2\in\mathbb{R}^{n\times n}$, $P_1>0$, $P_2>0$ and $\tau_0>0$ such that

$$\begin{aligned}\hat{A}_i{}^TP+P\hat{A}_i+\tau_0PA_d(P_1+P_2)A_d^TP+2\tau_0P &\\ <0, \quad i\in[1,2^m] &\qquad (20)\\ \left[A+B(D_iF+D_i^-H)\right]^TP_1^{-1}\left[A+B\left(D_iF+D_i^-H\right)\right] &\\ \le P, \quad i\in[1,2^m] &\qquad (21)\\ A_d^TP_2^{-1}A_d \le P &\qquad (22)\end{aligned}$$

and $\Omega(P,\rho)\subset\mathcal{L}(H)$, *then* $x(t)\equiv 0$ *of the system (3) is delay-dependent asymptotically stable. Moreover, for any time-delay* $\tau\le\tau_0$ *and any initial condition* ψ, $\psi(\theta)\in\Omega(P,\rho)$, $\forall\theta\in[-\tau,0]$, *we have* $\lim_{t\to\infty}x(t)=0$.

Proof: Since $x(t)$ is continuously differentiable for $t\ge 0$, using the Leibniz-Newton formula, one can write

$$\begin{aligned}x(t-\tau)&=x(t)-\int_{t-\tau}^{t}\dot{x}(s)ds\\ &=x(t)-\int_{t-\tau}^{t}\Big[Ax(s)+A_dx(s-\tau)\\ &\qquad +B\sigma(Fx(s))\Big]ds \qquad (23)\end{aligned}$$

for $t\ge\tau$. Thus the system (3) can be rewritten as

$$\begin{aligned}\dot{x}(t)=&(A+A_d)x(t)\\ &-A_d\int_{t-\tau}^{t}\Big[Ax(s)+A_dx(s-\tau)+B\sigma(Fx(s))\Big]ds\\ &+B\sigma(Fx(t))\\ x(t)=&\psi(t), \quad \in[-2\tau,0] \qquad (24)\end{aligned}$$

where $\psi\in\mathcal{C}_{n,2\tau}$. By [9], [21], the asymptotic stability of the above system will ensure the asymptotic stability of the original time-delay system (3).

Choose the Lyapunov functional candidate as $V(x(t)) = x^T(t)Px(t)$. To prove the theorem, it suffices to show that $x(t)\equiv 0$ is asymptotically stable for the system (24) and that the set

$$M_V(\rho)=\{\psi\in\mathcal{C}_{n,2\tau}\colon \psi(\theta)\in\Omega(P,\rho),\ \forall\theta\in[-2\tau,0]\}$$

is an invariant set inside the domain of attraction. Here we use x_t to denote the restriction of $x(t)$ to the interval $[t-2\tau,t]$ translated to $[-2\tau,0]$, that is, $x_t(\theta)=x(t+\theta)$, $\theta\in[-2\tau,0]$.

We are interested in $x_t\in M_V(\rho)$. In this case, $x(t)\in\Omega(P,\rho)$ and we have

$$\begin{aligned}\dot{V}(x(t))\le&2\max_{i\in[1,2^m]}x^T(t)P\hat{A}_ix(t)\\ &+\tau x^T(t)PA_d(P_1+P_2)A_d^TPx(t)\\ &+\int_{-\tau}^{0}\left[Ax(t+s)+B\sigma(Fx(t+s))\right]^TP_1^{-1}\\ &\quad\times\left[Ax(t+s)+B\sigma(Fx(t+s))\right]ds\\ &+\int_{-\tau}^{0}\Big[x^T(t-\tau+s)A_d^TP_2^{-1}A_dx(t-\tau+s)ds.\end{aligned}$$

By the convexity of the function $x^TP_1^{-1}x$ and Lemma 1, we have

$$\begin{aligned}&[Ax(t)+B\sigma(Fx(t))]^TP_1^{-1}[Ax(t)+B\sigma(Fx(t))]\\ &\quad\le\max_{i\in[1,2^m]}x(t)^T(A+B(D_iF+D_i^-H))^T\\ &\qquad\cdot P_1^{-1}(A+B(D_iF+D_i^-H))x(t).\end{aligned}$$

