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Calculus of Variations

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SERG

Calculus of Variations

- Background
- Maximum and Minimum of Functions
- Maximum and Minimum of Functionals
- The Variational Notation
- Constraints and Lagrange Multiplier
- Applications
- Approximate Methods
- Variational Methods
- Examples
- Summary
- References

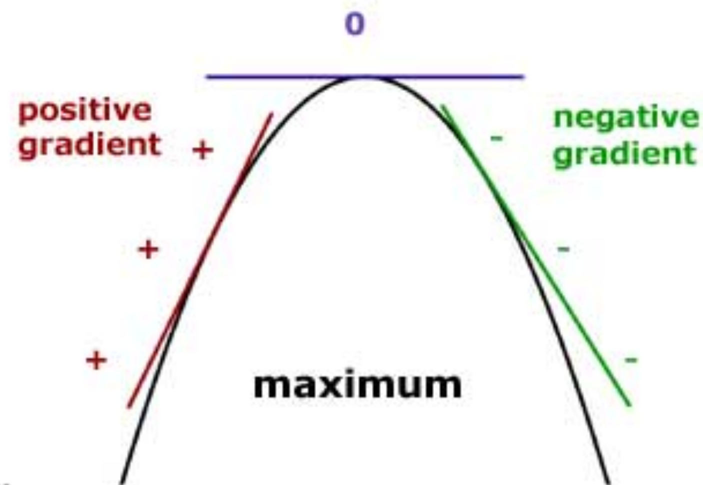
Background

- Definition
  - A *function* is a mapping of single values to single values.
  - A *functional* is a mapping of function values to single or function values. It usually contains single or multiple variables and their derivatives.
  - *Dirichlet Principle*: There exists one stationary ground state for energy.
  - *Euler's Equation* defines the condition for finding the extrema of functionals. → An *extremal* is the maximum or minimum integral curves of Euler's equation of a functional.
  - *Calculus of Functionals*: Determining the properties of functionals.
  - *Calculus of Variations*: Finding the extremals of functionals.

Background

- Single value calculus:
  - Functions take extreme values on bounded domain. Necessary condition for extremum at x_0, if *f* is differentiable:

$$f'(x_0)=0$$

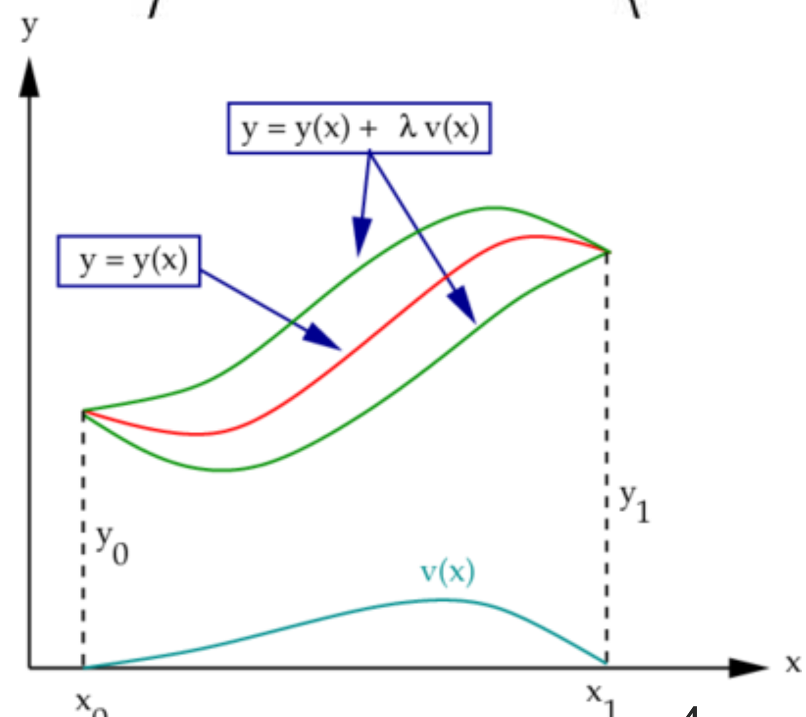


- Calculus of variations
  - Test function v(x), which vanishes at endpoints, used to find extremal:

$$w(x)=u(x)+\varepsilon v(x) \qquad I[\varepsilon]=\int_a^b F(x,w,w_x)\,dx$$

  - Necessary condition for extremal:

$$\frac{dI}{d\varepsilon}=0$$



Maximum and Minimum of Functions

Maximum and minimum

(a) If f(x) is twice continuously differentiable on $[x_0 , x_1]$ i.e.

