Calculus of Variations

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Calculus of Variations

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Background

• Definition
  • A function is a mapping of single values to single values.
  • A functional is a mapping of function values to single or function values. It usually contains single or multiple variables and their derivatives.
  • Dirichlet Principle: There exists one stationary ground state for energy.
  • Euler’s Equation defines the condition for finding the extrema of functionals. → An extremal is the maximum or minimum integral curves of Euler’s equation of a functional.
  • Calculus of Functionals: Determining the properties of functionals.
  • Calculus of Variations: Finding the extremals of functionals.
Background

• Single value calculus:
  • Functions take extreme values on bounded domain. Necessary condition for extremum at $x_0$, if $f$ is differentiable:
    $$f'(x_0) = 0$$

• Calculus of variations
  • Test function $v(x)$, which vanishes at endpoints, used to find extremal:
    $$w(x) = u(x) + \varepsilon v(x) \quad I[\varepsilon] = \int_a^b F(x, w, w_x) \, dx$$
  • Necessary condition for extremal:
    $$\frac{dI}{d\varepsilon} = 0$$
Maximum and Minimum of Functions

Maximum and minimum

(a) If \( f(x) \) is twice continuously differentiable on \([x_0, x_1]\) i.e.

Nec. condition for a max. (min.) of \( f(x) \) at \( x \in [x_0, x_1] \) is that \( F'(x) = 0 \)

Suff. condition for a max (min.) of \( f(x) \) at \( x \in [x_0, x_1] \) are that \( F'(x) = 0 \) and \( F''(x) \leq 0 \) or \( F''(x) \geq 0 \)

(b) If \( f(x) \) over closed domain \( D \). Then nec. and suff. condition for a max. (min.) of \( f(x) \) at \( x_0 \in D - \partial D \) are that \( \frac{\partial f}{\partial x_i} \bigg|_{x=x_0} = 0 \) \( i = 1, 2 \ldots n \) and also that \( \frac{\partial^2 f}{\partial x_i \partial x_j} \bigg|_{x=x_0} \) is a negative infinite.
Maximum and Minimum of Functions

(c) If $f(x)$ on closed domain $D$
If we want to extremize $f(x)$ subject to the constraints
$$g_i(x_1, K, x_n) = 0 \quad i=1,2,\ldots,k \quad (k < n)$$

EX: Find the extrema of $f(x,y)$ subject to $g(x,y) = 0$

i) Approach One: Direct differentiation of $g(x, y)$
$$dg = g_x dx + g_y dy = 0$$
$$\Rightarrow \quad dy = -\frac{g_x}{g_y} dx$$

To extremize $f$
$$df = f_x dx + f_y dy = 0$$
$$\Rightarrow \quad (f_x - f_y \frac{g_x}{g_y}) dx = 0$$

**Maximum and Minimum of Functions**

We have

\[ f_x g_y - f_y g_x = 0 \quad \text{and} \quad g = 0 \]

to find \((x_0, y_0)\) which is to extremize \(f\) subject to \(g = 0\)

ii) Approach Two: Lagrange Multiplier

Let \(v(x, y, \lambda) = f(x, y) + \lambda g(x, y)\)

\[ \Rightarrow \text{extrema of } v \text{ without any constraint} \]

\[ \iff \text{extrema of } f \text{ subject to } g = 0 \]

To extremize \(v\)

\[ \Rightarrow \left\{ \begin{array}{l} \frac{\partial v}{\partial x} = f_x + \lambda g_x = 0 \\ \frac{\partial v}{\partial y} = f_y + \lambda g_y = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} f_x g_y - f_y g_x = 0 \\ \frac{\partial v}{\partial \lambda} = g = 0 \end{array} \right\} \]

We obtain the same equations by extremizing \(v\). where \(\lambda\) is called the Lagrange Multiplier.
Maximum and Minimum of Functionals

- **Functionals** are function’s function.