It follows from (21) that

$$\begin{aligned}&[Ax(t)+B\sigma(Fx(t))]^T\\ &\qquad\cdot P_1^{-1}[Ax(t)+B\sigma(Fx(t))]\le x^T(t)Px(t)\end{aligned}$$

and from (22), we have

$$x^T(t)A_d^TP_2^{-1}A_dx(t)\le x^T(t)Px(t).$$

$$\begin{aligned}&\sup_{P>0,\rho,H}\ \alpha,\quad \text{s.t.}\\ &a)\ \alpha\mathcal{X}_{\mathrm{R}}\subset\Omega\left(P,\rho_1\right).\\ &b)\ \begin{bmatrix}\left(A+B\left(D_iF+D_i^-H\right)\right)^TP+P\left(A+B\left(D_iF+D_i^-H\right)\right)+W & PA_d\\ A_d^TP & -W\end{bmatrix}<0,\quad i\in[1,2^m].\\ &c)\ |h_ix|\le 1,\ \forall x\in\Omega(P,\rho),\ i\in[1,m].\end{aligned} \qquad (18)$$

$$\begin{aligned}&\inf_{Q>0,X>0,G,\rho}\gamma,\ \text{s.t.}\\ &a)\ \begin{bmatrix}\gamma & (1+\tau\varepsilon)x_i^T\\ (1+\tau\varepsilon)x_i & (1+\tau\varepsilon)Q\end{bmatrix}>0,\ Q\le\varepsilon X,\ i\in[1,l].\\ &b)\ \begin{bmatrix}QA^T+AQ+B\left(D_iFQ+D_i^-G\right)+\left(D_iFQ+D_i^-G\right)^TB^T+A_dXA_d^T & Q\\ Q & -X\end{bmatrix}<0, i\in[1,2^m].\\ &c)\ \text{Constraint } (13c).\end{aligned} \qquad (19)$$

Hence

$$\begin{aligned}\dot{V}(x(t)) \leq & 2 \max_{i\in[1,2^m]} x^T(t)P\hat{A}_i x(t) \\ & + \tau x^T(t)PA_d(P_1+P_2)A_d^T P x(t) \\ & + \int_{-\tau}^{0} V(x(t+s))ds \\ & + \int_{-\tau}^{0} V(x(t-\tau+s))ds. \end{aligned} \tag{25}$$

By Razumikhin Theorem, to show that $M_V(\rho)$ is an invariant set inside the domain of attraction, it suffices to construct an $\varepsilon > 1$ and a $\delta > 0$ such that

$$\begin{aligned} &\dot{V}(x(t)) < -\delta V(x(t)), \\ \text{if } & V(x(t+\theta)) < \varepsilon V(x(t)) \qquad \forall \theta \in [-2\tau, 0]. \end{aligned} \tag{26}$$

Under the condition of (20), there exists a $\delta_1 > 0$ such that

$$\hat{A}_i{}^T P + P\hat{A}_i + \tau_0 PA_d\left(P_1+P_2\right)A_d^T P + 2\tau_0\left(1+2\delta_1\right)P < 0, \quad i \in [1, 2^m].$$

Let $\varepsilon = 1 + \delta_1$. Suppose that $V(x(t-\theta)) < \varepsilon V(x(t)) \quad \forall \theta \in [-2\tau, 0]$. Then from (25), we have

$$\begin{aligned}\dot{V}(x(t)) \leq & 2 \max_{i\in[1,2^m]} x^T(t)P\hat{A}_i x(t) \\ & + \tau_0 x^T(t)PA_d(P_1+P_2)A_d^T P x(t) \\ & + 2\tau_0 \varepsilon x(t)^T P x(t) \\ = & \max_{i\in[1,2^m]} x^T \left(\hat{A}_i^T P + P\hat{A}_i + \tau_0 PA_d(P_1+P_2)A_d^T P \right. \\ & \left. + 2\tau_0\varepsilon P\right) x(t) \\ < & -2\tau_0\delta_1 x(t)^T P x(t). \end{aligned}$$

This completes our proof. ■

Remark 3: By letting $Q = \rho P^{-1}, G = HQ$ and $\bar{A}_i = A + A_d$, we see that the matrix inequalities (20) to (22) are equivalent to the LMIs shown in (27)–(29) at the bottom of the page where we have replaced P_1 and P_2 with P_1/ρ and P_2/ρ.

Theorem 5 provides a delay-dependent condition for regional stability of linear time-delay systems with input saturation in terms of the feasibility of several linear matrix inequalities. This result can also be easily extended to systems with multiple time-varying time-delays in state [3]. Note that in the proof the transformation (23) is used to transform the time-delay system with single time delay to a system with distributed delay. It is shown in [8] that such a transformation may incur some additional dynamics that can be characterized by appropriate additional eigenvalues. And hence, if the smallest of such delays is less than the stability delay limit of the original system, then any stability criteria obtained using such transformation will be conservative.