Nec. condition for a max. (min.) of f(x) at $x \in [x_0, x_1]$ is that $F'(x) = 0$

Suff. condition for a max (min.) of f(x) at $x \in [x_0, x_1]$ are that $F'(x) = 0$ and

$F''(x) \le 0$ or $F''(x) \ge 0$

(b) If f(x) over closed domain D. Then nec. and suff. condition for a max. (min.)

of f(x) at $x_0 \in D - \partial D$ are that $\left.\frac{\partial f}{\partial x_i}\right|_{x=x_0} = 0$ i = 1,2…n and also

that $\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{x=x_0}$ is a negative infinite .

Maximum and Minimum of Functions

(c) If *f(x)* on closed domain D

If we want to extremize $f(x)$ subject to the constraints

$$g_i(x_1, \ldots, x_n) = 0$$ i=1,2,…k (k < n)

EX: Find the extrema of $f(x,y)$ subject to $g(x,y) = 0$

i) Approach One: Direct differentiation of $g(x, y)$

$$dg = g_x dx + g_y dy = 0$$

$$\Rightarrow \quad dy = -\frac{g_x}{g_y} dx$$

To extremize f

$$df = f_x dx + f_y dy = 0$$

$$\Rightarrow \quad (f_x - f_y \frac{g_x}{g_y}) dx = 0$$

Maximum and Minimum of Functions

We have

$$f_x g_y - f_y g_x = 0 \quad \text{and} \quad g = 0$$

to find (x_0, y_0) which is to extremize f subject to $g = 0$

ii) Approach Two: Lagrange Multiplier

Let $v(x, y, \lambda) = f(x, y) + \lambda g(x, y)$

$\Longrightarrow$ extrema of v without any constraint

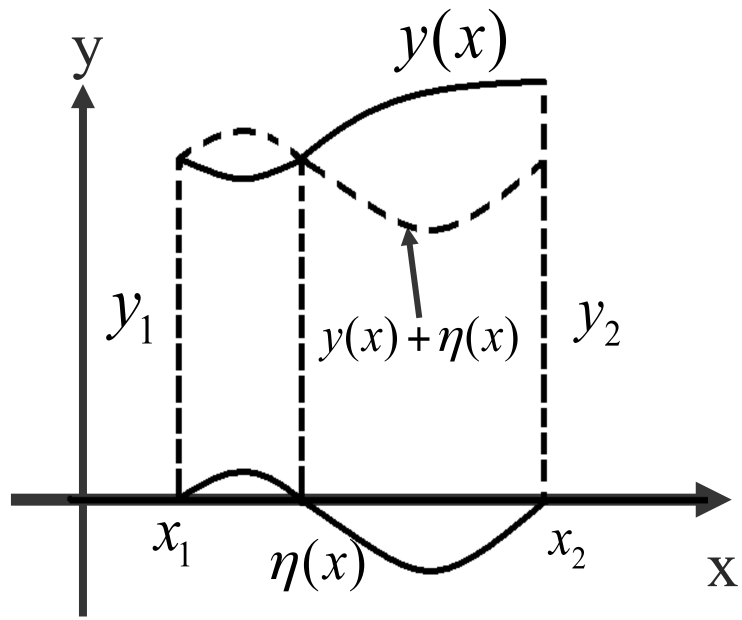
$\Longleftrightarrow$ extrema of f subject to $g = 0$

To extremize v $\Longrightarrow \left\{ \begin{aligned} \frac{\partial v}{\partial x} &= f_x + \lambda g_x = 0 \\ \frac{\partial v}{\partial y} &= f_y + \lambda g_y = 0 \end{aligned} \right\} \Longrightarrow \left\{ \begin{aligned} & f_x g_y - f_y g_x = 0 \\ & \frac{\partial v}{\partial \lambda} = g = 0 \end{aligned} \right.$

We obtain the same equations by extremizing v. where λ is called the Lagrange Multiplier.