- The basic problem in calculus of variations. Determine $y(x) \in c^2[x_0, x_1]$ such that the functional:

$$I\left(y(x)\right) = \int_{x_0}^{x_1} F\left(x, y(x), y'(x)\right) dx$$

as an extrema

where $F \in c^2$ over its entire domain, subject to $y(x_0) = y_0$, $y(x_1) = y_1$ at the end points.
Maximum and Minimum of Functionals

Using integrating by parts of the 2\textsuperscript{nd} term, it leads to

\[ \Rightarrow [F_y(x, y, y') \eta]_{x_0}^{x_1} - \int_{x_0}^{x_1} \left[ \frac{d}{dx} F_y(x, y, y') - F_y(x, y, y') \right] \eta dx = 0 \quad \text{----------(1)} \]

Since \( \eta(x_0) = \eta(x_1) = 0 \) and \( \eta(x) \) is arbitrary,

\[ \Rightarrow \frac{d}{dx} \left[ F_y(x, y, y') \right] - F_y(x, y, y') = 0 \quad \text{---------- (2) (Euler's Equation)} \]

Natural B.C's

\[ \begin{bmatrix} \frac{\partial F}{\partial y'} \end{bmatrix}_{x_0} = 0 \quad \text{or/and} \quad \begin{bmatrix} \frac{\partial F}{\partial y'} \end{bmatrix}_{x=x_1} = 0 \]

The above requirements are called national b.c's.
The Variational Notation

Variations

Imbed $u(x)$ in a “parameter family” of function $\phi(x, \varepsilon) = u(x) + \varepsilon \eta(x)$ the variation of $u(x)$ is defined as

$$\delta u = \varepsilon \eta(x)$$

The corresponding variation of $F$, $\delta F$ to the order in $\varepsilon$ is,

since

$$\delta F = F(x + y + \varepsilon \eta, y' + \varepsilon \eta') - F(x, y, y')$$

$$= \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y'$$

and

$$I(u + \varepsilon \eta) = \int_{x_0}^{x_1} F(x, u + \varepsilon \eta, u', \varepsilon \eta') dx = G(\varepsilon)$$

Then

$$\delta I = \delta \int_{x_0}^{x_1} F(x, y, y') dx$$

$$= \int_{x_0}^{x_1} \delta F(x, y, y') dx$$

$$= \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx$$
The Variational Notation

\[ I = \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] \delta y dx + \left[ \frac{\partial F}{\partial y} \delta y \right]_{x_0}^{x_1} \]

Thus, a stationary function for a functional is one for which the first variation.

\[ \frac{\partial F}{\partial y} = 0 \]

For more general cases

(a) Several dependent variables

\[ \text{EX: } I = \int_{x_0}^{x_1} F(x, y, z ; y', z') dx \]

Euler’s Eq. \[ \Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0 \], \[ \frac{\partial F}{\partial z} - \frac{d}{dx} \left( \frac{\partial F}{\partial z'} \right) = 0 \]

(b) Several Independent variables

\[ \text{EX: } I = \iint_R F(x, y, u, u_x, u_y) dx dy \]

Euler’s Eq. \[ \Rightarrow \frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial u_y} \right) = 0 \]
The Variational Notation

(c) High Orders

**EX:** \( I = \int_{x_0}^{x_1} F(x, y, y', y'')dx \)

Euler’s Eq. \( \Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \)

\[
\left\{ \begin{array}{c}
\text{Variables} \uparrow \\
\text{Order} \uparrow \\
\end{array} \right. \rightarrow \text{Causing more equations} \rightarrow \text{Causing longer equations}
\]
Lagrange Multiplier

Lagrange multiplier can be used to find the extreme value of a multivariate function \( f \) subjected to the constraints.

**EX:**
(a) Find the extreme value of 
\[
I = \int_{x_0}^{x_1} F(x, u, v, u_x, v_x) dx
\]

where 
\[
u(x_1) = u_1 \quad u(x_2) = u_2
\]
\[
v(x_1) = v_1 \quad v(x_2) = v_2
\]

and subject to the constraints
\[
G(x, u, v) = 0
\]

From 
\[
\delta I = \int_{x_1}^{x_2} \left\{ \left[ \frac{\partial F}{\partial v} - \frac{d}{dx} \left( \frac{\partial F}{\partial u_x} \right) \right] \delta u + \left[ \frac{\partial F}{\partial v} - \frac{d}{dx} \left( \frac{\partial F}{\partial v_x} \right) \right] \delta v \right\} dx = 0
\]
Constraints and Lagrange Multiplier

Because of the constraints, we don’t get two Euler’s equations.