Remark 4: Theorems 4 and 5 can also be strengthened when A_d is rank deficiency as in Remark 2.

As in Section III-A, we can propose an LMI optimization method for estimating the domain of attraction for any given time-delay for the system (3). If $\mathcal{X}_{\mathrm{R}}$ is a polyhedron, then we have the following optimization problem for estimating the domain of attraction for systems with $\tau \leq \tau_0$

$$\begin{aligned} & \inf_{Q>0, P_1>0, P_2>0, G} \gamma, \quad \text{s.t.} \\ & a) \begin{bmatrix} \gamma & x_i^T \\ x_i & Q \end{bmatrix} \geq 0, \quad i = 1, 2, \ldots, l \\ & b) \text{ LMI } (27)\text{–}(29), \quad i \in [1, 2^m], \\ & c) \text{ Constrain } (13c). \end{aligned} \tag{30}$$

As usual, the analysis problem can be easily modified for controller design by taking F as an optimizing parameter.

V. Numerical Examples

Example 1: Consider the example given in [22]. The system is described by (1) with

$$A = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 10 \\ 1 \end{bmatrix}, \quad \tau = 1, \quad u_{\max} = 15.$$

In [22], a feedback matrix

$$F = [-0.3592 \quad -0.1421]$$

is obtained with local stability in the ball $\mathcal{B}(\delta) = \{x \in \mathbb{R}^n \colon \|x\|^2 \leq \delta\}$ with $\delta = 1.7919 \times 10^3$. As in [22], we require that the origin of the saturated system be asymptotically stable and that the unsaturated system is β-stable with $\beta = 1$. By Theorem 3 and solving optimization problem (14) with the above control law and a unit ball as the reference set, we obtain

$$\alpha = 47.0626, \quad P = \begin{bmatrix} 0.1324 & 0.0283 \\ 0.0283 & 0.4489 \end{bmatrix} \times 10^{-3}.$$

This means that the asymptotic stability of the saturated system and β-stability $(\beta = 1)$ of the unsaturated system are guaranteed in the ellipsoid $\Omega(P, 1)$ which include the ball $\mathcal{B}(\delta)$ with $\delta = \alpha^2 = 2.2149 \times 10^3$. Obviously, this estimation is less conservative than the result of [22].

If we only require that the saturated system be asymptotically stable, *i.e.,*, $\beta = 0$, by Theorem 3, we obtain

$$\alpha = 67.0618 \quad P = \begin{bmatrix} 0.2223 & 0.0000 \\ 0.0000 & 0.2223 \end{bmatrix} \times 10^{-3}.$$

This means that the asymptotic stability of the saturated time-delay system is guaranteed in the ellipsoid $\Omega(P, 1)$ which includes the ball $\mathcal{B}(\delta)$ with $\delta = \alpha^2 = 4.4973 \times 10^3$. This is an estimate of the domain of attraction of the saturated time-delay system. Note that this estimate of the domain of attraction is delay-independent, *i.e.,*, it holds for any size of time-delay. This ellipsoid $\Omega(P, 1)$ is shown in Fig. 1. The dot-

$$Q\bar{A}_i^T + \bar{A}_i Q + B(D_i FQ + D_i^- G) + (D_i FQ + D_i^- G)^T B^T + \tau_0 A_d(P_1+P_2)A_d^T + 2\tau_0 Q < 0 \tag{27}$$

$$\begin{bmatrix} Q & [AQ + B(D_i FQ + D_i^- G)]^T \\ AQ + B(D_i FQ + D_i^- G) & P_1 \end{bmatrix} \geq 0 \tag{28}$$

$$\begin{bmatrix} Q & QA_d^T \\ A_d Q & P_2 \end{bmatrix} \geq 0 \tag{29}$$

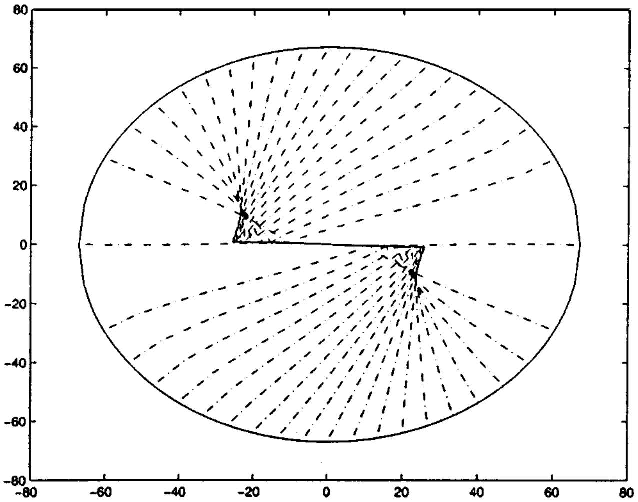

Fig. 1. Estimation of domain of attraction of Example 1.