Maximum and Minimum of Functionals

- **Functionals are function's function.**



- **The basic problem in calculus of variations.**

Determine $y(x) \in c^2[x_0, x_1]$ such that the functional：

$$I\left(y\left(x\right)\right) = \int_{x_0}^{x_1} F\left(x, y\left(x\right), y'\left(x\right)\right) dx \qquad \text{as an extrema}$$

where $F \in c^2$ over its entire domain, subject to $y(x_0) = y_0$, $y(x_1) = y_1$ at the end points.

Maximum and Minimum of Functionals

Using integrating by parts of the 2nd term, it leads to

$$\Rightarrow [F_{y'}(x,y,y')\eta]_{x_0}^{x_1} - \int_{x_0}^{x_1} [\frac{d}{dx} F_{y'}(x,y,y') - F_y(x,y,y')]\eta dx = 0 \qquad \text{-----------(1)}$$

Since $\eta(x_0) = \eta(x_1) = 0$ and $\eta(x)$ is arbitrary,

$$\Rightarrow \frac{d}{dx}\left[F_{y'}(x,y,y')\right] - F_y(x,y,y') = 0 \qquad \text{----------- (2)} \quad \text{(Euler's Equation)}$$

Natural B.C's

$$\left[\frac{\partial F}{\partial y'}\right]_{x_0} = 0 \qquad \text{or/and} \qquad \left[\frac{\partial F}{\partial y'}\right]_{x=x_1} = 0$$

The above requirements are called national b.c's.

The Variational Notation

Variations

Imbed $u(x)$ in a "parameter family" of function $\phi(x,\varepsilon) = u(x) + \varepsilon\eta(x)$ the variation of $u(x)$ is defined as

$$\delta u = \varepsilon\eta(x)$$

The corresponding variation of F, δF to the order in ε is ,

since $$\delta F = F(x + y + \varepsilon\eta, y' + \varepsilon\eta') - F(x, y, y')$$

$$= \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

and $$I(u + \varepsilon\eta) = \int_{x_0}^{x_1} F(x, u + \varepsilon\eta, u' + \varepsilon\eta')dx = G(\varepsilon)$$

Then $$\delta I = \delta\int_{x_0}^{x_1} F(x, y, y')dx$$

$$= \int_{x_0}^{x_1} \delta F(x, y, y')dx$$

$$= \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)dx$$

The Variational Notation

$$= \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \delta y dx + \left[\frac{\partial F}{\partial y'} \delta y \right]_{x_0}^{x_1}$$

Thus, a stationary function for a functional is one for which the first variation.

$$\frac{\partial F}{\partial y} = 0$$

For more general cases

(a) Several dependent variables

EX: $$I = \int_{x_0}^{x_1} F(x, y, z\,; y', z')dx$$

Euler's Eq. $\Longrightarrow$ $$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0 \quad , \quad \frac{\partial F}{\partial z} - \frac{d}{dx}(\frac{\partial F}{\partial z'}) = 0$$

(b) Several Independent variables

EX: $$I = \iint_R F(x, y, u, u_x, u_y)dxdy$$

Euler's Eq. $\Longrightarrow$ $$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y}) = 0$$

The Variational Notation

(c) High Orders

EX: $I = \int_{x_0}^{x_1} F(x, y, y', y'')dx$

Euler's Eq. $\Longrightarrow$ $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

Variables $\Uparrow$ $\Longrightarrow$ Causing more equations

Order $\Uparrow$ $\Longrightarrow$ Causing longer equations

Constraints and Lagrange Multiplier

Lagrange multiplier

Lagrange multiplier can be used to find the extreme value of a multivariate function f subjected to the constraints.