From

\[ \delta G = \frac{\partial G}{\partial u} \delta u + \frac{\partial G}{\partial v} \delta v = 0 \quad \Rightarrow \quad -\frac{G_v}{G_u} \delta v = \delta u \]

Therefore, Eq. (4) becomes

\[ \Rightarrow \delta I = \int_{x_0}^{x_1} \left\{ -\frac{G_v}{G_u} \left[ \frac{\partial F}{\partial u} - \frac{d}{dx} \left( \frac{\partial F}{\partial u_x} \right) \right] + \left[ \frac{\partial F}{\partial v} - \frac{d}{dx} \left( \frac{\partial F}{\partial v_x} \right) \right] \right\} \delta v dx = 0 \quad \text{---------(5)} \]

\[ \Rightarrow \frac{\partial G}{\partial v} \left[ \frac{\partial F}{\partial u} - \frac{d}{dx} \left( \frac{\partial F}{\partial u_x} \right) \right] - \frac{\partial G}{\partial v} \left[ \frac{\partial F}{\partial v} - \frac{d}{dx} \left( \frac{\partial F}{\partial v_x} \right) \right] = 0 \quad \text{---------(6)} \]

The above equations, together with Eq. (3), are used to solve for \( u, v \).
Constraints and Lagrange Multiplier

(b) Simple Isoparametric Problem

To extremize \( I = \int_{x_1}^{x_2} F(x, y, y') dx \), subject to the constraint:

i) \( J = \int_{x_1}^{x_2} G(x, y, y') dx = \text{const.} \)

ii) \( y(x_1) = y_1, \quad y(x_2) = y_2 \)

Take the variation of two-parameter family:

\[ y + \delta y = y + \varepsilon_1 \eta_1(x) + \varepsilon_2 \eta_2(x) \]

(\( \eta_1(x) \) and \( \eta_2(x) \) are some equations which satisfy

\[ \eta_1(x_1) = \eta_2(x_1) = \eta_1(x_2) = \eta_2(x_2) = 0 \)\)

Then,

\[ I(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} F(x, y + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2, y' + \varepsilon_1 \eta_1' + \varepsilon_2 \eta_2') dx \]

\[ J(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} G(x, y + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2, y' + \varepsilon_1 \eta_1' + \varepsilon_2 \eta_2') dx \]

To base on the Lagrange Multiplier Method, we can get:
Constraints and Lagrange Multiplier

\[ \frac{\partial}{\partial \varepsilon_1} (I + \lambda J) \big|_{\varepsilon_1=\varepsilon_2=0} = 0 \]
\[ \frac{\partial}{\partial \varepsilon_2} (I + \lambda J) \big|_{\varepsilon_1=\varepsilon_2=0} = 0 \]

\[ \Rightarrow \int_{x_1}^{x_2} \left\{ \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] + \lambda \left[ \frac{\partial G}{\partial y} - \frac{d}{dx} \left( \frac{\partial G}{\partial y'} \right) \right] \right\} \eta_i dx = 0 \quad i = 1,2 \]

The Euler’s equation becomes

\[ \frac{\partial}{\partial y} (F + \lambda G) - \frac{d}{dx} \left[ \frac{\partial}{\partial y'} (F + \lambda G) \right] = 0 \]

when \[ \frac{\partial G}{\partial y} - \frac{d}{dx} \left( \frac{\partial G}{\partial y'} \right) = 0 \] , \( \lambda \) is arbitrary numbers.

\[ \Rightarrow \] The constraint is trivial, we can ignore \( \lambda \).
Applications