TABLE I
COMPUTATION RESULTS OF EXAMPLE 1 BY LYAPUNOV-KRASOVSKII APPROACH

β	τ	α	ε	P
1	0.01	94.7215	2.3225	$\begin{bmatrix} 0.8703 & 0.2046 \\ 0.2046 & 0.8980 \end{bmatrix} \times 10^{-4}$
1	0.1	86.5436	2.1114	$\begin{bmatrix} 0.8895 & 0.1949 \\ 0.1949 & 0.9240 \end{bmatrix} \times 10^{-4}$
1	1	54.8073	1.7449	$\begin{bmatrix} 0.0835 & 0.0166 \\ 0.0166 & 0.1139 \end{bmatrix} \times 10^{-3}$
1	10	no solution		
0	0.01	97.6418	4.1145	$\begin{bmatrix} 0.7231 & 0.3103 \\ 0.3103 & 0.6684 \end{bmatrix} \times 10^{-4}$
0	0.1	86.5569	2.3225	$\begin{bmatrix} 0.8567 & 0.2143 \\ 0.2143 & 0.8801 \end{bmatrix} \times 10^{-4}$
0	1	59.8417	0.9850	$\begin{bmatrix} 0.1305 & 0.0060 \\ 0.0060 & 0.1371 \end{bmatrix} \times 10^{-3}$
0	10	27.8434	0.4177	$\begin{bmatrix} 0.1823 & 0.0155 \\ 0.0155 & 0.2456 \end{bmatrix} \times 10^{-3}$

TABLE II
COMPUTATION RESULTS OF EXAMPLE 2

τ_0	α	P	F
0.1	1.5685	$\begin{bmatrix} 0.2418 & -0.0097 \\ -0.0097 & 0.0530 \end{bmatrix}$	$[\,-1.6523 \quad 0.8092\,]$
0.2	1.2597	$\begin{bmatrix} 0.3631 & -0.0260 \\ -0.0260 & 0.0431 \end{bmatrix}$	$[\,-2.0026 \quad 0.8110\,]$
0.3	1.2557	$\begin{bmatrix} 0.5067 & -0.0373 \\ -0.0373 & 0.0311 \end{bmatrix}$	$[\,-2.3284 \quad 0.7827\,]$
0.35	0.9680	$\begin{bmatrix} 0.9033 & -0.0393 \\ -0.0393 & 0.0256 \end{bmatrix}$	$[\,-2.6383 \quad 0.7204\,]$

dashed curves are the state trajectories with the initial conditions on this ellipsoid and $\tau = 10$. Obviously, all trajectories converge to the origin.

If we use the LMI optimization (19) by Lyapunov-Krasovskii approach with the above control law, the computational results are shown in Table I. From this table, we find that our result is less conservative than that of [22] because our estimate of domain of attraction when $\tau = 1$ and $\beta = 1$ includes the ball $\mathcal{B}(\delta)$ with $\delta = \alpha^2 = 3.004 \times 10^3$ which is much bigger than the ball given in [22]. We can also find that the estimation of the domain of attraction by Lyapunov-Krasovskii approach becomes smaller as the size of time-delay becomes larger.

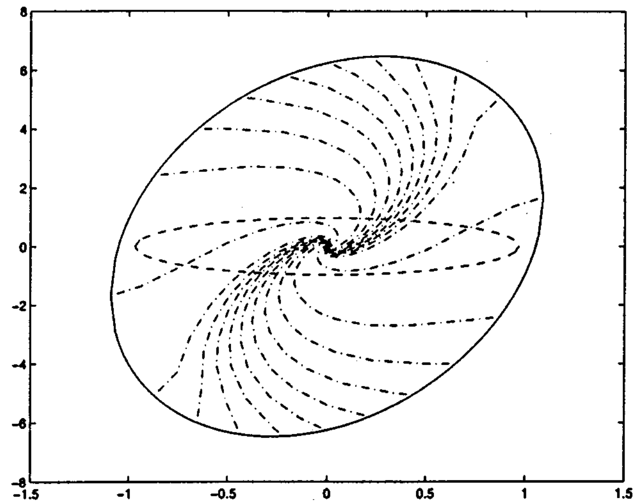

Fig. 2. Estimate of domain of attraction of Example 2.

