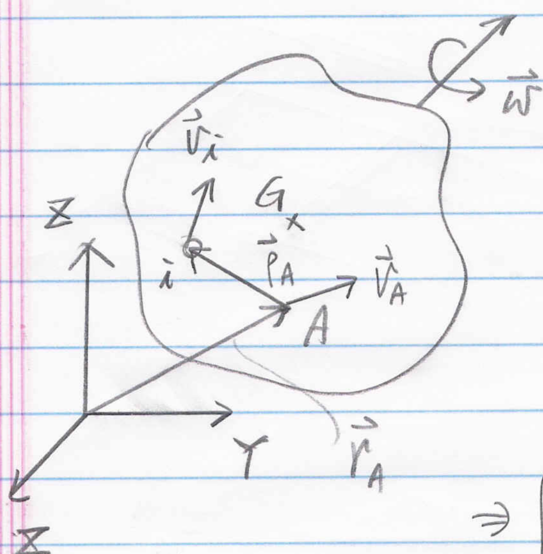
Angular Momentum



<u>Def:</u> $(\vec{H}_A)_i = \vec{\rho}_A \times m_i \vec{v}_i$

where $\vec{v}_i = \vec{v}_A + \vec{\omega} \times \vec{\rho}_A$

$$\Rightarrow (\vec{H}_A)_i = (\vec{\rho}_A m_i) \times \vec{v}_A + \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A) m_i$$

$$\Rightarrow \int_1^n (\vec{H}_A)_i \, di = \left(\int_m \vec{\rho}_A \, dm \right) \times \vec{v}_A + \int_m \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A) \, dm = \vec{H}_A$$

Since $\vec{\rho}_A = \vec{\rho}_G + \vec{\rho}_{G/A}$

$$\Rightarrow \vec{H}_A = \left[\int_m (\vec{\rho}_G + \vec{\rho}_{G/A}) \, dm \right] \times \vec{v}_A + \int_m (\vec{\rho}_G + \vec{\rho}_{G/A}) \times \left[\vec{\omega} \times (\vec{\rho}_G + \vec{\rho}_{G/A}) \right] dm$$

$$= \int_m \vec{\rho}_G \, dm \times \vec{v}_A + \int_m \vec{\rho}_{G/A} \, dm \times \vec{v}_A + \int_m \vec{\rho}_G \times \left[\vec{\omega} \times \vec{\rho}_G \right] dm + \int_m \vec{\rho}_G \times \left[\vec{\omega} \times \vec{\rho}_{G/A} \right] dm + \int_m \vec{\rho}_{G/A} \times \left[\vec{\omega} \times \vec{\rho}_G \right] dm + \int_m \vec{\rho}_{G/A} \times \left[\vec{\omega} \times \vec{\rho}_{G/A} \right] dm$$

$\because \int_m \vec{\rho}_G \, dm = 0$,

$$\int_m \vec{\rho}_{G/A} \, dm \times \vec{v}_A = \int_m dm \left(\vec{\rho}_{G/A} \times \vec{v}_A \right) = \left(\vec{\rho}_{G/A} \times \vec{v}_A \right) \cdot m$$

($\because$ $\vec{\rho}_{G/A}$, $\vec{v}_A$ are const.)

$$\Rightarrow \vec{H}_A = \left(\vec{\rho}_{G/A} \times \vec{v}_A \right) m + \int_m \vec{\rho}_G \times \left[\vec{\omega} \times \vec{\rho}_G \right] dm + \cancelto{0}{\int_m \vec{\rho}_G \, dm} \times \left[\vec{\omega} \times \underbrace{\vec{\rho}_{G/A}}_{\text{const.}} \right]$$
$$+ \underbrace{\vec{\rho}_{G/A}}_{} \times \left[\vec{\omega} \times \cancelto{0}{\int_m \vec{\rho}_G \, dm} \right] + \vec{\rho}_{G/A} \times \left[\vec{\omega} \times \vec{\rho}_{G/A} \right] \int_m dm$$

$$\Rightarrow \vec{H}_A = \left(\vec{\rho}_{G/A} \times \vec{v}_A \right) m + \int_m \vec{\rho}_G \times \left(\vec{\omega} \times \vec{\rho}_G \right) dm + \vec{\rho}_{G/A} \times \left(\vec{\omega} \times \vec{\rho}_{G/A} \right) m$$

$$= \vec{\rho}_{G/A} \times \underbrace{\left[\vec{v}_A + \left(\vec{\omega} \times \vec{\rho}_{G/A} \right) \right]}_{= \vec{v}_G} \cdot m + \int_m \vec{\rho}_G \times \left(\vec{\omega} \times \vec{\rho}_G \right) dm$$

$$\Rightarrow \boxed{\vec{H}_A = \left(\vec{\rho}_{G/A} \times m \vec{v}_G \right) + \vec{H}_G}$$

where $\boxed{\vec{H}_G = \int_m \vec{\rho}_G \times \left(\vec{\omega} \times \vec{\rho}_G \right) dm}$

or $\vec{H}_A = \vec{\rho}_A \times m \vec{v}_A$

Express $\vec{H}$ in Cartesian coordinate systems,

$$\vec{H} = \int_m \vec{\rho} \times (\vec{\omega} \times \vec{\rho})\, dm$$

$$\Rightarrow \begin{cases} H_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{cases}$$

When ① principal axes of inertia are used, or
② symmetrical objects are encountered,

$\Rightarrow$ ① $H_x = I_x \omega_x$, $H_y = I_y \omega_y$, $H_z = I_z \omega_z$

② $H_x = I_{xx}\omega_x$, $H_y = I_{yy}\omega_y$, $H_z = I_{zz}\omega_z$

In summary, we have

$$\begin{cases} m(\vec{V}_G)_1 + \sum \int_{t_1}^{t_2} \vec{F}\, dt = m(\vec{V}_G)_2 \longrightarrow \text{Conservation of linear momentum} \\ (\vec{H}_0)_1 + \sum \int_{t_1}^{t_2} \vec{M}_0\, dt = (\vec{H}_0)_2 \longrightarrow \text{Conservation of angular momentum} \end{cases}$$