- **Helmholtz Equation**

**EX:** Force vibration of a membrane

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(x, y, t) \tag{7}
\]

if the forcing function \( f \) is of the form

\[
f(x, y, t) = P(x, y) \sin(\omega t + \alpha)
\]

we may write the steady state displacement \( u \) in the form

\[
u = v(x, y) \sin(\omega t + \alpha)
\tag{8}
\]

\[
\Rightarrow c^2 \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \omega^2 v + p = 0
\]
Consider

\[ c^2 \int_R v_{xx} \, \delta v \, dx \, dy = c^2 \int_R [(v_x \delta v)_x - v_x \delta v_x] \, dx \, dy \]

Note that \((v_x \delta v)_x = v_{xx} \delta v + v_x \delta v_x\)

\[ V = v_x i + v_y j \quad V_x = v_x \delta v \quad V_y = v_y \delta v \]

\[ \vec{n} = \cos \theta i + \sin \theta j \]

\[ \nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial (v_x \delta v)}{\partial x} + \frac{\partial (v_y \delta v)}{\partial y} \]
\[ \int_{\gamma} (\nabla \cdot V) da = \int_{\gamma} V \cdot n \, ds = \int (v_x \delta v \cos \theta + v_y \delta v \sin \theta) \, ds \]

\[ c^2 \int_{R} v_{xx} \delta v \, dx \, dy + \int_{R} c^2 v_{yy} \delta v \, dx \, dy \]

\[ = c^2 \int_{\gamma} v_x \delta v \cos \theta \, ds - \int_{R} \frac{1}{2} c^2 \delta (v_x)^2 \, dx \, dy \]

\[ + c^2 \int_{\gamma} v_y \delta v \sin \theta \, ds - \int_{R} \frac{1}{2} c^2 \delta (v_y)^2 \, dx \, dy \]

\[ \Rightarrow \int_{\gamma} c^2 (v_x \cos \theta + v_y \sin \theta) \delta v \, ds - \int_{R} \frac{1}{2} c^2 \delta [(v_x)^2 + (v_y)^2] \, dx \, dy \]

\[ + \int_{R} \frac{1}{2} \omega^2 \delta (v^2) \, dx \, dy + \int_{R} P \delta v \, dx \, dy = 0 \]

\[ \Rightarrow \int_{\gamma} c^2 \frac{\partial v}{\partial n} \delta v \, ds - \delta \int_{R} \left[ \frac{1}{2} c^2 (\nabla v)^2 - \frac{1}{2} \omega^2 v^2 - P v \right] \, dx \, dy = 0 \]
Applications

Hence,

i) if \( v = f(x, y) \) is given on \( \gamma \)
   
i.e. \( \delta v = 0 \) on \( \gamma \)
   
   then the variational problem

\[
\Rightarrow \delta \int_R \left[ \frac{c^2}{2} (\nabla v)^2 - \frac{1}{2} \omega^2 v^2 - pv \right] dxdy = 0
\]

ii) if \( \frac{\partial v}{\partial n} = 0 \) is given on \( \gamma \)
   
   the variation problem is same as Eq. (10)

iii) if \( \frac{\partial v}{\partial n} = \psi(s) \) is given on \( \gamma \)

\[
\Rightarrow \delta \left[ \int_R \left\{ \frac{1}{2} c^2 (\nabla v)^2 - \frac{1}{2} \omega^2 v^2 - pv \right\} dxdy - \int_\tau c^2 \psi vdx \right] = 0
\]
Applications

- **Diffusion Equation**

  **EX:** Steady State Heat Condition

  \[ \nabla \cdot (k \nabla T) = f(x, T) \quad \text{in} \quad D \]

  B.C’s:

  \[ T = T_1 \quad \text{on} \quad B_1 \]

  \[ -kn \cdot \nabla T = q_2 \quad \text{on} \quad B_2 \]

  \[ -kn \cdot \nabla T = h(T - T_3) \quad \text{on} \quad B_3 \]

  Multiply the equation by \( \delta T \), and integrate over the domain \( D \). After integrating by parts, we find the variational problem as follow.