Example 2: Consider the following delay system (1) with

$$A = \begin{bmatrix} 0.5 & -1 \\ 0.5 & -0.5 \end{bmatrix}$$
$$A_d = \begin{bmatrix} 0.6 & 0.4 \\ 0 & -0.5 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u_{\max} = 5.$$

For simplicity, we also use the unit ball as our reference set. We are not able to obtain a feasible solution to LMI optimization problem (13). This means that this system may not be delay-independently stabilizable by a saturated memoryless state feedback law. Fortunately, the optimization problem (30) is feasible for $\tau_0 \leq 0.35$. This means that this saturated system is delay-dependently stabilizable with a memoryless state feedback. Table II shows the computational results with different time-delay. From Table II, we find that α increases when the system time-delay τ_0 decreases. Fig. 2 illustrates the estimate of the domain of attraction and the state trajectories for $\tau = 0.35$. The outer ellipsoid is $\Omega(P, 1)$ and the inner ellipsoid is the ball $\mathcal{B}(\alpha)$. The dot-dashed curves are the state trajectories with initial conditions inside this ellipsoid $\Omega(P, 1)$.

VI. CONCLUSIONS

In this paper, the domain of attraction of time-delay system subject to input saturation is addressed by applying Lyapunov-Razumihkin and -Krasovskii functional approach. An estimation of the domain of attraction is proposed by using the linear matrix inequality optimization. We also proposed a memoryless state feedback design method for the systems with time-delay in state and subject to input saturation to enlarge the domain of attraction. Both the delay-independent and delay-dependent local stabilizing controllers are discussed. Numerical examples show the effectiveness of the proposed method.

REFERENCES

[1] D. S. Bernstein and A. N. Michel, "A chronological bibliography on saturating actuators," *Int. J. Robust Nonlinear Cont.*, vol. 5, pp. 375–380, 1995.
[2] Y.-Y. Cao and Y.-X. Sun, "Robust stabilization of uncertain systems with time-varying multi-state-delay," *IEEE Trans. Automat. Control*, vol. 43, pp. 1484–1488, Oct. 1998.
[3] Y.-Y. Cao, Y.-X. Sun, and C. Cheng, "Delay-dependent robust stabilization of uncertain systems with multiple state delays," *IEEE Trans. Automat. Control*, vol. 43, pp. 1608–1612, Nov. 1998.

[4] Y.-Y. Cao, Y.-X. Sun, and J. Lam, "Delay-dependent robust H_∞ control for uncertain systems with time-varying delays," *IEE Proc. D: Contr. Theory Appl.*, vol. 145, no. 3, pp. 338–344, 1998.

[5] B.-S. Chen, S.-S. Wang, and H.-C. Lu, "Stabilization of time-delay systems containing saturating actuators," *Int. J. Contr.*, vol. 47, pp. 867–881, 1988.

[6] J.-H. Chou, I.-R. Horng, and B.-S. Chen, "Dynamical feedback compensator for uncertain time-delay systems containing saturating actuator," *Int. J. Contr.*, vol. 49, pp. 961–968, 1989.

[7] K. Gu, "Discretized Lyapunov functional for uncertain systems with multiple time-delay," *Int. J. Contr.*, vol. 72, no. 16, pp. 1436–1445, 1999.

[8] K. Gu and S. I. Niculescu, "Additional dynamics in transformed time-delay systems," *IEEE Trans. Automat. Control*, vol. 45, pp. 572–575, Mar. 2000.

[9] J. Hale, *Theory of Functional Differential Equations*. New York: Springer, 1977.

[10] Q. L. Han and B. Ni, "Delay-dependent robust stabilization for uncertain constrained systems with pointwise and distributed time-varying delays," in *Proc. IEEE 38th Conf. DC*, 1999, pp. 215–220.

[11] T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*. Boston, MA: Birkhäuser, 2001.

[12] T. Hu, Z. Lin, and B. M. Chen, "An analysis and design method for linear systems subject to actuator saturation and disturbance," in *Proc. ACC'00*, 2000, pp. 725–729.

[13] B. Lehman, J. Bentsman, S. V. Lunel, and E. I. Verriest, "Vibrational control of nonlinear time lag systems with bounded delay: Averaging theory, stabilizability and transient behavior," *IEEE Trans. Automat. Control*, vol. 39, pp. 898–912, May 1994.