EX:

(a) Find the extreme value of $I = \int_{x_0}^{x_1} F(x, u, v, u_x, v_x) dx$

where $u(x_1) = u_1 \qquad u(x_2) = u_2$

$v(x_1) = v_1 \qquad v(x_2) = v_2$

and subject to the constraints

$$G(x, u, v) = 0 \qquad \text{------------(3)}$$

From $$\delta I = \int_{x_1}^{x_2} \left\{ \left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x} \right) \right] \delta u + \left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x} \right) \right] \partial v \right\} dx = 0 \qquad \text{-----------(4)}$$

Constraints and Lagrange Multiplier

Because of the constraints, we don't get two Euler's equations.

From

$$\delta G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial v}\delta v = 0 \quad \Longrightarrow \quad -\frac{G_v}{G_u}\delta v = \delta u$$

Therefore, Eq. (4) becomes

$$\Longrightarrow \quad \delta I = \int_{x_0}^{x_1}\left\{-\frac{G_v}{G_u}\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\right] + \left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\right]\right\}\delta v dx = 0 \qquad \text{---------(5)}$$

$$\Longrightarrow \quad \frac{\partial G}{\partial v}\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\right] - \frac{\partial G}{\partial u}\left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\right] = 0 \qquad \text{---------(6)}$$

The above equations, together with Eq. (3), are used to solve for u, v.

Constraints and Lagrange Multiplier

(b) Simple Isoparametric Problem

To extremize $I = \int_{x_1}^{x_2} F\left(x, y, y'\right) dx$, subject to the constraint：

i) $J = \int_{x_1}^{x_2} G\left(x, y, y'\right) dx = const.$

ii) $y\ (x_1) = y_1$, $y\ (x_2) = y_2$

Take the variation of two-parameter family： $y + \delta y = y + \varepsilon_1 \eta_1\left(x\right) + \varepsilon_2 \eta_2\left(x\right)$
(where $\eta_1\left(x\right)$ and $\eta_2\left(x\right)$ are some equations which satisfy

$\eta_1\left(x_1\right) = \eta_2\left(x_1\right) = \eta_1\left(x_2\right) = \eta_2\left(x_2\right) = 0$)

Then , $I(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} F(x, y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, y' + \varepsilon_1\eta_1{}' + \varepsilon_2\eta_2{}')dx$

$J(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} G(x, y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, y' + \varepsilon_1\eta_1{}' + \varepsilon_2\eta_2{}')dx$

To base on the Lagrange Multiplier Method, we can get :

Constraints and Lagrange Multiplier

$$\frac{\partial}{\partial \varepsilon_1}(I + \lambda J)\bigg|_{\varepsilon_1=\varepsilon_2=0} = 0$$

$$\frac{\partial}{\partial \varepsilon_2}(I + \lambda J)\bigg|_{\varepsilon_1=\varepsilon_2=0} = 0$$

$$\Rightarrow \quad \int_{x_1}^{x_2} \left\{ \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] + \lambda \left[\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) \right] \right\} \eta_i \, dx = 0 \qquad i = 1,2$$

The Euler's equation becomes

$$\frac{\partial}{\partial y}(F + \lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial y'}(F + \lambda G) \right] = 0$$

when $\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) = 0$, λ is arbitrary numbers.

$\Longrightarrow$ The constraint is trivial, we can ignore λ .

