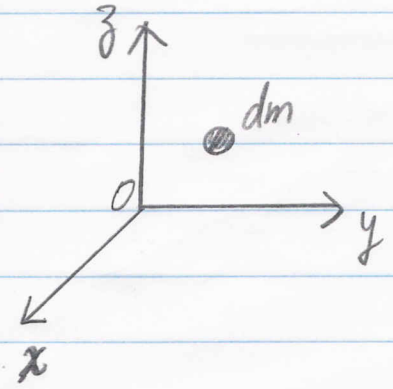
Moments and Products of Inertia

* Moment of inertia



$$I_{xx} = \int_m r_x^2 \, dm = \int_m (y^2 + z^2) \, dm$$

$$I_{yy} = \int_m r_y^2 \, dm = \int_m (x^2 + z^2) \, dm$$

$$I_{zz} = \int_m r_z^2 \, dm = \int_m (x^2 + y^2) \, dm$$

* Product of inertia

$$dI_{xy} = xy \, dm$$

$$I_{xy} = I_{yx} = \int_m xy \, dm$$

$$I_{yz} = I_{zy} = \int_m yz \, dm$$

$$I_{xz} = I_{zx} = \int_m xz \, dm$$

* Parallel-axis and parallel-axis theorems

$$I_{xx} = (I_{x'x'})_G + m(y_G^2 + z_G^2)$$

$$I_{yy} = (I_{y'y'})_G + m(x_G^2 + z_G^2)$$

$$I_{zz} = (I_{z'z'})_G + m(x_G^2 + y_G^2)$$

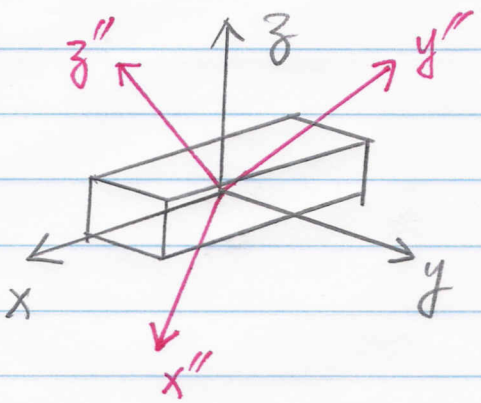
$$I_{xy} = (I_{x'y'})_G + m x_G y_G$$

$$I_{yz} = (I_{y'z'})_G + m y_G z_G$$

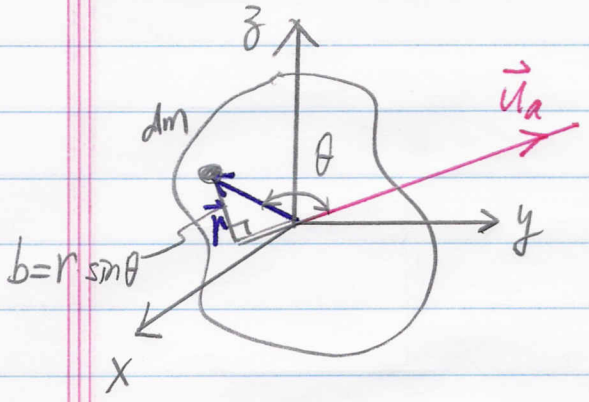
$$I_{zx} = (I_{z'x'})_G + m z_G x_G$$

$$\Rightarrow \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}_{x''y''z''}$$

$$\Rightarrow \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}_{xyz}$$



* Moment of inertia about an arbitrary axis



$$I_{oa} = \int_m b^2 \, dm$$

$\vec{b} \perp \vec{u}_a$

$$I_{oa} = \int_m \left| (\vec{u}_a \times \vec{r}) \right|^2 dm$$

$$= \int_m (\vec{u}_a \times \vec{r}) \cdot (\vec{u} \times \vec{r}) \, dm$$

With $\vec{u}_a = u_x \vec{i} + u_y \vec{j} + u_z \vec{k}$

$\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$

$$\Rightarrow \boxed{I_{oa} = I_{xx} u_x^2 + I_{yy} u_y^2 + I_{zz} u_z^2 - 2 I_{xy} u_x u_y - 2 I_{yz} u_y u_z - 2 I_{zx} u_z u_x}$$