  \[
  \delta \left[ \int_D \left\{ \frac{1}{2} k (\nabla T)^2 + \int_{T_0}^T f(x, T')dT' \right\}d\tau + \int_{B_2} q_2 T d\sigma + \frac{1}{2} \int_{B_3} h(T - T_3)^2 d\sigma \right] = 0
  \]

  with \( T = T_1 \) \quad \text{on} \quad B_1. \]
Applications

- Poisson’s Equation
  
  EX: Torsion of a Prismatic Bar
  
  \[ \nabla^2 \psi = -2 \quad \text{in} \quad R \]
  \[ \psi = 0 \quad \text{on} \quad \gamma \]

  where \( \psi \) is the Prandtl stress function and

  \[ \sigma_{xz} = G \alpha \frac{\partial \psi}{\partial y}, \quad \sigma_{zy} = \sigma_{\alpha} \frac{\partial \psi}{\partial x} \]

  The variation problem becomes

  \[ \delta \left\{ \int_{R} [(D\psi)^2 - 4\psi] dxdy \right\} = 0 \]

  with \( \psi = 0 \) on \( \gamma \).
Approximate Methods

I) Method of Weighted Residuals (MWR)

\[ L[u] = 0 \quad \text{in} \quad D \]

with homogeneous b.c.'s in \( B \).
Assume an approximate solution.

\[ u = u_n = \sum_{i=1}^{n} C_i \phi_i \]

where each trial function \( \phi_i \) satisfies the b.c.'s. The residual is

\[ R_n = L[u_n] \]

In this method (MWR), \( C_i \) are chosen such that \( R_n \) is forced to be zero in an average sense.

i.e. \( < w_j, R_n > = 0, \quad j = 1,2,\ldots,n \)

where \( w_j \) are the weighting functions..
Approximate Methods

II) Galerkin Method

\( w_j \) are chosen to be the trial functions \( \phi_j \) hence the trial functions is chosen as members of a complete set of functions.

Galerkin method force the residual to be zero w.r.t. an orthogonal complete set.

**EX**: Torsion of a Square Shaft

\[
\nabla^2 \psi = -2
\]

\[
\psi = 0 \quad \text{on} \quad x = \pm a, \quad y = \pm a
\]

i) One – term approximation

\[
\psi_1 = c_1 (x^2 - a^2)(y^2 - a^2)
\]

\[
R_i = \nabla^2 \psi_1 + 2 = 2c_1[(x-a)^2 + (y-a)^2] + 2
\]

\[
\phi_1 = (x^2 - a^2)(y^2 - a^2)
\]
Approximate Methods

From \[\int_{-a}^{a} \int_{-a}^{a} R_{1} \phi_{1} dx \, dy = 0\]

\[\Rightarrow c_{1} = \frac{5}{8} \frac{1}{a^{2}}\]

Therefore,

\[\psi_{1} = \frac{5}{8a^{2}}(x^{2} - a^{2})(y^{2} - a^{2})\]

The torsional rigidity is determined by

\[D_{1} = 2G \int_{R} \psi \, dx \, dy = 0.1388G(2a)^{4}\]

The exact value of \(D\) is

\[D_{a} = 0.1406G(2a)^{4}\]

The approximation error is -1.2\%. 

ii) Two - term approximation

\[ \psi_2 = (x^2 - a^2)(y^2 - a^2)[c_1 + c_2(x^2 + y^2)] \]

By symmetry \( \Rightarrow R_2 = \nabla \psi_2 + 2 \)

\[ \phi_1 = (x^2 - a^2)(y^2 - a^2) \]

\[ \phi_2 = (x^2 - a^2)(y^2 - a^2)(x^2 + y^2) \]

From \( \int_R R_2 \phi_1 dxdy = 0 \)

\[ \phi_2 = (x^2 - a^2)(y^2 - a^2)(x^2 + y^2) \]

and \( \int_R R_2 \phi_2 dxdy = 0 \)

We obtain

\[ c_1 = \frac{1295}{2216} \frac{1}{a^2}, \quad c_2 = \frac{525}{4432} \frac{1}{a^2} \]

Therefore

\[ D_2 = 2G \int_R \psi_2 dxdy = 0.1404G(2a)^4 \quad \Rightarrow \text{The error is -0.14%}. \]
Variational Methods

I) **Kantorovich Method** [Kantorovich (1948)]

Assuming the approximate solution as: \( u = \sum_{i=1}^{n} C_i(x_n)U_i \)

where \( U_i \) is a known function decided by b.c. condition.