Applications

- **<u>Helmholtz Equation</u>**

EX: Force vibration of a membrane

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y, t) \qquad \text{-----------(7)}$$

if the forcing function f is of the form

$$f(x, y, t) = P(x, y)\sin(\omega t + \alpha)$$

we may write the steady state displacement u in the form

$$u = v(x, y)\sin(\omega t + \alpha) \qquad \text{------------(8)}$$

$$\Rightarrow \quad c^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \omega^2 v + p = 0$$

Applications

$$\int_R \left[c^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \omega^2 v + p \right] \delta v dx dy = 0$$

Consider

$$c^2 \int_R v_{xx} \delta v dx dy$$
$$= c^2 \int_R [(v_x \delta v)_x - v_x \delta v_x] dx dy$$

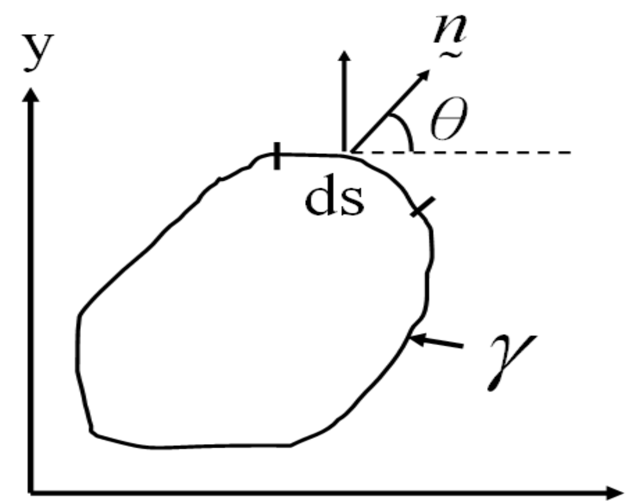
$$c^2 \int_R v_{yy} \delta v dx dy$$
$$= c^2 \int_R [(v_y \delta v)_y - v_y \delta v_y] dx dy$$

Note that $(v_x \delta v)_x = v_{xx} \delta v + v_x \delta v_x$ $\quad (v_y \delta v)_y = v_{yy} \delta v + v_y \delta v_y$

$$V = V_x i + V_y j \qquad V_x = v_x \delta v \qquad V_y = v_y \delta v$$

$$\underset{\sim}{n} = \cos\theta i + \sin\theta j$$

$$\nabla \bullet V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial (v_x \delta \mathrm{v})}{\partial x} + \frac{\partial (v_y \delta \mathrm{v})}{\partial y}$$



Applications

$$\int_{\Re} (\nabla \bullet V) da = \oint_{\gamma} V \bullet n \, ds \; = \oint (v_x \, \delta v \cos\theta + v_y \delta v \sin\theta) \mathrm{ds}$$

$$c^2 \int_R v_{xx} \delta v dxdy + \int_R c^2 v_{yy} \delta v dxdy$$

$$= c^2 \oint_{\gamma} v_x \delta v \cos\theta ds - \int_R \frac{1}{2} c^2 \delta (v_x)^2 dxdy$$

$$+ c^2 \oint_{\gamma} v_y \delta v \sin\theta ds - \int_R \frac{1}{2} c^2 \delta (v_y)^2 dxdy$$

$$\Rightarrow \oint_{\gamma} c^2 (v_x \cos\theta + v_y \sin\theta) \delta v ds - \int_R \frac{1}{2} c^2 \delta [(v_x)^2 + (v_y)^2] dxdy$$

$$+ \int_R \frac{1}{2} \omega_2 \delta (v^2) dxdy + \int_R P \delta v dxdy = 0$$

------------(9)

$$\Rightarrow \int_{\gamma} c^2 \frac{\partial v}{\partial n} \delta v ds - \delta \int_R [\frac{1}{2} c^2 (\nabla v)^2 - \frac{1}{2} \omega^2 v^2 - Pv] dxdy = 0$$

Applications

Hence,

i) if $v = f(x, y)$ is given on γ

i.e. $\delta v = 0$ on γ

then the variational problem

$$\Rightarrow \delta \int_R [\frac{c^2}{2}(\nabla v)^2 - \frac{1}{2}\omega^2 v^2 - pv]dxdy = 0 \qquad \text{---------(10)}$$

ii) if $\frac{\partial v}{\partial n} = 0$ is given on γ

the variation problem is same as Eq. (10)

iii) if $\frac{\partial v}{\partial n} = \psi(s)$ is given on γ

$$\Rightarrow \delta[\int_R \left\{\frac{1}{2}c^2(\nabla v)^2 - \frac{1}{2}\omega^2 v^2 - pv\right\}dxdy - \int_\tau c^2 \psi v dx] = 0 \qquad \text{----------(11)}$$