[14] Z. Lin, *Low Gain Feedback*. London, U.K.: Springer, 1998.

[15] D. Liu and A. Michel, *Dynamical Systems With Saturation Nonlinearities: Analysis and Design*. London, U.K.: Springer-Verlog, 1994.

[16] S. I. Niculescu, J. M. Dion, and L. Dugard, "Robust stabilization for uncertain time-delay systems containing saturating actuators," *IEEE Trans. Automat. Control*, vol. 41, pp. 742–747, May 1996.

[17] S.-I. Niculescu, E. I. Verriest, L. Dugard, and J.-D. Dion, "Stability and robust stability of time-delay systems: A guided tour," in *Stability and Control of Time-Delay Systems*, L. Dugard and E. I. Verriest, Eds. London, U.K.: Springer-Verlag, 1997, vol. 228, pp. 1–71.

[18] S. Oucheriah, "Global stabilization of a class of linear continuous time-delay systems with saturating controls," *IEEE Trans. Circuits Syst. I*, vol. 43, pp. 1012–1015, 1996.

[19] J.-K. Park, C.-H. Choi, and H. Choo, "Dynamic anti-windup method for a class of time-delay control systems with input saturation," *Int. J. Robust Nonlin. Contr.*, vol. 10, no. 6, pp. 457–488, 2000.

[20] G. Stepan, *Retarded Dynamical Systems: Stability and Characteristic Functions* Harlow, U.K, 1989, Pitman Research Notes in Mathematics, Longman Scientific and Technical.

[21] T.-J. Su and C.-G. Huang, "Robust stability of delay dependence for linear uncertain systems," *IEEE Trans. Automat. Control*, vol. 37, pp. 1656–1659, Oct. 1992.

[22] S. Tarbouriech and J. M. Gomes da Silva Jr., "Synthesis of controllers for continuous-time delay systems with saturating controls via LMI's," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 105–111, Jan. 2000.

Multiple Resonance Networks

Antonio Carlos M. de Queiroz

***Abstract*—This brief shows how "multiple resonance networks" of any order and with many possible structures can be systematically designed using standard lossless impedance synthesis techniques. These networks are composed of linear lumped or distributed capacitors, inductors, and transformers, with a switch separating one of the capacitors from the remaining circuit. They have the property of transferring completely the energy initially stored in the capacitor insulated by the switch, to another, much smaller, capacitor in the circuit, through a linear transient when the switch is closed. These circuits find applications in the production of very high voltages for pulsed power systems.**

***Index Terms*—Linear network synthesis, power converters, resonance.**

I. Introduction

"Multiple resonance networks" [1] is a name that generalizes the "double resonance" [2], [3], "triple resonance" [4]–[6], and the higher order networks discussed in this brief. These circuits are usually composed of a transformer and some extra capacitors and inductors and work by transferring the energy initially stored in a capacitor at one side of the transformer to another, much smaller, capacitor at the other side of the transformer, through a linear transient composed (in the ideal lossless case) of a sum of several cosinusoidal waveforms (Fig. 1).

The "double resonance" case is long known [2], [7] as the "Tesla coil" [3]. In this case, only two capacitors and one transformer are used, resulting in a fourth-order system with a transient formed by two oscillatory modes (Fig. 2). With the system properly designed, after some cycles all the initial energy in C_1 is transferred to C_2, and the obtained voltage is given, by energy conservation, by

$$v_{\mathrm{out}}\max = v_{\mathrm{in}}(0)\sqrt{\frac{C_1}{C_p}} \tag{1}$$

(with $p = 2$). This same equation fixes the maximum output voltage for all the systems of this type.

More recently, triple resonance systems were developed [4]–[6] for instrumentation used in high-energy physics. An additional capacitor and an inductor were added to the output side (Fig. 3), with the aim of reducing the voltage stress over the transformer and of taking into consideration the output capacitance of the transformer. With only the extra inductor added, the system is still a double resonance system, long known as the "Tesla magnifier." With the extra capacitor the system is of sixth order and the transient has three oscillatory modes, but operation with complete energy transfer is equally possible.

In all the cases found in the literature, the design of these systems is based on the analysis of a fixed structure. The following sections show that the design can be made by synthesis, can be applied to a wide range of structures, and can be extended to systems of any order.

II. Synthesis Approach

The transformer can be left out of the problem, because it can be inserted after the synthesis of a "ladder" structure composed of series

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