\( C_i \) is a unknown function decided by minimal “\( I \)”. \( \Rightarrow \) Euler Equation of \( C_i \)

**EX** : The Torsional Problem with a Functional “\( I \)”.  

\[
I(u) = \int_{-a}^{a} \int_{-b}^{b} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 - 4u \right] dx \, dy
\]
Assuming the one-term approximate solution as:

\[ u(x, y) = (b^2 - y^2)C(x) \]

Then,

\[ I(C) = \int_{-a}^{a} \int_{-b}^{b} \left\{ (b^2 - y^2)^2 [C'(x)]^2 + 4y^2C^2(x) - 4(b^2 - y^2)C(x) \right\} dxdy \]

Integrate by \( y \)

\[ I(C) = \int_{-a}^{a} \left[ \frac{16}{15} b^5 C'' + \frac{8}{3} b^3 C^2 - \frac{16}{3} b^3 C \right] dx \]

Euler’s equation is

\[ C''(x) - \frac{5}{2b^2} C(x) = -\frac{5}{2b^2} \]

where b.c. condition is \( C(\pm a) = 0 \)

General solution is

\[ C(x) = A_1 \cosh\left(\sqrt{\frac{5}{2b}} x\right) + A_2 \sinh\left(\sqrt{\frac{5}{2b}} x\right) + 1 \]
Variational Methods

where \( A_1 = -\frac{1}{\cosh\left(\sqrt{\frac{5}{2}} \frac{a}{b}\right)} \), \( A_2 = 0 \)

and

\[
C(x) = \begin{cases} 
\cosh\left(\sqrt{\frac{5}{2}} \frac{x}{b}\right) \\
1 - \frac{\cosh\left(\sqrt{\frac{5}{2}} \frac{a}{b}\right)}{\cosh\left(\sqrt{\frac{5}{2}} \frac{a}{b}\right)} 
\end{cases}
\]

Therefore, the one-term approximate solution is

\[
u = \begin{cases} 
\cosh\left(\sqrt{\frac{5}{2}} \frac{x}{b}\right) \\
1 - \frac{\cosh\left(\sqrt{\frac{5}{2}} \frac{a}{b}\right)}{\cosh\left(\sqrt{\frac{5}{2}} \frac{a}{b}\right)} (b^2 - y^2)
\end{cases}
\]
II) Rayleigh-Ritz Method

This is used when the exact solution is impossible or difficult to obtain.

First, we assume the approximate solution as:  
\[ u = \sum_{i=1}^{n} C_i U_i \]
where \( U_i \) are some approximate function which satisfy the b.c’s. Then, we can calculate extreme \( I \).

\[ I = I(c_1, K, ... , c_n) \quad \text{Choose} \quad c_1 \sim c_n \quad \text{i.e.} \quad \frac{\partial I}{\partial c_1} = K = \frac{\partial I}{\partial c_n} = 0 \]

\textbf{EX:} \quad y'' + xy = -x \quad y(0) = y(1) = 0

Its solution can be obtained from

\[ \int_0^1 (y'' + xy + x) \delta y \, dx = 0 \quad \Rightarrow \quad I = \int_0^1 \left[ \frac{1}{2} (y')^2 - \frac{1}{2} xy^2 - xy \right] \, dx \]
Variational Methods

Assuming that

\[ y = x(1-x)(c_1 + c_2 x + c_3 x^2 K) \]

i) One-term approximation

\[ y = c_1 x(1-x) = c_1(x-x^2) \quad y' = c_1(1-2x) \]