Applications

- **Diffusion Equation**

EX: Steady State Heat Condition

$$\nabla \cdot (k \nabla T) = f(x, T) \quad \text{in } D$$

B.C's :

$$\begin{aligned} T &= T_1 && \text{on } B_1 \\ -kn \cdot \nabla T &= q_2 && \text{on } B_2 \\ -kn \cdot \nabla T &= h(T - T_3) && \text{on } B_3 \end{aligned}$$

Multiply the equation by δT , and integrate over the domain D. After integrating by parts, we find the variational problem as follow.

$$\delta\Big[\int_D \left\{ \frac{1}{2} k (\nabla T)^2 + \int_{T_0}^{T} f(x, T') dT' \right\} d\tau + \int_{B_2} q_2 T d\sigma + \frac{1}{2} \int_{B_3} h (T - T_3)^2 d\sigma \Big] = 0$$

with $T = T_1$ on B_1.

Applications

- **Poisson's Equation**

EX: Torsion of a Prismatic Bar

$$\nabla^2 \psi = -2 \qquad \text{in} \quad R$$
$$\psi = 0 \qquad \text{on} \quad \gamma$$

where ψ is the Prandtl stress function and

$$\sigma_{\tau z} = G_\alpha \frac{\partial \psi}{\partial y} \quad , \quad \sigma_{zy} = \sigma_\alpha \frac{\partial \psi}{\partial x}$$

The variation problem becomes

$$\delta \left\{ \int_R [(D\psi)^2 - 4\psi] dxdy \right\} = 0$$

with $\psi = 0$ on γ .

Approximate Methods

I) Method of Weighted Residuals (MWR)

$$L[u] = 0 \quad \text{in} \quad D$$

with homogeneous b.c's in B.
Assume an approximate solution.

$$u = u_n = \sum_{i=1}^{n} C_i \phi_i$$

where each trial function ϕ_i satisfies the b.c's. The residual is

$$R_n = L[u_n]$$

In this method (MWR), C_i are chosen such that R_n is forced to be zero in an average sense.

i.e. $< w_j, R_n > = 0, \qquad j = 1,2,\ldots,n$

where w_j are the weighting functions..

Approximate Methods

II) Galerkin Method

w_j are chosen to be the trial functions ϕ_j hence the trial functions is chosen as members of a complete set of functions.

Galerkin method force the residual to be zero w.r.t. an orthogonal complete set.

EX: Torsion of a Square Shaft

$$\nabla^2\psi = -2$$

$$\psi = 0 \quad on \quad x = \pm a, \quad y = \pm a$$

i) One – term approximation

$$\psi_1 = c_1(x^2 - a^2)(y^2 - a^2)$$

$$R_i = \nabla^2\psi_1 + 2 = 2c_1[(x-a)^2 + (y-a)^2] + 2$$

$$\phi_1 = (x^2 - a^2)(y^2 - a^2)$$

Approximate Methods

From $\int_{-a}^{a}\int_{-a}^{a} R_1\phi_1 dxdy = 0$

$$\Rightarrow \; c_1 = \frac{5}{8}\frac{1}{a^2}$$

Therefore,

$$\psi_1 = \frac{5}{8a^2}(x^2 - a^2)(y^2 - a^2)$$

The torsional rigidity is determined by

$$D_1 = 2G\int_R \psi\, dxdy = 0.1388G(2a)^4$$

The exact value of D is

$$D_a = 0.1406G(2a)^4$$

The approximation error is -1.2%.

Approximate Methods

ii) Two – term approximation

$$\psi_2 = (x^2 - a^2)(y^2 - a^2)[c_1 + c_2(x^2 + y^2)]$$

By symmetry $\Longrightarrow$ $R_2 = \nabla \psi_2 + 2$

$$\phi_1 = (x^2 - a^2)(y^2 - a^2)$$

$$\phi_2 = (x^2 - a^2)(y^2 - a^2)(x^2 + y^2)$$

From $\int_R R_2 \phi_1 dxdy = 0$

and $\int_R R_2 \phi_2 dxdy = 0$

We obtain

$$c_1 = \frac{1295}{2216} \frac{1}{a^2}, \quad c_2 = \frac{525}{4432} \frac{1}{a^2}$$

Therefore

$$D_2 = 2G \int_R \psi_2 dxdy = 0.1404 G (2a)^4$$

→ The error is -0.14%.

Variational Methods

I) Kantorovich Method [Kantorovich (1948)]

Assuming the approximate solution as : $u = \sum_{i=1}^{n} C_i(x_n) U_i$

where U_i is a known function decided by b.c. condition.

C_i is a unknown function decided by minimal "I". $\Longrightarrow$ Euler Equation of C_i

EX : The Torsional Problem with a Functional "I".

$$I(u) = \int_{-a}^{a}\int_{-b}^{b}[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 - 4u]dxdy$$

Variational Methods

Assuming the one-term approximate solution as :

$$u(x,y) = (b^2 - y^2)C(x)$$

Then,

$$I(\mathrm{C}) = \int_{-a}^{a}\int_{-b}^{b} \{(b^2 - y^2)^2[\mathrm{C}'(x)]^2 + 4y^2\mathrm{C}^2(x) - 4(b^2 - y^2)\mathrm{C}(x)\}dxdy$$

Integrate by y

$$I(\mathrm{C}) = \int_{-a}^{a} [\frac{16}{15}b^5\mathrm{C}'^2 + \frac{8}{3}b^3\mathrm{C}^2 - \frac{16}{3}b^3\mathrm{C}]dx$$

Euler's equation is

$$\mathrm{C}''(x) - \frac{5}{2b^2}\mathrm{C}(x) = -\frac{5}{2b^2}$$ where b.c. condition is $\mathrm{C}(\pm a) = 0$

General solution is

$$\mathrm{C}(x) = \mathrm{A}_1 \cosh(\sqrt{\frac{5}{2}}\frac{x}{b}) + \mathrm{A}_2 \sinh(\sqrt{\frac{5}{2}}\frac{x}{b}) + 1$$

Variational Methods

where $A_1 = -\dfrac{1}{\cosh(\sqrt{\frac{5}{2}}\frac{a}{b})}, \quad A_2 = 0$

and

$$C(x) = \left\{1 - \frac{\cosh(\sqrt{\frac{5}{2}}\frac{x}{b})}{\cosh(\sqrt{\frac{5}{2}}\frac{a}{b})}\right\}$$

Therefore, the one-term approximate solution is

$$u = \left\{1 - \frac{\cosh(\sqrt{\frac{5}{2}}\frac{x}{b})}{\cosh(\sqrt{\frac{5}{2}}\frac{a}{b})}\right\}(b^2 - y^2)$$

Variational Methods

II) Rayleigh-Ritz Method

This is used when the exact solution is impossible or difficult to obtain.

First, we assume the approximate solution as : $u = \sum_{i=1}^{n} C_i U_i$

where U_i are some approximate function which satisfy the b.c's. Then, we can calculate extreme I .

$I = I(c_1, \ldots, c_n)$ Choose $c_1 \sim c_n$ i.e. $\frac{\partial I}{\partial c_1} = \ldots = \frac{\partial I}{\partial c_n} = 0$

EX: $y'' + xy = -x$ $y(0) = y(1) = 0$

Its solution can be obtained from

$$\int_0^1 \left(y'' + xy + x \right) \delta y dx = 0 \Rightarrow I = \int_0^1 \left[\frac{1}{2} \left(y' \right)^2 - \frac{1}{2} xy^2 - xy \right] dx$$