Then, \[ I(c_1) = \int_0^1 \left[ \frac{1}{2} c_1^2 \left(1-4x+4x^2\right) - \frac{x}{2} c_1^2 \left(x^2-2x^3+x^4\right) - c_1 x \left(x-x^2\right) \right] dx \]

\[ = \frac{c_1^2}{2} \left(1-2+\frac{4}{3}\right) - \frac{c_1^2}{2} \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) - c_1 \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{19}{120} c_1^2 - \frac{c_1}{12} \]

\[ \frac{\partial I}{\partial c_1} = 0 \quad \Rightarrow \quad \frac{19}{60} c_1 - \frac{1}{12} = 0 \quad \Rightarrow \quad c_1 = 0.263 \quad \Rightarrow \quad y(1) = 0.263x(1-x) \]

ii) Two-term approximation

\[ y = x(1-x)(c_1 + c_2 x) = c_1(x-x^2) + c_2(x^2-x^3) \]
Variational Methods

Then \[ y' = c_1(1 - 2x) + c_2(2x - 3x^2) \]

\[ I(c_1, c_2) = \int_0^1 \left[ \frac{1}{2} \left\{ c_1^2 \left( 1 - 4x + 4x^2 \right) + 2c_1c_2 \left( 2x - 7x^2 + 6x^3 \right) \right. \right. \]
\[ + c_3^2 \left( 4x^2 - 12x^3 + 9x^4 \right) \left. \left. \right. \right\} - \frac{1}{2} \left\{ c_1^2 \left( x^3 - 2x^4 + x^5 \right) + 2c_1c_2 \left( x^4 - 2x^5 + x^6 \right) \right. \right. \]
\[ + c_2^2 \left( x^5 - 2x^6 + x^7 \right) \right\} - \left\{ c_1 \left( x^2 - x^3 \right) + c_2 \left( x^3 - x^4 \right) \right\} \right\} \right\} dx \]
\[ = \frac{c_1^2}{2} \left( 1 - 2 + \frac{4}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{6} \right) + c_1c_2 \left( 1 - 7 \frac{7}{3} + 3 \frac{3}{2} - 5 \frac{5}{3} + 6 \frac{6}{3} - 7 \frac{7}{3} \right) \]
\[ - \frac{c_1^2}{2} \left( \frac{4}{3} - 3 + \frac{9}{5} - \frac{1}{6} + \frac{2}{7} - \frac{9}{8} \right) \]
\[ - \frac{c_1}{12} - \frac{c_2}{20} \]
\[ = \frac{19}{120} c_1^2 + \frac{11}{70} c_1c_2 + \frac{107}{1680} c_2^2 - \frac{c_1}{12} - \frac{c_2}{20} \]
Variational Methods

\[
\frac{\partial I}{\partial c_1} = 0 \quad \Rightarrow \quad \frac{19}{60} c_1 + \frac{11}{70} c_2 = \frac{1}{12}
\]

\[
\frac{\partial I}{\partial c_2} = 0 \quad \Rightarrow \quad \frac{11}{70} c_1 + \frac{109}{840} c_2 = \frac{1}{20}
\]

0.317 c_1 + 0.127 c_2 = 0.05 \quad \Rightarrow \quad c_1 = 0.177 , \ c_2 = 0.173

\Rightarrow \quad y(2) = (0.177x - 0.173x^2)(1-x)
Examples

I) The Brachistochrone (fastest descent) Curve Problem – Proposed by Johann Bernoulli (1696)

II) Structural Dynamics – Formulation of Governing Equation of Motion
Summary

- Many governing equations in physics and chemistry can be formulated by functionals. Finding the extremals of functionals can lead to the solutions in many problems in science and engineering.

- Euler’s Equations provide the necessary condition for evaluating problems involving functionals. However, analyzing Euler’s equations sometimes can be difficult.

- Calculus of Variations determines the extremals of functionals, even though the solution is only approximated.

- Calculus of Variations provides the theoretical basis for many methods in engineering, such as the Principle of Virtual Displacement (PVD) and the Finite Element Method (FEM).
References

- C. Lanezos (1949), *The Variational Principles of Mechanics*, University of Toronto Press, Toronto, Canada.