Variational Methods

Assuming that

$$y = x(1-x)(c_1 + c_2 x + c_3 x^2 \ldots)$$

i) One-term approximation

$$y = c_1 x(1-x) = c_1\left(x - x^2\right) \qquad y' = c_1(1-2x)$$

Then, $$I(c_1) = \int_0^1 \left[\frac{1}{2}c_1^{\,2}\left(1-4x+4x^2\right) - \frac{x}{2}c_1^{\,2}\left(x^2-2x^3+x^4\right) - c_1 x\left(x-x^2\right)\right]dx$$

$$= \frac{c_1^{\,2}}{2}\left(1-2+\frac{4}{3}\right) - \frac{c_1^{\,2}}{2}\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right) - c_1\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{19}{120}c_1^{\,2} - \frac{c_1}{12}$$

$$\frac{\partial I}{\partial c_1} = 0 \;\Rightarrow\; \frac{19}{60}c_1 - \frac{1}{12} = 0 \;\Rightarrow\; c_1 = 0.263 \;\Rightarrow\; y(1) = 0.263x(1-x)$$

ii) Two-term approximation

$$y = x(1-x)(c_1 + c_2 x) = c_1(x - x^2) + c_2(x^2 - x^3)$$

Variational Methods

Then $y' = c_1(1-2x) + c_2(2x-3x^2)$

$$I(c_1,c_2) = \int_0^1 [\frac{1}{2}\left\{c_1^{\ 2}\left(1-4x+4x^2\right)+2c_1c_2\left(2x-7x^2+6x^3\right)\right.$$

$$\left.+c_3^{\ 2}(4x^2-12x^3+9x^4)\right\}-\frac{1}{2}\left\{c_1^{\ 2}\left(x^3-2x^4+x^5\right)+2c_1c_2\left(x^4-2x^5+x^6\right)\right.$$

$$\left.+c_2^{\ 2}(x^5-2x^6+x^7)\right\}-\left\{c_1\left(x^2-x^3\right)+c_2\left(x^3-x^4\right)\right\}]dx$$

$$=\frac{c_1^{\ 2}}{2}\left(1-2+\frac{4}{3}-\frac{1}{4}+\frac{2}{5}-\frac{1}{6}\right)+c_1c_2\left(1-\frac{7}{3}+\frac{3}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}\right)$$

$$-\frac{c_1^{\ 2}}{2}\left(\frac{4}{3}-3+\frac{9}{5}-\frac{1}{6}+\frac{2}{7}-\frac{9}{8}\right)-\frac{c_1}{12}-\frac{c_2}{20}$$

$$=\frac{19}{120}c_1^{\ 2}+\frac{11}{70}c_1c_2+\frac{107}{1680}c_2^{\ 2}-\frac{c_1}{12}-\frac{c_2}{20}$$

Variational Methods

$$\frac{\partial I}{\partial c_1} = 0 \qquad \Longrightarrow \quad \frac{19}{60}c_1 + \frac{11}{70}c_2 = \frac{1}{12}$$

$$\frac{\partial I}{\partial c_2} = 0 \qquad \Longrightarrow \quad \frac{11}{70}c_1 + \frac{109}{840}c_2 = \frac{1}{20}$$

0.317 c_1 + 0.127 c_2 = 0.05 $\quad \Longrightarrow \quad$ c_1 = 0.177 , c_2 = 0.173

$$\Longrightarrow \quad y(2) = (0.177x - 0.173x^2)(1 - x)$$

Examples

I) The Brachistochrone (fastest descent) Curve Problem – Proposed by Johann Bernoulli (1696)

II) Structural Dynamics – Formulation of Governing Equation of Motion

Summary

- Many governing equations in physics and chemistry can be formulated by functionals. Finding the extremals of functionals can lead to the solutions in many problems in science and engineering.

- Euler's Equations provide the necessary condition for evaluating problems involving functionals. However, analyzing Euler's equations sometimes can be difficult.

- Calculus of Variations determines the extremals of functionals, even though the solution is only approximated.

- Calculus of Variations provides the theoretical basis for many methods in engineering, such as the Principle of Virtual Displacement (PVD) and the Finite Element Method (FEM).

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